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# SCHUR–CONVEXITY, SCHUR GEOMETRIC AND SCHUR HARMONIC CONVEXITIES OF DUAL FORM OF A CLASS SYMMETRIC FUNCTIONS

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*Abstract.* By the properties of Schur-convex function, Schur geometrically convex function and Schur harmonically convex function, Schur-convexity, Schur geometric and Schur harmonic convexities of the dual form for a class of symmetric functions are simply proved. As an application, several inequalities are obtained, some of which extend the known ones.

## 1. Introduction

Throughout the article, $\mathbb{R}$ denotes the set of real numbers, $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ denotes $n$-tuple ($n$-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\mathbf{x} = (x_1, \cdots, x_n) : x_i \in \mathbb{R}, i = 1, \cdots, n\},$$

$$\mathbb{R}^n_+ = \{\mathbf{x} = (x_1, \cdots, x_n) : x_i > 0, i = 1, \cdots, n\}.$$

In particular, the notations $\mathbb{R}$ and $\mathbb{R}_+$ denote $\mathbb{R}^1$ and $\mathbb{R}^1_+$, respectively.

For convenience, we introduce some definitions as follows.

DEFINITION 1. [9, 14] Let $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$.

$(i)$ $\mathbf{x} \geqslant \mathbf{y}$ means $x_i \geqslant y_i$ for all $i = 1, 2, \cdots, n$.

$(ii)$ Let $\Omega \subset \mathbb{R}^n$, $\varphi \colon \Omega \to \mathbb{R}$ is said to be increasing if $\mathbf{x} \geqslant \mathbf{y}$ implies $\varphi(\mathbf{x}) \geqslant \varphi(\mathbf{y})$. $\varphi$ is said to be decreasing if and only if $-\varphi$ is increasing.

DEFINITION 2. [9, 14] Let $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$.

$(i)$ $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (in symbols $\mathbf{x} \prec \mathbf{y}$) if $\sum_{i=1}^k x_{[i]} \leqslant \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \cdots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geqslant \cdots \geqslant x_{[n]}$ and $y_{[1]} \geqslant \cdots \geqslant y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order.

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$(ii)$ Let $\Omega \subset \mathbb{R}^n$, $\varphi\colon\ \Omega \to \mathbb{R}$ is said to be a Schur-convex function on $\Omega$ if $\mathbf{x} \prec \mathbf{y}$ on $\Omega$ implies $\varphi(\mathbf{x}) \leqslant \varphi(\mathbf{y})$. $\varphi$ is said to be a Schur-concave function on $\Omega$ if and only if $-\varphi$ is Schur-convex function on $\Omega$.

DEFINITION 3. [9, 14] Let $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$.

$(i)$ $\Omega \subset \mathbb{R}^n$ is said to be a convex set if $\mathbf{x}, \mathbf{y} \in \Omega$, $0 \leqslant \alpha \leqslant 1$ implies $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} = (\alpha x_1 + (1-\alpha)y_1, \cdots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.

$(ii)$ Let $\Omega \subset \mathbb{R}^n$ be a convex set. A function $\varphi\colon\ \Omega \to \mathbb{R}$ is said to be a convex function on $\Omega$ if

$$\varphi(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leqslant \alpha\varphi(\mathbf{x}) + (1-\alpha)\varphi(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \Omega$, and all $\alpha \in [0,1]$. $\varphi$ is said to be a concave function on $\Omega$ if and only if $-\varphi$ is a convex function on $\Omega$.

$(iii)$ Let $\Omega \subset \mathbb{R}^n$. A function $\varphi\colon\ \Omega \to \mathbb{R}$ is said to be a log-convex function on $\Omega$ if the function $\log\varphi$ is convex.

THEOREM A. (Schur-Convex Function Decision Theorem) [9, p. 84] *Let $\Omega \subset \mathbb{R}^n$ be symmetric and have a nonempty interior convex set. $\Omega^0$ is the interior of $\Omega$. $\varphi : \Omega \to \mathbb{R}$ is continuous on $\Omega$ and differentiable in $\Omega^0$. Then $\varphi$ is the $Schur-convex$ $(Schur-concave)$ function if and only if $\varphi$ is symmetric on $\Omega$ and*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \geqslant 0 (\leqslant 0) \tag{1}$$

*holds for any $\mathbf{x} \in \Omega^0$.*

The Schur-convexity described the ordering of majorization, the order-preserving functions were first comprehensively studied by Issai Schur in 1923. It has important applications in combinatorial analysis, analytic inequalities, matrix theory, numerical analysis, and so on. See [9], [11], [4], [12], [24].

DEFINITION 4. [23] Let $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n_+$ and $\mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n_+$.

$(i)$ $\Omega \subset \mathbb{R}^n_+$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, \cdots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\mathbf{x}, \mathbf{y} \in \Omega$ and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$.

$(ii)$ Let $\Omega \subset \mathbb{R}^n_+$. The function $\varphi\colon\ \Omega \to \mathbb{R}_+$ is said to be a Schur geometrically convex function on $\Omega$ if $(\log x_1, \cdots, \log x_n) \prec (\log y_1, \cdots, \log y_n)$ on $\Omega$ implies $\varphi(\mathbf{x}) \leqslant \varphi(\mathbf{y})$. The function $\varphi$ is said to be a Schur geometrically concave function on $\Omega$ if and only if $-\varphi$ is a Schur geometrically convex function.

THEOREM B. (Schur Geometrically Convex Function Decision Theorem) [23] *Let* $\Omega \subset \mathbb{R}_+^n$ *be a symmetric and geometrically convex set with a nonempty interior* $\Omega^0$. *Let* $\varphi : \Omega \to \mathbb{R}_+$ *be continuous on* $\Omega$ *and differentiable in* $\Omega^0$. *If* $\varphi$ *is symmetric on* $\Omega$ *and*

$$(\log x_1 - \log x_2)\left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2}\right) \geqslant 0 \quad (\leqslant 0) \tag{2}$$

*holds for any* $\mathbf{x} = (x_1, \cdots, x_n) \in \Omega^0$, *then* $\varphi$ *is a Schur geometrically convex* (*Schur geometrically concave*) *function.*

The Schur geometrically convex function was proposed by Zhang [23] in 2004, and was investigated by Chu et al. [6], Guan [7], Jiang [8], Sun et al. [13], Xia et al. [18], and so forth. We also note that some authors use the term "Schur multiplicative convexity".

In 2009, Chu ([2], [3], [1], [20]) introduced the notion of Schur harmonically convex function. Some interesting inequalities were obtained, see e.g. [5], [17], [19], [21], [22].

DEFINITION 5. [2] Let $\Omega \subset \mathbb{R}_+^n$.

(*i*) A set $\Omega$ is said to be harmonically convex if $\frac{\mathbf{xy}}{\lambda \mathbf{x} + (1-\lambda)\mathbf{y}} \in \Omega$ for every $\mathbf{x}, \mathbf{y} \in \Omega$ and $\lambda \in [0,1]$, where $\mathbf{xy} = \sum_{i=1}^n x_i y_i$ and $\frac{1}{\mathbf{x}} = \left(\frac{1}{x_1}, \cdots, \frac{1}{x_n}\right)$.

(*ii*) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be Schur harmonically convex on $\Omega$ if $\frac{1}{\mathbf{x}} \prec \frac{1}{\mathbf{y}}$ implies $\varphi(\mathbf{x}) \leqslant \varphi(\mathbf{y})$. A function $\varphi$ is said to be a Schur harmonically concave function on $\Omega$ if and only if $-\varphi$ is a Schur harmonically convex function.

THEOREM C. (Schur Harmonically Convex Function Decision Theorem) [2] *Let* $\Omega \subset \mathbb{R}_+^n$ *be a symmetric and harmonically convex set with inner points and let* $\varphi : \Omega \to \mathbb{R}_+$ *be a continuously symmetric function which is differentiable on* $\Omega^0$. *Then* $\varphi$ *is Schur harmonically convex* (*Schur harmonically concave*) *on* $\Omega$ *if and only if*

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \varphi(\mathbf{x})}{\partial x_1} - x_2^2 \frac{\partial \varphi(\mathbf{x})}{\partial x_2}\right) \geqslant 0 \quad (\leqslant 0), \quad \mathbf{x} \in \Omega^0. \tag{3}$$

Let interval $I \subset \mathbb{R}$ and let $f : I \to \mathbb{R}_+$ be a log-convex function. Define the symmetric function $F_k$ by

$$F_k(\mathbf{x}) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \prod_{j=1}^k f(x_{i_j}), \quad k = 1, \cdots, n. \tag{4}$$

In 2010, for 1,2 and $n-1$, Rovenţa [10] proved that $F_k(\mathbf{x})$ is a Schur-convex function on $I^n$, but without discuss the case of $2 < k < n-1$. In 2011, Shu-hong Wang et al. [15] studied completely Schur-convexity, Schur geometric and Schur harmonic convexities of $F_k(\mathbf{x})$ on $I^n$, using the above decision theorems, i.e. Theorem A, Theorem B and Theorem C respectively proved the following three theorems.

THEOREM D. *Let $I \subset \mathbb{R}$ is a symmetric convex set with nonempty interior and let $f : I \to \mathbb{R}$ be continuous on $I$ and differentiable in the interior of $I$. If $f$ is a log-convex function, then for any $k = 1, 2, \cdots, n$, $F_k(\mathbf{x})$ is a Schur-convex function on $I^n$*

THEOREM E. *Let $I \subset \mathbb{R}_+$ is a symmetric convex set with nonempty interior and let $f : I \to \mathbb{R}_+$ be continuous on $I$ and differentiable in the interior of $I$. If $f$ is an increasing log-convex function, then for any $k = 1, 2, \cdots, n$, $F_k(\mathbf{x})$ is a Schur geometrically convex function on $I^n$.*

THEOREM F. *Let $I \subset \mathbb{R}_+$ is a symmetric convex set with nonempty interior and let $f : I \to \mathbb{R}_+$ be continuous on $I$ and differentiable in the interior of $I$. If $f$ is an increasing log-convex function, then for any $k = 1, 2, \cdots, n$, $F_k(\mathbf{x})$ is a Schur harmonically convex function on $I^n$.*

In this paper, we study the dual form of $F_k(\mathbf{x})$:

$$F_k^*(\mathbf{x}) = \prod_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \sum_{j=1}^{k} f(x_{i_j}), \quad k = 1, \cdots, n. \tag{5}$$

By some properties of the Schur-convex function, Schur geometrically convex function and Schur harmonically convex function, we obtained the following results:

THEOREM 1. *Let $I \subset \mathbb{R}$ is a symmetric convex set with nonempty interior and let $f : I \to \mathbb{R}$ be continuous on $I$ and differentiable in the interior of $I$. If $f$ is a log-convex function, then for any $k = 1, 2, \cdots, n$, $F_k^*(\mathbf{x})$ is a Schur-convex function on $I^n$*

THEOREM 2. *Let $I \subset \mathbb{R}_+$ is a symmetric convex set with nonempty interior and let $f : I \to \mathbb{R}_+$ be continuous on $I$ and differentiable in the interior of $I$. If $f$ is an increasing log-convex function, then for any $k = 1, 2, \cdots, n$, $F_k^*(\mathbf{x})$ is a Schur geometrically convex function on $I^n$.*

THEOREM 3. *Let $I \subset \mathbb{R}_+$ is a symmetric convex set with nonempty interior and let $f : I \to \mathbb{R}_+$ be continuous on $I$ and differentiable in the interior of $I$. If $f$ is an increasing log-convex function, then for any $k = 1, 2, \cdots, n$, $F_k^*(\mathbf{x})$ is a Schur harmonically convex function on $I^n$.*

## 2. Lemmas

To prove the above three theorems, we need the following lemmas.

LEMMA 1. [9, p. 97], [14] *If $\varphi$ is symmetric and convex (concave) on a symmetric convex set $\Omega$, then $\varphi$ is Schur-convex (Schur-concave) on $\Omega$.*

LEMMA 2. [14, p. 64] *Let $\Omega \subset \mathbb{R}^n$, $\varphi$: $\Omega \to \mathbb{R}_+$. Then* $\log\varphi$ *is Schur-convex (Schur-concave) if and only if $\varphi$ is Schur-convex (Schur-concave).*

LEMMA 3. [9, p. 642], [14] *Let* $\Omega \subset \mathbb{R}^n$ *be an open convex set,* $\varphi : \Omega \to \mathbb{R}$. *For* $\mathbf{x}, \mathbf{y} \in \Omega$, *define one variable function* $g(t) = \varphi(t\mathbf{x} + (1-t)\mathbf{y})$ *on the interval* $(0,1)$. *Then* $\varphi$ *is convex* (*concave*) *on* $\Omega$ *if and only if* $g$ *is convex* (*concave*) *on* $[0,1]$ *for all* $\mathbf{x}, \mathbf{y} \in \Omega$.

LEMMA 4. *Let* $\mathbf{x} = (x_1, \cdots, x_m)$ *and* $\mathbf{y} = (y_1, \cdots, y_m) \in \mathbb{R}^m$. *If* $f$ *is a log-convex function, then the functions* $p(t) = \log g(t)$ *is convex on* $[0,1]$, *where*

$$g(t) = \sum_{j=1}^{m} f(tx_j + (1-t)y_j).$$

*Proof.*

$$p'(t) = \frac{g'(t)}{g(t)},$$

where

$$g'(t) = \sum_{j=1}^{m} (x_j - y_j) f'(tx_j + (1-t)y_j).$$

$$p''(t) = \frac{g''(t)g(t) - (g'(t))^2}{g^2(t)},$$

where

$$g''(t) = \sum_{j=1}^{m} (x_j - y_j)^2 f''(tx_j + (1-t)y_j).$$

By the Cauchy inequality, we have

$$\begin{aligned}
& g''(t)g(t) - (g'(t))^2 \\
& = \left( \sum_{j=1}^{m} (x_j - y_j)^2 f''(tx_j + (1-t)y_j) \right) \left( \sum_{j=1}^{m} f(tx_j + (1-t)y_j) \right) \\
& - \left( \sum_{j=1}^{m} (x_j - y_j) f'(tx_j + (1-t)y_j) \right)^2 \\
& \geqslant \left( \sum_{j=1}^{m} |x_j - y_j| \sqrt{f''(tx_j + (1-t)y_j)} \cdot \sqrt{f(tx_j + (1-t)y_j)} \right)^2 \\
& - \left( \sum_{j=1}^{m} (x_j - y_j) f'(tx_j + (1-t)y_j) \right)^2
\end{aligned}$$

From the log-convexity of $f$ it follows that $(\log f(u))'' = \frac{f''(u)f(u) - (f'(u))^2}{f^2(u)} \geqslant 0$, hence

$$\sqrt{f''(tx_j + (1-t)y_j)} \cdot \sqrt{f(tx_j + (1-t)y_j)} \geqslant f'(tx_j + (1-t)y_j),$$

and then $g''(t)g(t) - (g'(t))^2 \geqslant 0$, i.e. $p''(t) \geqslant 0$, that is, $p(t) = \log g(t)$ is convex on $[0,1]$.

The proof of Lemma 4 is completed. □

LEMMA 5. *Let*

$$f(t) = \frac{x^t - 1}{t}.$$

*If* $x > 1$*, then* $f(t)$ *is a log-convex function on* $\mathbb{R}_+$*.*

*Proof.* By computing, we have

$$(\log f(t))'' = -\frac{x^t(\log x)^2}{(x^t - 1)^2} + \frac{1}{t^2}.$$

We need only prove $(\log f(t))'' \geqslant 0$. It equivalent to

$$t^2 x^t (\log x)^2 \leqslant (x^t - 1)^2. \tag{6}$$

In both sides the inequality (6), dividing by $x^t$ and extracting the square root, then the inequality (6) equivalent to

$$g(t) := x^{\frac{t}{2}} - x^{-\frac{t}{2}} - t\log x \geqslant 0.$$

When $x > 1$, $g'(t) = \frac{1}{2}\log x\left(x^{\frac{t}{2}} + x^{-\frac{t}{2}} - 2\right) = \frac{1}{2}\log(x)\left(x^{\frac{t}{2}} - 1\right)^2 x^{-\frac{t}{2}} \geqslant 0$, hence $g(t)$ is increasing on $\mathbb{R}_+$, and then $g(t) \geqslant g(0) = 0$, that is $(\log f(t))'' \geqslant 0$.

The proof of Lemma 5 is completed. □

## 3. Proof of Main Results

*Proof of Theorem 1.* For any $1 \leqslant i_1 < \cdots < i_k \leqslant n$, by Lemma 3 and Lemma 4, it follows that $\log \sum_{j=1}^{k} f(x_{i_j})$ is convex on $I^k$. Obviously, $\log \sum_{j=1}^{k} f(x_{i_j})$ is also convex on $I^n$, and then $\log F_k^*(\mathbf{x}) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \log \sum_{j=1}^{k} f(x_{i_j})$ is convex on $I^n$. Furthermore, it is clear that $\log F_k^*(\mathbf{x})$ is symmetric on $I^n$, by Lemma 1, it follows that $\log F_k^*(\mathbf{x})$ is Schur-convex on $I^n$, and then from Lemma 2 we conclude that $F_k^*(\mathbf{x})$ is also Schur-convex on $I^n$.

The proof of Theorem 1 is completed. □

*Proof of Theorem 2.* For $\mathbf{x} \in I^n \subset \mathbb{R}_+^n$ and $x_1 \neq x_2$, we have

$$\begin{aligned}
\Delta &= (\log x_1 - \log x_2)\left(x_1 \frac{\partial F_k^*}{\partial x_1} - x_2 \frac{\partial F_k^*}{\partial x_2}\right) \\
&= (\log x_1 - \log x_2)\left(x_1 \frac{\partial F_k^*}{\partial x_1} - x_1 \frac{\partial F_k^*}{\partial x_2} + x_1 \frac{\partial F_k^*}{\partial x_2} - x_2 \frac{\partial F_k^*}{\partial x_2}\right) \\
&= x_1 \frac{\log x_1 - \log x_2}{x_1 - x_2}(x_1 - x_2)\left(\frac{\partial F_k^*}{\partial x_1} - \frac{\partial F_k^*}{\partial x_2}\right) + \frac{\partial F_k^*}{\partial x_2}(x_1 - x_2)(\log x_1 - \log x_2).
\end{aligned}$$

Since $F_k^*(\mathbf{x})$ is Schur-convex on $I^n$, by Theorem A, we have

$$(x_1 - x_2)\left(\frac{\partial F_k^*}{\partial x_1} - \frac{\partial F_k^*}{\partial x_2}\right) \geqslant 0.$$

Notice that $f$ and $\log t$ is increasing, we have

$$\frac{\partial F_k^*}{\partial x_2} \geqslant 0,$$

$$\frac{\log x_1 - \log x_2}{x_1 - x_2} \geqslant 0$$

and

$$(x_1 - x_2)(\log x_1 - \log x_2) \geqslant 0,$$

so that $\Delta \geqslant 0$, by Theorem B, it follows that $F_k^*(\mathbf{x})$ is Schur geometric convex on $I^n$. □

*Proof of Theorem 3.* The proof of Theorem 3 similar to Theorem 2, the detailed proof is left to the reader. □

REMARK 1. If using the decision theorems, i.e. Theorem A, Theorem B and Theorem C respectively direct to prove Theorem 1, Theorem 2 and Theorem 3, I am afraid not above proofs are simple, interested readers may wish to try.

## 4. Applications

THEOREM 4. *The symmetric function*

$$Q_k(\mathbf{x}) = \prod_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \sum_{j=1}^{k} \frac{1 + x_{i_j}}{1 - x_{i_j}}, \quad k = 1, \cdots, n. \tag{7}$$

*is Schur-convex function, Schur geometrically and Schur harmonically convex function on* $(0,1)^n$. *And for* $\mathbf{x} \in (0,1)^n$, *we have*

$$\prod_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \sum_{j=1}^{k} \frac{1 + x_{i_j}}{1 - x_{i_j}} \geqslant \left( \frac{k(n+s)}{n-s} \right)^{\binom{n}{k}}, \quad k = 1, \cdots, n. \tag{8}$$

*where* $s = \sum_{i=1}^{n} x_i$ *and* $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

*Proof.* Let $f(x) = \frac{1+x}{1-x}, x \in (0,1)$. By computing, we have $f'(x) = \frac{2}{(1-x)^2} > 0$ and $(\log f(x))'' = \frac{4x}{(1+x)^2(1-x)^2} \geqslant 0$, that is, $f$ is an increasing log-convex function. By Theorem 1, Theorem 2 and Theorem 3, it follows that $Q_k(\mathbf{x})$ is respectively Schur-convex function, Schur geometrically and Schur harmonically convex function on $(0,1)^n$.

Since $\mathbf{y} = \left(\frac{s}{n}, \frac{s}{n}, \cdots, \frac{s}{n}\right) \prec \mathbf{x} = (x_1, x_2, \cdots, x_n)$, from Schur-convexity of $Q_k(\mathbf{x})$, it follows that $Q_k(\mathbf{y}) \leqslant Q_k(\mathbf{x})$, i.e. inequality (8) holds.

The proof of Theorem 4 is completed. □

REMARK 2. Specially, taking $k=1,\ s=1$, from the inequality (8) we can get the known Klamkin inequality:

$$\prod_{i=1}^{n}\frac{1+x_i}{1-x_i}\geqslant\left(\frac{n+1}{n-1}\right)^{n}. \tag{9}$$

By analogous proof with Theorem 4, we can obtain the following theorem.

THEOREM 5. *The symmetric function*

$$R_k(\mathbf{x})=\prod_{1\leqslant i_1<\cdots<i_k\leqslant n}\sum\nolimits_{j=1}^{k}\frac{x_{i_j}}{1-x_{i_j}},\quad k=1,\cdots,n. \tag{10}$$

*is Schur-convex function, Schur geometrically and Schur harmonically convex function on* $[\frac{1}{2},1)^n$. *And for* $\mathbf{x}\in[\frac{1}{2},1)^n$, *we have*

$$\prod_{1\leqslant i_1<\cdots<i_k\leqslant n}\sum\nolimits_{j=1}^{k}\frac{x_{i_j}}{1-x_{i_j}}\geqslant\left(\frac{ks}{n-s}\right)^{\binom{n}{k}},\quad k=1,\cdots,n. \tag{11}$$

*where* $s=\sum_{i=1}^{n}x_i$ *and* $\binom{n}{k}=\frac{n!}{k!(n-k)!}$.

THEOREM 6. *The symmetric function*

$$D_k(\mathbf{x})=\prod_{1\leqslant i_1<\cdots<i_k\leqslant n}\sum\nolimits_{j=1}^{k}x_{i_j}^{x_{i_j}},\quad k=1,\cdots,n. \tag{12}$$

*is Schur-convex on* $\mathbb{R}_+^n$ *and Schur geometric and Schur harmonic convex on* $[e^{-1},\infty)^n$. *And for* $\mathbf{x}\in\mathbb{R}_+^n$, *we have*

$$\prod_{1\leqslant i_1<\cdots<i_k\leqslant n}\sum\nolimits_{j=1}^{k}x_{i_j}^{x_{i_j}}\geqslant\left(k[A(\mathbf{x})]^{A(\mathbf{x})}\right)^{\binom{n}{k}},\quad k=1,\cdots,n. \tag{13}$$

*where* $A(\mathbf{x})=\frac{1}{n}\sum_{i=1}^{n}x_i$ *and* $\binom{n}{k}=\frac{n!}{k!(n-k)!}$.

*Proof.* It is not difficult to verify that $x^x$ is log-convex function on $(0,\infty)$ and increasing on $[e^{-1},\infty)$. By Theorem 1, Theorem 2 and Theorem 3, it follows that $D_k(\mathbf{x})$ is Schur-convex on $\mathbb{R}_+^n$, is Schur geometric and Schur harmonic convex on $[e^{-1},\infty)^n$.

Since $\mathbf{y}=(A(\mathbf{x}),A(\mathbf{x}),\cdots,A(\mathbf{x}))\prec\mathbf{x}=(x_1,x_2,\cdots,x_n)$, from Schur-convexity of $D_k(\mathbf{x})$, it follows that $D_k(\mathbf{y})\leqslant D_k(\mathbf{x})$, i.e. inequality (13) holds.

The proof of Theorem 6 is completed. □

From Lemma 5 and Theorem 1, we can obtain the following Theorem 7.

THEOREM 7. *Let* $x>1$.

$$P_k(\mathbf{t})=\prod_{1\leqslant i_1<\cdots<i_k\leqslant n}\sum\nolimits_{j=1}^{k}\frac{x^{t_{i_j}}-1}{t_{i_j}},\quad k=1,\cdots,n. \tag{14}$$

*is Schur-convex on* $\mathbb{R}_+^n$.

*And for* $\mathbf{p},\mathbf{q} \in \mathbb{R}_+^n$ *and* $\mathbf{p} \prec \mathbf{q}$, *we have*

$$\prod_{1\leqslant i_1<\cdots<i_k\leqslant n} \sum_{j=1}^k \frac{x^{p_{i_j}}-1}{p_{i_j}} \leqslant \prod_{1\leqslant i_1<\cdots<i_k\leqslant n} \sum_{j=1}^k \frac{x^{q_{i_j}}-1}{q_{i_j}}, \quad k=1,\cdots,n. \tag{15}$$

REMARK 3.

$(i)$ Taking $n=2, k=1$ and $\mathbf{p}=(m,m)$, $\mathbf{q}=(m+r,m-r)$, from the inequality (15) we can get the known inequality:

$$(x^{m-r}-1)(x^{m+r}-1) \geqslant \left(1-\frac{r^2}{m^2}\right)(x^m-1)^2, \tag{16}$$

where $r \in \mathbb{N}, m \geqslant 2, r < m$.

$(ii)$ Taking $k=1$, from the inequality (15) we can get the inequality (3) in [16].

$$\prod_{j=1}^n q_j\,(x^{p_j}-1) \leqslant \prod_{j=1}^n p_j\,(x^{q_j}-1). \tag{17}$$

$(iii)$ Taking $k=n$, from the inequality (15) we can get the inequality:

$$\sum_{i=1}^n \frac{x^{p_i}-1}{p_i} \leqslant \sum_{i=1}^n \frac{x^{q_i}-1}{q_i}. \tag{18}$$

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