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n–EXPONENTIAL CONVEXITY OF SOME DYNAMIC HARDY–TYPE FUNCTIONALS

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Abstract. Recently, some dynamic Hardy-type inequalities with certain kernels are studied in [5] with the help of arbitrary time scales. We use the positive linear functionals obtained from the results of [5] to give non trivial examples of n-exponentially convex functions.

1. Introduction and preliminary results

The theory of time scales can be studied in [1, 2, 3] and the well-known Hardy inequality as presented in [7] is investigated in [6, 11, 12, 13] under more general settings. Some of Hardy-type inequalities are extended on time scales (see [15, 16, 20]).

We start with some notions of time scales. Any nonempty closed subset of $\mathbb{R}$ is called a time scale $\mathbb{T}$. A time scale $\mathbb{T}$ may or may not be connected, keeping in mind the disconnection of time scales the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}.$$

In general, $\sigma(t) \geqslant t$ and $\rho(t) \leqslant t$. The mappings $\mu, \nu : \mathbb{T} \to [0, +\infty)$ defined by

$$\mu(t) = \sigma(t) - t$$

and

$$\nu(t) = t - \rho(t)$$

are called, respectively, the *forward* and *backward graininess functions*. For further properties including the concept of delta differentiation, we refer the reader to [3, 4].

Let $n \in \mathbb{N}$ be fixed. For each $i \in \{1, 2, \ldots, n\}$, let $\mathbb{T}_i$ denote a time scale and let σ_i, ρ_i and Δ_i denote the forward jump operator, the backward jump operator, and the delta differentiation operator, respectively. Let us set

$$\Omega^n = \{a = (a_1, a_2, \ldots, a_n) : \ a_i \in \mathbb{T}_i, \ 1 \leqslant i \leqslant n\}.$$

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We call Ω^n an n-dimensional time scale.

If $a \in \mathbb{T}$, where $\mathbb{T}$ is an arbitrary time scale, then the set $[a,\infty) = \{t \in \mathbb{T} : a \leqslant t\}$ is Δ-measurable.

An extended real-valued function $f : \Omega^n \to \overline{\mathbb{R}} := [-\infty,\infty]$ is Δ-measurable if for every $\alpha \in \mathbb{R}$, the set

$$f^{-1}([-\infty,\alpha)) = \{t = (t_1,t_2,\ldots,t_n) \in \Omega^n : f(t) < \alpha\}$$

is Δ-measurable. Note that f is Δ-measurable iff for each open set $G \subset \mathbb{R}$, the set $f^{-1}(G) = \{t \in \Omega^n : f(t) \in G\}$ is Δ-measurable. Moreover, if $f : \Omega^n \to \mathbb{R}$ is Δ-measurable and $g : I \to \mathbb{R}$ with $I \subset \mathbb{R}$ is a continuous function, then $g \circ f : \Omega^n \to \mathbb{R}$ is Δ-measurable.

Let $V = [a,b)$ be an n-dimensional time scale interval in Ω^n and let f be a bounded real-valued function on V. If f is Riemann Δ-integrable over V, then f is Lebesgue Δ-integrable over V and

$$R\int_V f(t)\Delta t = L\int_V f(t)\Delta t,$$

where R and L indicate the Riemann and Lebesgue Δ-integrals, respectively. In particular, if $[a,b) \subset \mathbb{T}$ contains only isolated points, then

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} (\sigma(t)-t)f(t),$$

where $\mathbb{T}$ is an arbitrary time scale.

Let $(\Omega,\mathscr{M},\mu_\Delta)$ and $(\Lambda,\mathscr{L},\lambda_\Delta)$ be two finite dimensional time scale measure spaces. We consider the measure space $(\Omega\times\Lambda,\mathscr{M}\times\mathscr{L},\mu_\Delta\times\lambda_\Delta)$, where $\mathscr{M}\times\mathscr{L}$ is σ-algebra product generated by the family $\{E\times F : E\in\mathscr{M}, F\in\mathscr{L}\}$ and

$$(\mu_\Delta\times\lambda_\Delta)(E\times F) = \mu_\Delta(E)\lambda_\Delta(F).$$

Recently in [5] the following extension of Hardy-type inequality for arbitrary time scales is constructed.

THEOREM 1.1. *Assume*

$$(\Omega,\mathscr{M},\mu_\Delta) \text{ and } (\Lambda,\mathscr{L},\lambda_\Delta) \text{ are two time scale measure spaces,} \tag{1}$$

$$k : \Omega\times\Lambda \to \mathbb{R}_+ \text{ is such that } K(x) := \int_\Lambda k(x,y)\Delta y < \infty,\ x\in\Omega \tag{2}$$

and

$$\xi : \Omega \to \mathbb{R}_+ \text{ is such that } w(y) := \int_\Omega \frac{k(x,y)\xi(x)}{K(x)}\Delta x < \infty,\ y\in\Lambda. \tag{3}$$

If $\Phi \in \mathrm{C}(I,\mathbb{R})$ *is convex, where* $I \subset \mathbb{R}$ *is an interval, then*

$$\int_\Omega \xi(x)\Phi((A_k f)(x))\Delta x \leqslant \int_\Lambda w(y)\Phi(f(y))\Delta y$$

holds for all λ_Δ*-integrable* $f:\Lambda\to\mathbb{R}$ *such that* $f(\Lambda)\subset I$*, where*

$$(A_k f)(x)=\frac{1}{K(x)}\int_\Lambda k(x,y)f(y)\Delta y.$$

In the following, the entries of a vector $x\in\mathbb{R}^n$ are called x_i, where $1\leqslant i\leqslant n$.

THEOREM 1.2. *Let* $\mathbb{T}$ *be a time scale and assume*

$$a_i,b_i\in\overline{\mathbb{T}},\ 0\leqslant a_i<b_i\leqslant\infty,\ 1\leqslant i\leqslant n,\ \Omega=\Lambda:=\times_{i=1}^n[a_i,b_i)_{\mathbb{T}}, \tag{4}$$

(2) *and*

$$u:\Omega\to\mathbb{R}_+ \text{ is such that } v(y):=\int_\Omega\frac{y_1\cdots y_n k(x,y)u(x)}{\sigma(x_1)\cdots\sigma(x_n)K(x)}\Delta x<\infty,\ y\in\Lambda. \tag{5}$$

If $\Phi\in\mathrm{C}(I,\mathbb{R})$ *is convex, where* $I\subset\mathbb{R}$ *is an interval, then*

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}u(x)\Phi((\hat{A}_k f)(x))\frac{\Delta x_1\cdots\Delta x_n}{\sigma(x_1)\cdots\sigma(x_n)}\leqslant\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}v(y)\Phi(f(y))\frac{\Delta y_1\cdots\Delta y_n}{y_1\cdots y_n} \tag{6}$$

holds for all λ_Δ*-integrable* $f:\Lambda\to\mathbb{R}$ *such that* $f(\Lambda)\subset I$*, where*

$$(\hat{A}_k f)(x):=\frac{1}{K(x)}\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}k(x,y)f(y)\Delta y_1\cdots\Delta y_n.$$

COROLLARY 1.3. *Assume* (4), (2) *and* (5) *with the kernel* k *such that*

$$k(x_1,\ldots,x_n,y_1,\ldots,y_n)=0\quad\text{if}\quad a_i\leqslant\sigma(x_i)\leqslant y_i\leqslant b,\ 1\leqslant i\leqslant n.$$

If $\Phi\in\mathrm{C}(I,\mathbb{R})$ *is convex, where* $I\subset\mathbb{R}$ *is an interval, then* (6) *holds for all* λ_Δ*-integrable* $f:\Lambda\to\mathbb{R}$ *such that* $f(\Lambda)\subset I$*, where*

$$K(x)=\int_{a_1}^{\sigma(x_1)}\cdots\int_{a_n}^{\sigma(x_n)}k(x_1,\ldots,x_n,y_1,\ldots,y_n)\Delta y_1\cdots\Delta y_n,$$

$$v(y)=y_1\cdots y_n\int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n}\frac{k(x_1,\ldots,x_n,y_1,\ldots,y_n)u(x_1,\ldots,x_n)}{\sigma(x_1)\cdots\sigma(x_n)K(x_1,\ldots,x_n)}\Delta x_1\cdots\Delta x_n$$

and

$$(\hat{A}_k f)(x)=\frac{1}{K(x)}\int_{a_1}^{\sigma(x_1)}\cdots\int_{a_n}^{\sigma(x_n)}k(x,y_1,\ldots,y_n)f(y_1,\ldots,y_n)\Delta y_1\cdots\Delta y_n.$$

THEOREM 1.4. ([5]) *Assume* (4) *and*

$$\xi:\Omega\to\mathbb{R}_+ \text{ is such that } \widetilde{w}(y):=\int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n}\frac{\xi(x_1,\ldots,x_n)}{\prod\limits_{i=1}^n(\sigma(x_i)-a_i)}\Delta x_1\cdots\Delta x_n<\infty,\ y\in\Lambda.$$

If $\Phi \in C(I,\mathbb{R})$ is convex, where $I \subset \mathbb{R}$ is an interval, then

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \xi(x_1,\ldots,x_n)\Phi\left((\widetilde{A}f)(x_1,\ldots,x_n)\right)\Delta x_1 \cdots \Delta x_n \leqslant \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \widetilde{w}(y_1,\ldots,y_n)\Phi(f(y_1,\ldots,y_n))\Delta y_1 \cdots \Delta y_n \quad (7)$$

holds for all λ_Δ-integrable $f : \Lambda \to \mathbb{R}$ such that $f(\Lambda) \subset I$, where

$$(\widetilde{A}f)(x) := \frac{1}{\prod_{i=1}^{n}(\sigma(x_i) - a_i)} \int_{a_1}^{\sigma(x_1)} \cdots \int_{a_n}^{\sigma(x_n)} f(y_1,\ldots,y_n)\Delta y_1 \cdots \Delta y_n.$$

REMARK 1.5. Under the considerations of Theorem 1.1, we have

$$\Upsilon_1(\Phi) := \int_\Lambda w(y)\Phi(f(y))\Delta y - \int_\Omega \xi(x)\Phi(A_k f)(x))\Delta x \geqslant 0.$$

From Theorem 1.2, we have

$$\Upsilon_2(\Phi) := \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} v(y)\Phi(f(y))\frac{\Delta y_1 \cdots \Delta y_n}{y_1 \cdots y_n} - \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(x)\Phi((\hat{A}_k f)(x))\frac{\Delta x_1 \cdots \Delta x_n}{\sigma(x_1)\cdots\sigma(x_n)} \geqslant 0.$$

From Theorem 1.4, we have

$$\Upsilon_3(\Phi) := \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \widetilde{w}(y_1,\ldots,y_n)\Phi(f(y_1,\ldots,y_n))\Delta y_1 \cdots \Delta y_n - \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \xi(x_1,\ldots,x_n)\Phi\left((\widetilde{A}f)(x_1,\ldots,x_n)\right)\Delta x_1 \cdots \Delta x_n \geqslant 0.$$

For simplicity, we use $\Upsilon(\Phi)$ instead of $\Upsilon_i(\Phi)$ $\forall$ $i \in \{1,2,3\}$.

Hence, for any convex function $\Phi \in C(I,\mathbb{R})$,

$$\Upsilon(\Phi) \geqslant 0.$$

2. n-exponential convexity

The notion of *n-exponentially convex* function and the following properties of *exponentially convex* function defined on an interval $I \subset \mathbb{R}$, are given in [17].

DEFINITION 1. A function $g : I \to \mathbb{R}$ is called *n-exponentially convex* in the Jensen sense if

$$\sum_{i,j=1}^{n} a_i a_j g\left(\frac{x_i + x_j}{2}\right) \geqslant 0$$

holds for every $a_i \in \mathbb{R}$ and every $x_i \in I$, $i \in \{1,2,\ldots,n\}$.

A function $g : I \to \mathbb{R}$ is *n-exponentially convex* if it is *n-exponentially convex* in the Jensen sense and continuous on I.

REMARK 2.1. From the definition it is clear that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are m-exponentially convex in the Jensen sense for every $m \in \mathbb{N}$, $m \leqslant n$.

DEFINITION 2. A function $g : I \to \mathbb{R}$ is exponentially convex in the Jensen sense, if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $g : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

REMARK 2.2. It is easy to see that a positive function $g : I \to \mathbb{R}$ is $\log$-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense, that is

$$a_1^2 g(x) + 2a_1 a_2 g\left(\frac{x+y}{2}\right) + a_2^2 g(y) \geqslant 0$$

holds for every $a_1, a_2 \in \mathbb{R}$ and $x, y \in I$.

Similarly, if g is 2-exponentially convex, then g is $\log$-convex. Conversely, if g is $\log$-convex and continuous, then g is 2-exponentially convex.

Divided differences are fertile to study functions having different degree of smoothness.

DEFINITION 3. The second order divided difference of a function $g : I \to \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by

$$[y_i; g] = g(y_i), \quad i \in \{0,1,2\}$$

$$[y_i, y_{i+1}; g] = \frac{g(y_{i+1}) - g(y_i)}{y_{i+1} - y_i}, \quad i \in \{0,1\}$$

$$[y_0, y_1, y_2; g] = \frac{[y_1, y_2; g] - [y_0, y_1; g]}{y_2 - y_0}. \tag{8}$$

REMARK 2.3. The value $[y_0, y_1, y_2; g]$ is independent of the order of the points y_0, y_1, and y_2. By taking limits this definition may be extended to include the cases in which any two or all three points coincide as follows: $\forall\ y_0,\ y_1,\ y_2\ \in I$ such that $y_2 \neq y_0$

$$\lim_{y_1 \to y_0} [y_0, y_1, y_2; g] = [y_0, y_0, y_2; g] = \frac{g(y_2) - g(y_0) - g'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}$$

provided that g' exists, and furthermore, taking the limits $y_i \to y_0, i \in \{1,2\}$ in (8), we get

$$[y_0,y_0,y_0;g] = \lim_{y_i \to y_0} [y_0,y_1,y_2;g] = \frac{g''(y_0)}{2} \text{ for } i \in \{1,2\}$$

provided that g'' exists on I.

In [9], the authors describe the n-exponential convexity for the functionals obtained from the inequalities of Hardy and Boas types.

In this paper we utilize the functional $\Upsilon(\Phi)$ given in Remark 1.5 to establish the n-exponential convexity via theory of time scales. Therefore our work is a continuation of results in [9].

THEOREM 2.4. *Assume $J \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t \mid t \in J\}$ is a family of continuous functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \to [y_0,y_1,y_2;\phi_t]$ $(t \in J)$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0,y_1,y_2 \in I$. Consider $\Upsilon(\Phi)$ as given in Remark 1.5. Then $t \to \Upsilon(\phi_t)$ $(t \in J)$ is an n-exponentially convex function in the Jensen sense on J.*

Also the function $t \to \Upsilon(\phi_t)$ (for any $t \in J$) is continuous, therefore it is n-exponentially convex on J.

Proof. Let $t_k,t_l \in J,\ t_{kl} := \dfrac{t_k+t_l}{2}$ and $b_k,b_l \in \mathbb{R}$ for $k,l \in \{1,2,\ldots,n\}$, and define the function ω on I by

$$\omega := \sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}.$$

Then ω is continuous on I being the linear combination of continuous functions. Also by hypothesis the function $t \to [y_0,y_1,y_2;\phi_t]$ $(t \in J)$ is n-exponentially convex in the Jensen sense, therefore we have

$$[y_0,y_1,y_2;\omega] = \sum_{k,l=1}^{n} b_k b_l [y_0,y_1,y_2;\phi_{t_{kl}}] \geqslant 0,$$

which implies that ω is a convex function on I. Therefore we have $\Upsilon(\omega) \geqslant 0$, which yields by the linearity of Υ, that

$$\sum_{k,l=1}^{n} b_k b_l \Upsilon(\phi_{t_{kl}}) \geqslant 0.$$

We conclude that the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is an n-exponentially convex function in the Jensen sense on J. □

As a consequence of the above theorem we can give the following corollaries.

COROLLARY 2.5. *Assume $J \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t \mid t \in J\}$ is a family of continuous functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \to [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Consider $\Upsilon(\Phi)$ as given in Remark 1.5. Then $t \to \Upsilon(\phi_t)$ $(t \in J)$ is an exponentially convex function in the Jensen sense on J.*

As the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is continuous, therefore exponentially convex on J.

COROLLARY 2.6. *Assume $J \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t : t \in J\}$ is a family of continuous functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \to [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Consider $\Upsilon(\Phi)$ as given in Remark 1.5. Then the following two statements hold:*

(i) As the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is continuous, therefore it is 2-exponentially convex on J, and thus $\log$-convex, i.e.,

$$\Upsilon^{(r-p)}(\phi_q) \leqslant \Upsilon^{(r-q)}(\phi_p)\Upsilon^{(q-p)}(\phi_r) \tag{9}$$

for $p, q, r \in J$ such that $p < q < r$.

(ii) If the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is positive, then for every $s, t, u, v \in J$, such that $s \leqslant u$ and $t \leqslant v$, we have

$$\mathfrak{u}_{s,t}(\Upsilon, \Lambda) \leqslant \mathfrak{u}_{u,v}(\Upsilon, \Lambda), \tag{10}$$

where

$$\mathfrak{u}_{s,t}(\Upsilon, \Lambda) := \begin{cases} \left(\dfrac{\Upsilon(\phi_s)}{\Upsilon(\phi_t)}\right)^{\frac{1}{s-t}}; & s \neq t, \\[2ex] \exp\left(\dfrac{\frac{d}{ds}\Upsilon(\phi_s)}{\Upsilon(\phi_s)}\right); & s = t \end{cases} \tag{11}$$

for $\phi_s, \phi_t \in \Lambda$. Also we consider that the function $t \to \Upsilon(\phi_t)$ is differentiable when $t = s$.

Proof.

(i) See Remark 2.2 and Theorem 2.4.

(ii) From the definition of a convex function ψ on J, we have the following inequality (see [18, page 2])

$$\frac{\psi(s) - \psi(t)}{s - t} \leqslant \frac{\psi(u) - \psi(v)}{u - v}, \tag{12}$$

$\forall s,t,u,v \in J$ such that $s \leqslant u$, $t \leqslant v$, $s \neq t$, $u \neq v$.
By (i), $s \to \Upsilon(\phi_s)$, $s \in J$ is log-convex, and hence using $\psi(s) = \log \Upsilon(\phi_s)$, $s \in J$ in (12), we have

$$\frac{\log \Upsilon(\phi_s) - \log \Upsilon(\phi_t)}{s-t} \leqslant \frac{\log \Upsilon(\phi_u) - \log \Upsilon(\phi_v)}{u-v} \tag{13}$$

for $s \leqslant u, t \leqslant v, s \neq t, u \neq v$, which is equivalent to (10). For $s = t$ or $u = v$, (10) follows from (13) by taking limit. □

REMARK 2.7. Note that the results from Theorem 2.4, Corollary 2.5, Corollary 2.6 are valid when two of the points $y_0, y_1, y_2 \in I$ coincide, say $y_1 = y_0$, for a family of differentiable functions ϕ_t such that the function $t \to [y_0, y_1, y_2; \phi_t]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and moreover, they are also valid when all three points coincide for a family of twice differentiable functions with the same property. The proofs can be obtained by recalling Remark 2.3 and suitable characterization of convexity.

The following result is given in [8].

THEOREM 2.8. *Assume $J \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t \mid t \in J\}$ is a family of twice differentiable functions defined on an interval $I \subset \mathbb{R}$ such that the function $t \mapsto \phi_t''(x)$ $(t \in J)$ is exponentially convex for every fixed $x \in I$. Then the function $t \mapsto [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is exponentially convex in the Jensen sense for any three points $y_0, y_1, y_2 \in I$.*

3. Applications to Cauchy means

In this section, first we give the mean value theorems corresponding to the Hardy-type functional $\Upsilon(\Phi)$ given in Remark 1.5.

THEOREM 3.1. *Let $[a,b] \subset \mathbb{R}$ and consider the linear functional $\Upsilon(\Phi)$ as defined in Remark 1.5 for $\Phi = g \in (C^2[a,b], \mathbb{R})$, then there exists $\xi \in [a,b]$ such that*

$$\Upsilon(g) = \frac{1}{2} g''(\xi) \Upsilon\left(x^2\right).$$

Proof. The idea of proof is same as given in [8]. □

THEOREM 3.2. *Let $[a,b] \subset \mathbb{R}$ and consider the linear functional $\Upsilon(\Phi)$ as defined in Remark 1.5 for $g, h \in C^2[a,b]$. Then there exists $\xi \in [a,b]$ such that*

$$\frac{\Upsilon(g)}{\Upsilon(h)} = \frac{g''(\xi)}{h''(\xi)}, \tag{14}$$

provided that $\Upsilon(h) \neq 0$.

Proof. The idea of proof is same as given in [8]. □

Suppose that $\frac{g''}{h''}$ has inverse function. Then (14) gives

$$\xi = \left(\frac{g''}{h''}\right)^{-1}\left(\frac{\Upsilon(g)}{\Upsilon(h)}\right). \tag{15}$$

Now, we generate new Cauchy means with the help of some classes of functions from [17].

EXAMPLE 3.3. Assume $I = \mathbb{R}$ and consider the class of continuous convex functions

$$\Lambda_1 := \{\phi_t : \mathbb{R} \to [0,\infty) \mid t \in \mathbb{R}\},$$

where

$$\phi_t(x) := \begin{cases} \frac{1}{t^2}e^{tx}; & t \neq 0, \\ \frac{1}{2}x^2; & t = 0. \end{cases}$$

Then $t \mapsto \phi_t''(x)$ $(t \in \mathbb{R})$ is exponentially convex for every fixed $x \in \mathbb{R}$ (see [10]), thus by Theorem 2.8, the function $t \mapsto [y_0, y_1, y_2; \phi_t]$, $t \in \mathbb{R}$ is exponentially convex in the Jensen sense for every three mutually different points $y_0, y_1, y_2 \in \mathbb{R}$.

By applying Corollary 2.5 with $\Lambda = \Lambda_1$, we get the exponential convexity of $t \mapsto \Upsilon(\phi_t)$ $(t \in \mathbb{R})$ in the Jensen sense. This mapping is also differentiable, therefore exponentially convex, and the expression in (11) has the form

$$\mathfrak{u}_{s,t}(\Upsilon, \Lambda_1) = \begin{cases} \left(\frac{\Upsilon(\phi_s)}{\Upsilon(\phi_t)}\right)^{\frac{1}{s-t}}; & s \neq t, \\ \exp\left(\frac{\Upsilon(id\,\phi_s)}{\Upsilon(\phi_s)} - \frac{2}{s}\right); & s = t \neq 0, \\ \exp\left(\frac{\Upsilon(id\,\phi_0)}{3\Upsilon(\phi_0)}\right); & s = t = 0, \end{cases}$$

where "id" means the identity function on $\mathbb{R}$.

From (10) we have the monotonicity of the functions $\mathfrak{u}_{s,t}(\Upsilon, \Lambda_1)$ in both parameters.

Suppose $\Upsilon(\phi_t) > 0$ $(t \in \mathbb{R})$, and let

$$\mathfrak{M}_{s,t}(\Upsilon, \Lambda_1) := \log \mathfrak{u}_{s,t}(\Upsilon, \Lambda_1); \quad s, t \in \mathbb{R}.$$

Then from (15) we have

$$a \leqslant \mathfrak{M}_{s,t}(\Upsilon, \Lambda_1) \leqslant b,$$

and thus $\mathfrak{M}_{s,t}(\Upsilon, \Lambda_1)$ $(s, t \in \mathbb{R})$ are means. The monotonicity of these means is followed by (10).

EXAMPLE 3.4. Assume $I = (0,\infty)$ and consider the class of continuous convex functions

$$\Lambda_2 = \{\psi_t : (0,\infty) \to \mathbb{R} \mid t \in \mathbb{R}\},$$

where

$$\psi_t(x) := \begin{cases} \dfrac{x^t}{t(t-1)}; & t \neq 0,1, \\ -\log x; & t = 0, \\ x\log x; & t = 1. \end{cases}$$

Then $t \mapsto \psi_t''(x) = x^{t-2} = e^{(t-2)\log x}$ $(t \in \mathbb{R})$ is exponentially convex for every fixed $x \in (0,\infty)$.

By similar arguments as given in Example 3.3 we get the exponential convexity of $t \mapsto \Upsilon(\psi_t)$ $(t \in \mathbb{R})$ in the Jensen sense. This mapping is differentiable too, therefore exponentially convex. From (11) we have the following Cauchy means

$$M_{s,t} = \left(\frac{t(t-1)}{s(s-1)} \frac{\int_\Lambda w(y)(f(y))^s \Delta y - \int_\Omega \xi(x)(R(x))^s \Delta x}{\int_\Lambda w(y)(f(y))^t \Delta y - \int_\Omega \xi(x)(R(x))^t \Delta x} \right)^{\frac{1}{s-t}}; \ s \neq t \neq 0,1,$$

$$M_{s,s} = \exp\left(\frac{1-2s}{s(s-1)} + \frac{\int_\Lambda w(y)(f(y))^s \log(f(y))\Delta y - \int_\Omega \xi(x)(R(x))^s \log(R(x))\Delta x}{\int_\Lambda w(y)(f(y))^s \Delta y - \int_\Omega \xi(x)(R(x))^s \Delta x} \right); \quad s \neq 0,1,$$

$$M_{s,0} = \left(\frac{-1}{s(s-1)} \frac{\int_\Lambda w(y)(f(y))^s \Delta y - \int_\Omega \xi(x)(R(x))^s \Delta x}{\int_\Lambda w(y)\log(f(y))\Delta y - \int_\Omega \xi(x)\log(R(x))\Delta x} \right)^{\frac{1}{s}}; \ s \neq 0,1,$$

$$M_{s,1} = \left(\frac{1}{s(s-1)} \frac{\int_\Lambda w(y)(f(y))^s \Delta y - \int_\Omega \xi(x)(R(x))^s \Delta x}{\int_\Lambda w(y)f(y)\log(f(y))\Delta y - \int_\Omega \xi(x)R(x)\log(R(x))\Delta x} \right)^{\frac{1}{s-1}}; \ s \neq 0,1,$$

$$M_{0,1} = -\frac{\int_\Lambda w(y)f(y)\log(f(y))\Delta y - \int_\Omega \xi(x)R(x)\log(R(x))\Delta x}{\int_\Lambda w(y)\log(f(y))\Delta y - \int_\Omega \xi(x)\log(R(x))\Delta x},$$

$$M_{1,1} = \exp\left(-1 + \frac{1}{2} \frac{\int_\Lambda w(y)f(y)(\log(f(y)))^2 \Delta y - \int_\Omega \xi(x)R(x)(\log(R(x)))^2 \Delta x}{\int_\Lambda w(y)f(y)\log(f(y))\Delta y - \int_\Omega \xi(x)R(x)\log(R(x))\Delta x} \right),$$

$$M_{0,0} = \exp\left(1 + \frac{1}{2} \frac{\int_\Lambda w(y)(\log(f(y)))^2 \Delta y - \int_\Omega \xi(x)(\log(R(x)))^2 \Delta x}{\int_\Lambda w(y)\log(f(y))\Delta y - \int_\Omega \xi(x)\log(R(x))\Delta x} \right).$$

where

$$R(x) \doteq (A_k f)(x) = \frac{1}{K(x)} \int_\Lambda k(x,y) f(y) \Delta y$$

and $w(y) := \displaystyle\int_\Omega \frac{k(x,y)\xi(x)}{K(x)} \Delta x$.

The means $M_{s,t}$ $(s,t \in \mathbb{R})$ are continuous, symmetric and monotone in both parameters (using (10)).

For the class Λ_2, we have

$$\Upsilon_1(\phi_p) = \begin{cases} \frac{1}{p(p-1)}\left(\int_\Lambda w(y) f^p(y)\Delta y - \int_\Omega \xi(x)(R(x))^p \Delta x\right); \ p \neq 0,1, \\ -\int_\Lambda w(y)\log f(y)\Delta y + \int_\Omega \xi(x)\log(R(x))\Delta x; \ p = 0, \\ \int_\Lambda w(y) f(y)\log f(y)\Delta y - \int_\Omega \xi(x)R(x)\log(R(x))\Delta x; \ p = 1. \end{cases} \tag{16}$$

For (16), (9) gives the following improvement result; for $p = 0 < q < 1 = r$, we have

$$\begin{aligned} &\frac{1}{q(q-1)}\left(\int_\Lambda w(y) f^q(y)\Delta y - \int_\Omega \xi(x)(R(x))^q \Delta x\right) \\ &\leqslant \left(-\int_\Lambda w(y)\log f(y)\Delta y + \int_\Omega \xi(x)\log(R(x))\Delta x\right)^{1-q} \\ &\quad \times \left(\int_\Lambda w(y) f(y)\log f(y)\Delta y - \int_\Omega \xi(x)R(x)\log(R(x))\Delta x\right)^q. \end{aligned} \tag{17}$$

If $q < 0 < 1$ or $0 < 1 < q$, then we have reverse inequality in (17).

Observe that (17) is a refinement of inequality given in [5, Corollary 3.3].

Similar results can be written for $i \in \{2,3\}$.

Particularly, for $i = 3$, $n = 1$, we can write

$$\Upsilon_3(\phi_p) = \begin{cases} \frac{1}{p(p-1)}\left(\int_a^b \widetilde{w}(y) f^p(y)\Delta y - \int_a^b \xi(x)\left(\widetilde{R}(x)\right)^p \Delta x\right); \ p \neq 0,1, \\ -\int_a^b \widetilde{w}(y)\log f(y)\Delta y + \int_a^b \xi(x)\log\left(\widetilde{R}(x)\right)\Delta x; \quad p = 0, \\ \int_a^b \widetilde{w}(y) f(y)\log f(y)\Delta y - \int_a^b \xi(x)\widetilde{R}(x)\log\left(\widetilde{R}(x)\right)\Delta x; \ p = 1. \end{cases} \tag{18}$$

where

$$\widetilde{R}(x) \doteq (\widetilde{A}_k f)(x) = \frac{1}{\sigma(x) - a}\int_a^{\sigma(x)} f(y)\Delta y$$

and $\widetilde{w}(y) = \int_y^\infty \frac{\xi(x)\Delta x}{\sigma(x) - a}$.

For $0 < q < 1$, using (18) in (9) we have the following inequality

$$\frac{1}{q(q-1)}\left(\int_a^b \widetilde{w}(y)f^q(y)\Delta y - \int_a^b \xi(x)\left(\widetilde{R}(x)\right)^q \Delta x\right)$$
$$\leqslant \left(-\int_a^b \widetilde{w}(y)\log f(y)\Delta y + \int_a^b \xi(x)\log\left(\widetilde{R}(x)\right)\Delta x\right)^{1-q} \tag{19}$$
$$\times\left(\int_a^b \widetilde{w}(y)f(y)\log f(y)\Delta y - \int_a^b \xi(x)\widetilde{R}(x)\log\left(\widetilde{R}(x))\right)\Delta x\right)^q.$$

If $q < 0 < 1$ or $0 < 1 < q$, then we have reverse inequality in (19).

4. Applications to time scales consists of isolated points

Now, we consider some particular cases corresponding to examples from [5].

Let us take $\Omega = \Lambda = [a,\infty)_{\mathbb{T}} \doteq [0,\infty)\cap\mathbb{T}$, $a \geqslant 0$. Further assume that time scale $\mathbb{T}$ consists of isolated points.

In this case (18) takes the form

$$\Upsilon_3(\phi_p) = \begin{cases} \frac{1}{p(p-1)}\left(\sum\limits_{[a,b)_{\mathbb{T}}} \widetilde{w}(y)(f(y))^p\mu(y) - \sum\limits_{[a,b)_{\mathbb{T}}} \xi(x)(\widetilde{R}(x))^p\mu(x)\right); & p \neq 0,1, \\ -\log\left(\prod\limits_{[a,b)_{\mathbb{T}}} (f(y))^{\widetilde{w}(y)\mu(y)}\right) + \log\left(\prod\limits_{[a,b)_{\mathbb{T}}} (\widetilde{R}(x))^{\xi(x)\mu(x)}\right); & p = 0, \\ \log\left(\prod\limits_{[a,b)_{\mathbb{T}}} (f(y))^{\widetilde{w}(y)f(y)\mu(y)}\right) - \log\left(\prod\limits_{[a,b)_{\mathbb{T}}} (\widetilde{R}(x))^{\xi(x)\widetilde{R}(x)\mu(x)}\right); & p = 1, \end{cases} \tag{20}$$

where

$$\widetilde{R}(x) = \left(\frac{1}{\sigma(x)-a}\sum_{y\in[a,x]_{\mathbb{T}}} f(y)\mu(y)\right)$$

and $\widetilde{w}(y) = \left(\sum\limits_{x\in[y,\infty)_{\mathbb{T}}} \xi(x)\dfrac{\mu(x)}{\sigma(x)-a}\right)$.

Also, for $0 < q < 1$, (19) takes the form

$$\frac{1}{q(q-1)}\left(\sum_{[a,b)_{\mathbb{T}}} \widetilde{w}(y)(f(y))^q\mu(y) - \sum_{[a,b)_{\mathbb{T}}} \xi(x)(\widetilde{R}(x))^q\mu(x)\right)$$
$$\leqslant \left(\log\left(\frac{\prod\limits_{[a,b)_{\mathbb{T}}} (R(x))^{\xi(x)\mu(x)}}{\prod\limits_{[a,b)_{\mathbb{T}}} (f(y))^{\widetilde{w}(y)\mu(y)}}\right)\right)^{1-q}\left(\log\left(\frac{\prod\limits_{[a,b)_{\mathbb{T}}} (f(y))^{\widetilde{w}(y)f(y)\mu(y)}}{\prod\limits_{[a,b)_{\mathbb{T}}} (\widetilde{R}(x))^{\xi(x)\widetilde{R}(x)\mu(x)}}\right)\right)^q. \tag{21}$$

For $\mathbb{T} = h\mathbb{N} = \{hn : n \in \mathbb{N}\}$ with $h > 0$, $a = h$, and $\xi(x) = \dfrac{1}{\sigma(x)}$, (20) takes the form

$$\Upsilon_3(\phi_p) = \begin{cases} \dfrac{1}{q(q-1)}\left(\sum\limits_{n=1}^{\infty}\dfrac{(f(nh))^p}{n} - \sum\limits_{n=1}^{\infty}\dfrac{1}{n+1}\left(\widetilde{R}(nh)\right)^p\right); \ p \neq 0,1, \\ -\log\left(\prod\limits_{n=1}^{\infty}(f(nh))^{\frac{1}{n}}\right) + \log\left(\prod\limits_{n=1}^{\infty}(\widetilde{R}(nh))^{\frac{1}{n+1}}\right); \ p = 0, \\ \log\left(\prod\limits_{n=1}^{\infty}(f(nh))^{\frac{(f(nh))}{n}}\right) - \log\left(\prod\limits_{n=1}^{\infty}(\widetilde{R}(nh))^{\frac{\widetilde{R}(nh)}{n+1}}\right); \ p = 1. \end{cases}$$

For $0 < q < 1$, (21) takes the form

$$\begin{aligned} &\frac{1}{q(q-1)}\left(\sum_{n=1}^{\infty}\frac{(f(nh))^q}{n} - \sum_{n=1}^{\infty}\frac{1}{n+1}\left(\widetilde{R}(nh)\right)^q\right) \\ &\leqslant \left(\log\left(\frac{\prod_{n=1}^{\infty}(\widetilde{R}(nh))^{\frac{1}{n+1}}}{\prod_{n=1}^{\infty}(f(nh))^{\frac{1}{n}}}\right)\right)^{1-q}\left(\log\left(\frac{\prod_{n=1}^{\infty}(f(nh))^{\frac{(f(nh))}{n}}}{\prod_{n=1}^{\infty}(\widetilde{R}(nh))^{\frac{\widetilde{R}(nh)}{n+1}}}\right)\right)^q, \end{aligned} \tag{22}$$

where

$$\widetilde{R}(nh) = \frac{1}{n}\sum_{k=1}^{n} f(kh).$$

For $\mathbb{T} = \mathbb{N}^2 = \{n^2 : n \in \mathbb{N}\}$ with $a = 1$ and

$$\xi(x) = \frac{2(\sigma(x)-1)}{(\sigma(x)-x)^2(2\sqrt{x}+3)},$$

(20) takes the form

$$\Upsilon_3(\phi_p) = \begin{cases} \frac{1}{p(p-1)}\left(\sum\limits_{n=1}^{\infty}(f(n^2))^p - \sum\limits_{n=1}^{\infty}\frac{2n(n+2)}{(2n+1)(2n+3)}\left(\widetilde{R}(n^2)\right)^p\right); \ p \neq 0,1, \\ -\log\left(\prod\limits_{n=1}^{\infty}f(n^2)\right) + \log\left(\prod\limits_{n=1}^{\infty}(\widetilde{R}(n^2))^{\frac{2n(n+2)}{(2n+1)(2n+3)}}\right); \ p = 0, \\ \log\left(\prod\limits_{n=1}^{\infty}f(n^2)^{f(n^2)}\right) - \log\left(\prod\limits_{n=1}^{\infty}(\widetilde{R}(n^2))^{\frac{2n(n+2)}{(2n+1)(2n+3)}\widetilde{R}(n^2)}\right); \ p = 1. \end{cases}$$

For $0<q<1$, (21) takes the form

$$\frac{1}{q(q-1)}\left(\sum_{n=1}^{\infty}(f(n^2))^q-\sum_{n=1}^{\infty}\frac{2n(n+2)}{(2n+1)(2n+3)}\left(\widetilde{R}(n^2)\right)^q\right)$$
$$\leqslant\left(\log\left(\frac{\prod_{n=1}^{\infty}(\widetilde{R}(n^2))^{\frac{2n(n+2)}{(2n+1)(2n+3)}}}{\prod_{n=1}^{\infty}f(n^2)}\right)\right)^{1-q}\left(\log\left(\frac{\prod_{n=1}^{\infty}f(n^2)^{f(n^2)}}{\prod_{n=1}^{\infty}(\widetilde{R}(n^2))^{\frac{2n(n+2)}{(2n+1)(2n+3)}\widetilde{R}(n^2)}}\right)\right)^q, \tag{23}$$

where

$$\widetilde{R}(n^2)=\frac{\sum_{k=1}^{n}(2k+1)f(k^2)}{n(n+2)}.$$

For $\mathbb{T}=\lambda^{\mathbb{N}}=\{\lambda^n : n\in\mathbb{N}\}$ with $\lambda>1$, $a=\lambda$ and

$$\xi(x)=\frac{\sigma(x)-a}{\sigma(x)(\sigma(x)-x)},$$

(20) takes the form

$$\Upsilon_3(\phi_p)=\begin{cases}\frac{1}{p(p-1)}\left(\sum_{n=1}^{\infty}(f(\lambda^n))^p-\sum_{n=1}^{\infty}\lambda^{-n}(\lambda^n-1)(\widetilde{R}(\lambda^n))^p\right); & p\neq 0,1,\\ -\log\left(\prod_{n=1}^{\infty}f(\lambda^n)\right)+\log\left(\prod_{n=1}^{\infty}(\widetilde{R}(\lambda^n))^{\lambda^{-n}(\lambda^n-1)}\right); & p=0,\\ \log\left(\prod_{n=1}^{\infty}(f(\lambda^n))^{f(\lambda^n)}\right)-\log\left(\prod_{n=1}^{\infty}(\widetilde{R}(\lambda^n))^{\lambda^{-n}(\lambda^n-1)\widetilde{R}(\lambda^n)}\right); & p=1.\end{cases}$$

For $0<q<1$, (21) takes the form

$$\frac{1}{q(q-1)}\left(\sum_{n=1}^{\infty}(f(\lambda^n))^q-\sum_{n=1}^{\infty}\lambda^{-n}(\lambda^n-1)(\widetilde{R}(\lambda^n))^q\right)$$
$$\leqslant\left(\log\left(\frac{\prod_{n=1}^{\infty}(\widetilde{R}(\lambda^n))^{\lambda^{-n}(\lambda^n-1)}}{\prod_{n=1}^{\infty}f(\lambda^n)}\right)\right)^{1-q}\left(\log\left(\frac{\prod_{n=1}^{\infty}(f(\lambda^n))^{f(\lambda^n)}}{\prod_{n=1}^{\infty}(\widetilde{R}(\lambda^n))^{\lambda^{-n}(\lambda^n-1)\widetilde{R}(\lambda^n)}}\right)\right)^q, \tag{24}$$

where

$$\widetilde{R}(\lambda^n)=\frac{(\lambda-1)\sum_{k=1}^{n}\lambda^{k-1}f(\lambda^k)}{\lambda^n-1}.$$

REMARK 4.1. (a) If $q<0<1$ or $0<1<q$, then we have reverse inequalities in (22), (23) and (24).

(b) The inequalities (22), (23) and (24) are refinements of (5.9), first inequality given in Example 5.11, and (5.11) of [5] respectively.

EXAMPLE 4.2. Assume $I=(0,\infty)$ and consider the class of continuous convex functions

$$\Lambda_3=\{\eta_t:(0,\infty)\to(0,\infty)\mid t\in(0,\infty)\},$$

where

$$\eta_t(x):=\begin{cases}\dfrac{t^{-x}}{\log^2 t}; & t\neq 1,\\ \dfrac{x^2}{2}; & t=1.\end{cases}$$

$t\mapsto \eta_t''(x)$ $(t\in(0,\infty))$ is exponentially convex for every fixed $x\in(0,\infty)$, being the restriction of the Laplace transform of a nonnegative function (see [10] or [19] page 210).

We can get the exponential convexity of $t\mapsto \Upsilon(\psi_t)$ $(t\in\mathbb{R}^+)$ as in Example 3.3. For the class Λ_3, (11) has the form

$$\mathfrak{u}_{s,t}(\Upsilon,\Lambda_3)=\begin{cases}\left(\dfrac{\Upsilon(\eta_s)}{\Upsilon(\eta_t)}\right)^{\frac{1}{s-t}}; & s\neq t,\\ \exp\left(-\dfrac{2}{s\log s}-\dfrac{\Upsilon(id\,\eta_s)}{s\Upsilon(\eta_s)}\right); & s=t\neq 1,\\ \exp\left(-\dfrac{\Upsilon(id\,\eta_1)}{3\Upsilon(\eta_1)}\right); & s=t=1.\end{cases}$$

The monotonicity of $\mathfrak{u}_{s,t}(\Upsilon,\Lambda_3)$ $(s,t\in(0,\infty))$ comes from (10).

Suppose $\Upsilon(\eta_t)>0$ $(t\in(0,\infty))$, and define

$$\mathfrak{M}_{s,t}(\Upsilon,\Lambda_3):=-L(s,t)\log\mathfrak{u}_{s,t}(\Upsilon,\Lambda_3),\quad s,t\in(0,\infty),$$

where $L(s,t)$ is the well known logarithmic mean

$$L(s,t):=\begin{cases}\dfrac{s-t}{\log s-\log t}; & s\neq t,\\ t; & s=t.\end{cases}$$

From (15) we have

$$a\leqslant \mathfrak{M}_{s,t}(\Upsilon,\Lambda_3)\leqslant b,\quad s,t\in(0,\infty),$$

and therefore we get means.

EXAMPLE 4.3. Assume $I=(0,\infty)$ and consider the class of continuous convex functions

$$\Lambda_4=\{\gamma_t:(0,\infty)\to(0,\infty)\mid t\in(0,\infty)\},$$

where

$$\gamma_t(x) := \frac{e^{-x\sqrt{t}}}{t}.$$

$t \mapsto \gamma_t''(x) = e^{-x\sqrt{t}}$, $t \in (0,\infty)$ is exponentially convex for every fixed $x \in (0,\infty)$, being the restriction of the Laplace transform of a non-negative function (see [10] or [19] page 214).

As before $t \mapsto \Upsilon(\psi_t)$ $(t \in \mathbb{R}^+)$ is exponentially convex and differentiable. For the class Λ_4, (11) becomes

$$\mathfrak{u}_{s,t}(\Upsilon,\Lambda_4) = \begin{cases} \left(\dfrac{\Upsilon(\gamma_s)}{\Upsilon(\gamma_t)}\right)^{\frac{1}{s-t}}; & s \neq t, \\ \exp\left(-\dfrac{1}{t} - \dfrac{\Upsilon(id\gamma_t)}{2\sqrt{t}\Upsilon(\gamma_t)}\right); & s = t, \end{cases}$$

where 'id' means the identity function on $(0,\infty)$. The monotonicity of $\mathfrak{u}_{s,t}(\Upsilon,\Lambda_4)$ $(s,t \in (0,\infty))$ is followed by (10).

Suppose $\Upsilon(\eta_t) > 0$ $(t \in (0,\infty))$ and define

$$\mathfrak{M}_{s,t}(\Upsilon,\Lambda_4) := -(\sqrt{s}+\sqrt{t})\log \mathfrak{u}_{s,t}(\Upsilon,\Lambda_4), \quad s,t \in (0,\infty).$$

Then (15) yields that

$$a \leqslant \mathfrak{M}_{s,t}(\Upsilon,\Lambda_4) \leqslant b,$$

thus we have new means.

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