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NOTE ON SOME UPPER BOUNDS FOR THE CONDITION NUMBER

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Abstract. In this letter, some lower bounds for the smallest singular value of the nonsingular matrix are established. In addition, we also proposed some upper bounds on the condition number of a matrix which are the better than the bound proposed by Guggenheimer et al. [College Math. J. 26(1) (1995) 2-5]. To illustrate our bounds, some examples are also given.

1. Introduction

Denote by $\mathscr{M}_n$ the set of all $n\times n$ nonsingular complex matrices. Let σ_i $(i=1,\cdots,n)$ be the singular values of $A\in\mathscr{M}_n$ and assume that $\sigma_1\geqslant\sigma_2\geqslant\cdots\geqslant\sigma_{n-1}\geqslant\sigma_n>0$. The condition number of $A\in\mathscr{M}_n$ is defined by

$$\mathscr{K}(A)=\frac{\sigma_1}{\sigma_n}.$$

The Frobenius norm of $A=(a_{ij})\in\mathscr{M}_n$ is defined by

$$\|A\|_F=\left(\sum_{i,j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}.$$

By the relationship between the Frobenius norm and singular value, we have

$$\|A\|_F^2=\sum_{i=1}^{n}\sigma_i^2.$$

Recently, some lower bounds for the smallest singular value of nonsingular matrix have been proposed [2-8], as well as some upper bounds for the condition number of the nonsingular matrix. Yu and Gu [7] established the following lower bound:

$$\sigma_n\geqslant|\mathrm{det}A|\left(\frac{n-1}{\|A\|_F^2}\right)^{\frac{n-1}{2}}>0, \tag{1}$$

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Inequality (1) is also seen in [5]. Zou [9] obtained a lower bound based on (1) as follows:

$$\sigma_n \geqslant |\det A| \left(\frac{n-1}{\|A\|_F^2 - l^2} \right)^{\frac{n-1}{2}} > 0, \tag{2}$$

where $l = |\det A| \left(\frac{n-1}{\|A\|_F^2} \right)^{\frac{n-1}{2}}$. Apparently, the bound of [9] is better than one of [7]. Guggenheimer et al. [1] gave the following upper bound:

$$\mathscr{K}(A) < \frac{2}{|\det A|} \left(\frac{\|A\|_F^2}{n} \right)^{\frac{n}{2}}.$$

In this paper, we propose some lower bounds for the smallest singular value σ_n and some upper bounds for the condition number of the nonsingular matrix.

2. Main results

To prove our main results, firstly we will give a lemma.

LEMMA 1. *If $A \in \mathscr{M}_n (n \geqslant 2)$, then*

$$\sigma_n > \frac{\sigma_k}{2} |\det A| \left(\frac{n}{\|A\|_F^2} \right)^{\frac{n}{2}} \left[1 + \frac{1}{n} \left(\frac{\sigma_k}{2} \right)^2 |\det A|^2 \left(\frac{n}{\|A\|_F^2} \right)^{n+1} \right], \tag{3}$$

and moreover,

$$\sigma_n > \frac{\sigma_k}{2} |\det A| \left(\frac{n}{\|A\|_F^2} \right)^{\frac{n}{2}}, \tag{4}$$

where $1 \leqslant k \leqslant n-1$.

Proof. Let

$$\Upsilon = \sigma_1^2 \cdots \frac{\sigma_k^2}{p_k} \frac{\sigma_k^2}{q_k} \sigma_{k+1}^2 \cdots \sigma_{n-1}^2, \quad 1 \leqslant k \leqslant n-1, \tag{5}$$

where

$$\frac{1}{p_k} + \frac{1}{q_k} = 1,\ p_k > 0,\ q_k > 0. \tag{6}$$

Applying the arithmetic-geometric mean inequality to (5) and (6), we get

$$\Upsilon \leqslant \left(\frac{1}{n} \sum_{i=1}^{n-1} \sigma_i^2 \right)^n = \left(\frac{\|A\|_F^2 - \sigma_n^2}{n} \right)^n, \tag{7}$$

and

$$2\sqrt{\frac{1}{p_k q_k}} \leqslant \frac{1}{p_k} + \frac{1}{q_k} = 1, \tag{8}$$

respectively.

By $|\det A|^2 = \sigma_1^2\sigma_2^2\cdots\sigma_n^2$, we have

$$\Upsilon = \frac{\sigma_k^2}{\sigma_n^2}\frac{1}{p_k q_k}|\det A|^2, \quad 1 \leqslant k \leqslant n-1. \tag{9}$$

Hence, by (7), (8) and (9),

$$\sigma_n^2 \geqslant \frac{\sigma_k^2}{p_k q_k}\left(\frac{n}{\|A\|_F^2 - \sigma_n^2}\right)^n |\det A|^2. \tag{10}$$

It follows that

$$\begin{aligned}\sigma_n^2 &\geqslant \frac{\sigma_k^2}{2^2}\left(\frac{n}{\|A\|_F^2 - \sigma_n^2}\right)^n |\det A|^2 \\ &= \frac{\sigma_k^2}{2^2}\left(\frac{n}{\|A\|_F^2}\right)^n \left(\frac{1}{1-\frac{\sigma_n^2}{\|A\|_F^2}}\right)^n |\det A|^2 \\ &\geqslant \frac{\sigma_k^2}{2^2}|\det A|^2\left(\frac{n}{\|A\|_F^2}\right)^n\left(1+\frac{\sigma_n^2}{\|A\|_F^2}\right)^n.\end{aligned}$$

Moreover,

$$\sigma_n \geqslant \frac{\sigma_k}{2}|\det A|\left(\frac{n}{\|A\|_F^2}\right)^{\frac{n}{2}}\left(1+\frac{\sigma_n^2}{\|A\|_F^2}\right)^{\frac{n}{2}} \tag{11}$$

$$> \frac{\sigma_k}{2}|\det A|\left(\frac{n}{\|A\|_F^2}\right)^{\frac{n}{2}}. \tag{12}$$

Using (12) into (11), we have

$$\sigma_n > \frac{\sigma_k}{2}|\det A|\left(\frac{n}{\|A\|_F^2}\right)^{\frac{n}{2}}\left[1+\frac{1}{n}\left(\frac{\sigma_k}{2}\right)^2|\det A|^2\left(\frac{n}{\|A\|_F^2}\right)^{n+1}\right]^{\frac{n}{2}},$$

and since $n \geqslant 2$, it is easy to see that

$$\sigma_n > \frac{\sigma_k}{2}|\det A|\left(\frac{n}{\|A\|_F^2}\right)^{\frac{n}{2}}\left[1+\frac{1}{n}\left(\frac{\sigma_k}{2}\right)^2|\det A|^2\left(\frac{n}{\|A\|_F^2}\right)^{n+1}\right], \ 1 \leqslant k \leqslant n-1.$$

In order to obtain a better lower bound, let $k=1$ in (3) and (4).

THEOREM 1. *If $A \in \mathscr{M}_n (n \geqslant 2)$, then*

$$\sigma_n > \frac{\sigma_1}{2}|\det A|\left(\frac{n}{\|A\|_F^2}\right)^{\frac{n}{2}}\left[1+\frac{1}{n}\left(\frac{\sigma_1}{2}\right)^2|\det A|^2\left(\frac{n}{\|A\|_F^2}\right)^{n+1}\right], \tag{13}$$

and moreover,

$$\sigma_n > \frac{\sigma_1}{2}|\det A|\left(\frac{n}{\|A\|_F^2}\right)^{\frac{n}{2}}. \tag{14}$$

THEOREM 2. *If $A \in \mathscr{M}_n (n \geqslant 2)$, then*

$$\mathscr{K}(A) < \frac{2}{|\det A|}\left(\frac{\|A\|_F^2}{n}\right)^{\frac{n}{2}} \frac{1}{1+\frac{1}{n}(\frac{\sigma_1}{2})^2|\det A|^2(\frac{n}{\|A\|_F^2})^{n+1}}, \tag{15}$$

and moreover,

$$\mathscr{K}(A) < \frac{2}{|\det A|}\left(\frac{\|A\|_F^2}{n}\right)^{\frac{n}{2}}. \tag{16}$$

REMARK 1. Since

$$\frac{2}{|\det A|}\left(\frac{\|A\|_F^2}{n}\right)^{\frac{n}{2}} \frac{1}{1+\frac{1}{n}(\frac{\sigma_1}{2})^2|\det A|^2(\frac{n}{\|A\|_F^2})^{n+1}} < \frac{2}{|\det A|}\left(\frac{\|A\|_F^2}{n}\right)^{\frac{n}{2}},$$

it is easy to show that the upper bound (15) of the condition number of A is better than the upper bound (16) of [1] from Theorem 2.

THEOREM 3. *If $A = (a_{ij}) \in \mathscr{M}_n (n \geqslant 2)$, then*

$$\sigma_n > \frac{\sigma_1}{2}|\det A|\left(\frac{n}{\|A\|_F^2 - l^2}\right)^{\frac{n}{2}}, \tag{17}$$

where $l = \frac{\sigma_1}{2}|\det A|(\frac{n}{\|A\|_F^2})^{\frac{n}{2}}$.

Proof. From (4) and (10), for $1 \leqslant k \leqslant n-1$, we have

$$\sigma_n^2 \geqslant \frac{\sigma_k^2}{p_k q_k}\left(\frac{n}{\|A\|_F^2 - \sigma_n^2}\right)^n |\det A|^2 \geqslant \frac{\sigma_k^2}{p_k q_k}\left(\frac{n}{\|A\|_F^2 - l^2}\right)^n |\det A|^2,$$

where $l = \frac{\sigma_1}{2}|\det A|(\frac{n}{\|A\|_F^2})^{\frac{n}{2}}$.

In order to obtain a better lower bound of the smallest singular value, let $k = 1$ and $p_1 = q_1 = 2$ in the above inequality, we can get

$$\sigma_n^2 \geqslant \frac{\sigma_1^2}{2^2}\left(\frac{n}{\|A\|_F^2 - l^2}\right)^n |\det A|^2.$$

Hence,

$$\sigma_n > \frac{\sigma_1}{2}|\det A|\left(\frac{n}{\|A\|_F^2 - l^2}\right)^{\frac{n}{2}}.$$

THEOREM 4. *If $A \in \mathscr{M}_n (n \geqslant 2)$, then*

$$\mathscr{K}(A) < \frac{2}{|\det A|}\left(\frac{\|A\|_F^2 - l^2}{n}\right)^{\frac{n}{2}}. \tag{18}$$

where $l = \frac{\sigma_1}{2}|\det A|(\frac{n}{\|A\|_F^2})^{\frac{n}{2}}$.

REMARK 2. Evidently, the lower bounds (17) and (18) are sharper than the lower bounds (14) and (16), respectively. From the above proof, we note that we can get the better bound inequalities than (17) and (18) if we repeat the above procedure using the new lower bounds in the proof.

3. Examples

In this section, we will consider two simple examples for validating our results.

EXAMPLE 1. [3] Consider a 3×3 matrix as follows:

$$A = \begin{bmatrix} 10 & 1 & 2 \\ 2 & 20 & 3 \\ 20 & 1 & 10 \end{bmatrix}.$$

By direct calculation, we have $\sigma_3 = 2.4909$ and $\mathscr{K}(A) = 10.1870$.

1. By Theorem 2 in [3], we have $\sigma_3 \geqslant 0.6227$.
2. By Theorem 3.1 in [8], we have $\sigma_3 \geqslant 2.0694$.
3. By Theorem 2.1 in [9], we have $\sigma_3 \geqslant 2.3961$.
4. By (14) in this paper, we have $\sigma_3 \geqslant 2.4604$.
5. By (17) in this paper, we have $\sigma_3 \geqslant 2.4825$.
6. By (16) in this paper (see also [1]), we have $\mathscr{K}(A) \leqslant 10.3131$.
7. By (18) in this paper, we have $\mathscr{K}(A) \leqslant 10.2214$.

EXAMPLE 2. [3] Consider a 3×3 matrix as follows:

$$A = \begin{bmatrix} 0.75 & 0.5 & 0.4 \\ 0.5 & 1 & 0.6 \\ 0 & 0.5 & 1 \end{bmatrix}.$$

By direct calculation, we have $\sigma_3 = 0.2977$ and $\mathscr{K}(A) = 6.0610$.

1. By Theorem 2 in [3], we have $\sigma_3 \geqslant 0.0560$.
2. By Theorem 3.1 in [8], we have $\sigma_3 \geqslant 0.1547$.
3. By Theorem 2.1 in [9], we have $\sigma_3 \geqslant 0.1977$.
4. By (14) in this paper, we have $\sigma_3 \geqslant 0.2343$.
5. By (17) in this paper, we have $\sigma_3 \geqslant 0.2395$.
6. By (16) in this paper (see also [1]), we have $\mathscr{K}(A) \leqslant 7.7009$.
7. By (18) in this paper, we have $\mathscr{K}(A) \leqslant 7.5359$.

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