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SOME HARDY TYPE INEQUALITIES WITH "BROKEN" EXPONENT p

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Abstract. Some new Hardy-type inequalities, where the parameter p is permitted to take different values in different intervals, are proved and discussed. The parameter can even be negative in one interval and greater than one in another. Moreover, a similar result is derived for a multi-dimensional case.

1. Introduction

Hardy's original inequality reads: If f is a non-negative and p-integrable on $(0,\infty)$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \quad p>1. \tag{1.1}$$

This result was stated in [4] and finally proved in [5]. One remarkable fact is that (1.1) in fact, via the substitution $f(x)=g(x^{1-\frac{1}{p}})x^{-\frac{1}{p}}$, is equivalent to the inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(t)dt\right)^p \frac{dx}{x} \leqslant 1\int_0^\infty g^p(x)\frac{dx}{x}, \quad p>1. \tag{1.2}$$

We note that

(a) the inequality (1.2) is just a simple consequence of Jensen's inequality (convexity) and Fubini's theorem.

(b) (1.2) holds also for $p=1$ (but (1.1) does not) and $p<0$ and, thus, also (1.1) holds for $p<0$.

(c) By using the more general substitution $f(x)=g(x^{\frac{p-1-\alpha}{p}})x^{-\frac{1+\alpha}{p}}$, we find that (1.2) is equivalent to also the following weighted version of (1.1): If f is a nonnegative measurable function on $(0,\infty)$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\alpha dx \leqslant \left(\frac{p}{p-1-\alpha}\right)^p \int_0^\infty f^p(x)x^\alpha dx, \tag{1.3}$$

whenever $p\geqslant 1$, $\alpha<p-1$ or $p<0$, $\alpha>p-1$.

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For the case $0 < p \leqslant 1$ (1.3) holds in the reversed direction with the same argument. See [22], where an even more general case was considered.

(d) Note that (1.3) was the first generalization of (1.1), which was also proved by Hardy himself (see [6]).

(e) Obviously Hardy himself did not discover this way to handle his inequalities in [4]–[6] via convexity. In fact, Jensen's inequality is from 1905 (see [8]–[9]) and was well-known to Hardy since he used it in many other situations.

(f) In fact, (1.2) and (1.3) are equivalent for all $p \geqslant 1$ and $p < 0$. They are also equivalent to the following "dual" version of (1.3):

$$\int_0^\infty \left(\frac{1}{x}\int_x^\infty f(t)dt\right)^p x^\alpha dx \leqslant \left(\frac{p}{\alpha+1-p}\right)^p \int_0^\infty f^p(x)x^\alpha dx \tag{1.4}$$

whenever $p \geqslant 1,\ \alpha > p-1$ or $p<0, \alpha < p-1$.

(g) This equivalence theorem was recently proved in [22] (see also [21]) in the more general case where the interval $(0,\infty)$ is replaced with the interval $(0,l),\ 0<l\leqslant\infty$. In this case (1.3) and (1.4) can be formulated in a little more precise way. E.g. (1.3) reads

$$\int_0^l \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^{\alpha_0} dx \leqslant \left(\frac{p}{p-1-\alpha_0}\right)^p \int_0^l f^p(x)x^{\alpha_0}\left[1-\left(\frac{x}{l}\right)^{\frac{p-\alpha_0-1}{p}}\right] dx \tag{1.5}$$

whenever $p \geqslant 1,\ \alpha_0 < p-1$ or $p<0,\ \alpha_0 > p-1$ (for the case $p<0$, we require that f is strictly positive). In this case, the basic inequality (1.2) reads

$$\int_0^l \left(\frac{1}{x}\int_0^x g(t)dt\right)^p \frac{dx}{x} \leqslant 1\int_0^l g^p(x)\left(1-\frac{x}{l}\right)dx, \tag{1.6}$$

which holds whenever $p \geqslant 1$ or $p<0$ (for the case $l=\infty,\ 1-\frac{x}{l}\equiv 1$ and a similar identification is used in (1.5)).

The main aim of this paper is to improve the basic inequality (1.6). We continue the research initiated in [22] by proving a new variant of (1.6) where the weight $\frac{1}{x}$ is replaced by a weight $x^{-\beta}$, where $\beta>0$. We even consider the more general case with "broken" exponent, i.e. when p is replaced by $p(x)$ which can take different values in different intervals (see Theorem 2.1).

In order to prove our multidimensional result in Section 3 (see Theorem 3.1) we even need to prove a more general case of this result, namely when also the parameter β is "broken" in a similar way (see Theorem 2.3). Also some dual versions of these results are proved (see Theorems 2.5 and 3.3, respectively). Finally, some concluding remarks can be found in Section 4.

2. The one-dimensional case

First we state the following slight improvement of a result stated (but not proved) in [22, Theorem 3.6]:

THEOREM 2.1. *Let $b>0,\ \beta>0,\ 0<l\leqslant\infty$ and*

$$p(x)=\begin{cases} p_0, & 0\leqslant x\leqslant b,\\ p_1, & x>b,\end{cases}$$

where $p_0,p_1\in\mathbb{R}\setminus\{0\}$. If f is nonnegative and measurable, then

$$\int_0^l\left(\frac{1}{x}\int_0^x f(t)dt\right)^{p(x)}x^{-\beta}dx\leqslant\frac{1}{\beta}\int_0^l(f(x))^{p(x)}x^{-\beta}\left(1-\left(\frac{x}{l}\right)^{\beta}\right)dx$$
$$+K\int_0^l\left((f(x))^{p_1}-(f(x))^{p_0}\right)dx, \tag{2.1}$$

$K=0$ if $l\leqslant b$ and $K=\frac{1}{\beta}\left(b^{-\beta}-l^{-\beta}\right)$ if $l>b$, whenever $p_0\geqslant 1,\ p_1\geqslant 1$ or $p_0\geqslant 1$, $p_1<0$ or $p_1\geqslant 1,\ p_0<0$ or $p_0<0,\ p_1<0$ (for the case with negative parameters, we assume that the function f is strictly positive on the corresponding interval). If $0<p(x)\leqslant 1$, then (2.1) holds in the reversed direction (for the case $l=\infty,\ 1-\left(\frac{x}{l}\right)^{\beta(x)}\equiv 1$ and $l^{-\beta}\equiv 0$).

REMARK 2.2. For the case $b=l=\infty$ $(p_0=p_1=p)$ and $\beta=1$, (2.1) coincides with (1.2), which as we have motivated in fact is equivalent with all variants (1.1), (1.3) and (1.4) of Hardy's inequality. Hence, (2.1) is more general than all inequalities mentioned in the introduction.

In order to be able to prove our announced multidimensional version of Hardy's inequality with "broken" exponent, we in fact need the following improvement of Theorem 2.1, where also the parameter β is broken in a similar way.

THEOREM 2.3. *Let b,l and $p(x)$ be defined as in Theorem 2.1 and, in addition, let*

$$\beta(x)=\begin{cases} \beta_0, & 0\leqslant x\leqslant b,\\ \beta_1, & x>b,\end{cases}$$

where $\beta_0,\beta_1\in\mathbb{R}\setminus\{0\}$. If f is nonnegative and measurable and $\beta(x)>0$, then

$$\int_0^l\left(\frac{1}{x}\int_0^x f(t)dt\right)^{p(x)}x^{-\beta(x)}dx$$
$$\leqslant\int_0^l\frac{1}{\beta(x)}(f(x))^{p(x)}x^{-\beta(x)}\left(1-\left(\frac{x}{l}\right)^{\beta(x)}\right)dx+I_0, \tag{2.2}$$

where $I_0 = 0$ *if* $l \leqslant b$ *(so that* $\beta(x) = \beta_0$ *and* $p(x) = p_0$*) and*

$$I_0 = \frac{1}{\beta_1}\left(b^{-\beta_1} - l^{-\beta_1}\right)\int_0^b (f(x))^{p_1}\,dx \\ -\frac{1}{\beta_0}\left(b^{-\beta_0} - l^{-\beta_0}\right)\int_0^b (f(x))^{p_0}\,dx,$$

if $l > b$. *If* $0 < p(x) \leqslant 1$, *then* (2.2) *holds in the reversed direction (for the case* $l = \infty$, $1 - \left(\frac{x}{l}\right)^{\beta(x)} \equiv 1$ *and* $l^{-\beta_0} = l^{-\beta_1} \equiv 0$*).*

Note that for the case $\beta_0 = \beta_1$ Theorem 2.1 coincides with Theorem 2.3, so it is sufficient to prove this Theorem.

Proof. Let $l \leqslant b,\ l < \infty$. Then, by Jensen's inequality and the Fubini theorem,

$$\begin{aligned}\int_0^l \left(\frac{1}{x}\int_0^x f(t)dt\right)^{p(x)} x^{-\beta(x)}dx &= \int_0^l \left(\frac{1}{x}\int_0^x f(t)dt\right)^{p_0} x^{-\beta_0}dx \\ &\leqslant \int_0^l \int_0^x (f(t))^{p_0}\,dt x^{-\beta_0-1}dx \\ &= \int_0^l (f(t))^{p_0}\left(\int_t^l x^{-\beta_0-1}dx\right)dt \\ &= \frac{1}{\beta_0}\int_0^l (f(t))^{p_0}\left(t^{-\beta_0} - l^{-\beta_0}\right)dt \\ &= \int_0^l \frac{1}{\beta(t)}(f(t))^{p(t)}t^{-\beta(t)}\left(1-\left(\frac{t}{l}\right)^{\beta(t)}\right)dt.\end{aligned}$$

Now let $l_0 < l < \infty$. By again using Jensen's inequality and the Fubini theorem we find that

$$\begin{aligned}&\int_0^l \left(\frac{1}{x}\int_0^x f(t)dt\right)^{p(x)} x^{-\beta(x)}dx \\ &= \int_0^b \left(\frac{1}{x}\int_0^x f(t)dt\right)^{p_0} x^{-\beta_0}dx + \int_b^l \left(\frac{1}{x}\int_0^x f(t)dt\right)^{p_1} x^{-\beta_1}dx \\ &\leqslant \int_0^b \left(\int_0^x (f(t))^{p_0}\,dt\right)x^{-\beta_0-1}dx + \int_b^l \left(\int_0^b (f(t))^{p_1}\,dt\right)x^{-\beta_1-1}dx \\ &\quad + \int_b^l \left(\int_b^x (f(t))^{p_1}\,dt\right)x^{-\beta_1-1}dx \\ &= \int_0^b (f(t))^{p_0}\left(\int_t^b x^{-\beta_0-1}dx\right)dt + \int_0^l (f(t))^{p_1}\,dt\int_b^l x^{-\beta_1-1}dx \\ &\quad + \int_b^l (f(t))^{p_1}\left(\int_t^l x^{-\beta_1-1}dx\right)dt\end{aligned}$$

$$
\begin{aligned}
&= \int_0^b \frac{1}{\beta_0}(f(t))^{\beta_0}\left(t^{-\beta_0}-l^{-\beta_0}\right)dt + \frac{1}{\beta_1}\left(b^{-\beta_1}-l^{-\beta_1}\right)\int_0^b (f(t))^{p_1}\,dt \\
&\quad + \int_b^l \frac{1}{\beta_1}(f(t))^{\beta_1}\left(t^{-\beta_1}-l^{-\beta_1}\right)dt \\
&= \int_0^b \frac{1}{\beta_0}(f(t))^{\beta_0}\left(t^{-\beta_0}-l^{-\beta_0}\right)dt + \int_b^l \frac{1}{\beta_1}(f(t))^{\beta_1}\left(t^{-\beta_1}-l^{-\beta_1}\right)dt + I_0 \\
&= \int_0^l \frac{1}{\beta(t)}(f(t))^{\beta(t)}t^{-\beta(t)}\left(1-\left(\frac{t}{l}\right)^{\beta(t)}\right)dt + I_0.
\end{aligned}
$$

For the case $0<p(x)\leqslant 1$ (so that $f(x)=u^{p_0}$ and $g(u)=u^{p_1}$ are concave functions) the only inequality holds in the reversed direction so also this case is proved. The proof of the case $l=\infty$ only consists of obvious modifications of the proof above so the proof is complete. $\square$

REMARK 2.4. By following the proof of Theorem 2.3 for the case $p_0=p_1=1$ we find that the only inequality holds with equality sign so inequality (2.2) reduces to an identity. This explains partly why the additional term I_0 appears.

Next, we also state a dual version of Theorem 2.3, namely when the Hardy operator

$$H : f(x) \to \frac{1}{x}\int_0^x f(t)dt$$

is replaced by the dual Hardy operator

$$H^\star : f(x) \to x\int_x^\infty f(t)\frac{dt}{t^2}.$$

THEOREM 2.5. *Let $b>0, 0\leqslant l<\infty$, and*

$$p(x)=\begin{cases} p_0, & x\leqslant b,\\ p_1, & x>b,\end{cases}\qquad \beta(x)=\begin{cases} \beta_0, & x\leqslant b,\\ \beta_1, & x>b,\end{cases}$$

where $p_0,p_1,\beta_0,\beta_1\in\mathbb{R}\setminus\{0\}$. Moreover, assume that $p_0\geqslant 1,\ p_1\geqslant 1$ or $p_0\geqslant 1,\ p_1<0$ or $p_1\geqslant 1,\ p_0<0$ or $p_0<0,\ p_1<0$ (for the case with negative parameters, we assume that the function f is strictly positive on the corresponding interval). If f is nonnegative and measurable and $\beta(x)>0$, then

$$
\begin{aligned}
&\int_l^\infty \left(x\int_x^\infty f(t)\frac{dt}{t^2}\right)^{p(x)} x^{\beta(x)}\frac{dx}{x^2} \\
&\leqslant \int_l^\infty \frac{1}{\beta(x)}(f(x))^{p(x)}x^{\beta(x)}\left(1-\left(\frac{l}{x}\right)^{\beta(x)}\right)\frac{dx}{x^2}+I_0, \qquad (2.3)
\end{aligned}
$$

where $I_0=0$ if $l\geqslant b$ and

$$I_0=\frac{1}{\beta_0}\left(b^{\beta_0}-l^{\beta_0}\right)\int_b^\infty (f(x))^{\beta_0}\frac{dx}{x^2}-\frac{1}{\beta_1}\left(b^{\beta_1}-l^{\beta_1}\right)\int_b^\infty (f(x))^{p_1}\frac{dx}{x^2},$$

when $l < b$ (for the case $l = 0$, $1 - \left(\frac{l}{x}\right)^{\beta(x)} \equiv 1$ and $l^{\beta_0} = l^{\beta_1} \equiv 0$). If $0 < p(x) \leqslant 1$, then (2.3) holds in the reversed direction.

Proof. Let $l \geqslant b,\ l > 0$. Then, by Jensen's inequality and the Fubini theorem,

$$\begin{aligned}
\int_l^\infty \left(x\int_x^\infty f(t)t^{-2}dt\right)^{p(x)} x^{\beta(x)}\frac{dx}{x^2} &= \int_l^\infty \left(x\int_x^\infty f(t)t^{-2}dt\right)^{p_1} x^{\beta_1}\frac{dx}{x^2}\\
&\leqslant \int_l^\infty x\left(\int_x^\infty (f(t))^{p_1}t^{-2}dt\right)x^{\beta_1}\frac{dx}{x^2}\\
&= \int_l^\infty (f(t))^{p_1}t^{-2}dt\left(\int_b^t x^{\beta_1-1}dx\right)dt\\
&= \int_l^\infty (f(t))^{p_1}t^{-2}dt\frac{1}{\beta_1}\left(t^{\beta_1}-b^{\beta_1}\right)dt\\
&= \int_l^\infty (f(t))^{p(t)}\frac{1}{\beta(t)}t^{\beta(t)}\left(1-\left(\frac{b}{t}\right)^{\beta(t)}\right)\frac{dt}{t^2}.
\end{aligned}$$

Now, let $0 < l < b$. Then, by again using Jensen's inequality and the Fubini theorem,

$$\begin{aligned}
&\int_l^\infty \left(x\int_x^\infty f(t)\frac{dt}{t^2}\right)^{p(x)} x^{\beta(x)}dx\\
&= \int_l^b \left(x\int_x^\infty f(t)\frac{dt}{t^2}\right)^{p_0} x^{\beta_0}\frac{dx}{x^2} + \int_b^\infty \left(x\int_x^\infty f(t)\frac{dt}{t^2}\right)^{p_1} x^{\beta_1}\frac{dx}{x^2}\\
&\leqslant \int_l^b x\left(\int_x^\infty (f(t))^{p_0}t^{-2}dt\right)x^{\beta_0}\frac{dx}{x^2} + \int_b^\infty x\left(\int_x^\infty (f(t))^{p_1}t^{-2}dt\right)x^{\beta_1}\frac{dx}{x^2}\\
&= \int_l^b x\left(\int_x^b (f(t))^{p_0}t^{-2}dt\right)x^{\beta_0}\frac{dx}{x^2} + \int_l^b x\left(\int_b^\infty (f(t))^{p_0}t^{-2}dt\right)x^{\beta_0}\frac{dx}{x^2}\\
&\quad + \int_b^\infty x\left(\int_x^\infty (f(t))^{p_1}t^{-2}dt\right)x^{\beta_1}\frac{dx}{x}\\
&= \int_l^b (f(t))^{p_0}t^{-2}\left(\int_l^t x^{\beta_0-1}dx\right)dt + \int_b^\infty (f(t))^{p_0}t^{-2}dt\int_l^b x^{\beta_0-1}dx\\
&\quad + \int_l^\infty (f(t))^{p_1}t^{-2}\left(\int_b^t x^{\beta_1-1}dx\right)dt\\
&= \int_l^b (f(t))^{p_0}\frac{1}{\beta_0}\left(t^{\beta_0}-l^{\beta_0}\right)\frac{dt}{t^2} + \beta_0\left(b^{\beta_0}-l^{\beta_0}\right)\int_l^\infty (f(t))^{p_0}\frac{dt}{t^2}\\
&\quad + \int_b^\infty (f(t))^{p_1}\frac{1}{\beta_1}\left(t^{\beta_1}-b^{\beta_1}\right)\frac{dt}{t^2}\\
&= \int_l^\infty (f(t))^{p(t)}\frac{1}{\beta(t)}t^{\beta(t)}\left(1-\left(\frac{l}{t}\right)^{\beta(t)}\right)dt + I_0.
\end{aligned}$$

The proof of the case $0 < p(x) \leqslant 1$ is the same since the only inequality holds in the reversed direction then. The proof of the case $l = 0$ only consists of obvious modifications of the proof above so the proof is complete. □

REMARK 2.6. Also in this case it is clear from the proof that if $p_0 = p_1 = 1$, then (2.3) reduces to an identity. This partly explains why the additional term I_0 in (2.3) appears.

3. The multidimensional case

Before we formulate our main results in this Section (Theorems 3.1 and 3.3) we need some notations and to recall some well-known relations. We let $n = 2,3,...,$ $\mathbf{x} = (x_1,...,x_n)$, $d\mathbf{x} = dx_1...dx_n$ and $r > 0$. Moreover, let $V_{r,n}$ denote the volume of the sphere $\{\mathbf{x} : |\mathbf{x}| \leqslant r\}$ and $S_{r,n-1}$ denote the surface area of the same sphere. For simplicity we also define $V_n = V_{1,n}$ and $S_{n-1} = S_{1,n-1}$. We also need the well-known relations $V_n = \frac{S_{n-1}}{n}$ and $V_{n,r} = \int_0^r S_{n-1}u^{n-1}du$.

THEOREM 3.1. *Let $p(x)$ be defined as in Theorem 2.3 with $b = R_0$ and $l = R$, $0 < R_0$, $R \leqslant \infty$, and where $p_0 \geqslant 1$, $p_1 \geqslant 1$ or $p_0 \geqslant 1$, $p_1 < 1$ or $p_1 \geqslant 0$, $p_0 < 0$ or $p_0 < 0$, $p_1 < 1$. Moreover, let $a(r)$ be defined by*

$$a(r) = \begin{cases} a_0, & 0 < r \leqslant R_0 \\ a_1, & r > R_0, \end{cases}$$

where $a_0, a_1 \in \mathbb{R} \setminus \{0\}$ and $\beta(r) := (n-1)(a(r)-1) + a(r) > 0$.

If $f = f(|x|)$ is a radially symmetric nonnegative function in $\mathbb{R}^n$, $n = 2,3,4,...,$ then

$$\begin{aligned} &\int_{|x|\leqslant R} \left(\frac{1}{V_{r,n}} \int_{|\mathbf{t}|\leqslant|\mathbf{x}|} f(|t|)dt \right)^{p(|x|)} |x|^{-a(|x|)}\, dx \\ &\leqslant \int_{|x|\leqslant R} \frac{1}{\beta(|x|)} n^{p(|x|)} \left(f(|x|)\right)^{p(|x|)} |x|^{-a(|x|)} \left(1 - \left(\frac{|x|}{R}\right)^{\beta(|x|)}\right) dx + J_0, \end{aligned} \tag{3.1}$$

where $J_0 = 0$ if $R \leqslant R_0$ and

$$\begin{aligned} J_0 = &\frac{1}{\beta_1}\left(R_0^{-\beta_1} - R^{-\beta_1}\right) \int_0^{R_0} \left(nf(r)r^{n-1}\right)^{p_1} dr \\ &- \frac{1}{\beta_0}\left(R_0^{-\beta_0} - R^{-\beta_0}\right) \int_0^{R_0} \left(nf(r)r^{n-1}\right)^{p_0} dr \end{aligned}$$

if $R > R_0$ with $\beta_i := (n-1)(p_i - 1) + a_i$, $i = 0,1$. If $0 < p(x) \leqslant 1$, then (3.1) holds in the reversed direction (for the case $R = \infty$, $1 - \left(\frac{|x|}{R}\right)^{\beta(|x|)} = 1$ and $R^{-\beta_0} = R^{-\beta_1} = 0$).

Proof. Let $R<\infty$. By using the notations and information above we find that

$$\begin{aligned} I_1 &:= \int_{|x|\leqslant R}\left(\frac{1}{V_{r,n}}\int_{|t|\leqslant|x|} f(|t|)dt\right)^{p(|x|)} |x|^{-a(|x|)}\,dx \\ &= \int_0^R\left(\frac{1}{V_{r,n}}\int_0^r f(u)S_{n-1}u^{n-1}du\right)^{p(r)} r^{-a(r)}S_{n-1}r^{n-1}dr \\ &= \int_0^R\left(\frac{S_{n-1}}{V_n.r^n}\int_0^r f(u)u^{n-1}du\right)^{p(r)} S_{n-1}r^{-\alpha(r)+n-1}dr \\ &= \int_0^R\left(\frac{1}{r}\int_0^r nf(r)r^{n-1}dr\right)^{p(r)} S_{n-1}r^{-(n-1)p(r)-a(r)+n-1}dr. \end{aligned}$$

Next we use Theorem 2.3 with $\beta(r):=(n-1)\,(p(r)-1)+a(r)$ and $f(r)$ replaced by $nf(r)r^{n-1}$ and find that

$$I_1\leqslant\int_0^R\frac{1}{\beta(r)}\,(nf(r))^{p(r)}\,r^{(n-1)p(r)}r^{-(n-1)p(r)-a(r)+n-1}S_{n-1}\left(1-\left(\frac{r}{R}\right)^{\beta(r)}\right)dr+I_0,$$

where I_0 is defined in Theorem 2.3, so that, by again using general spherical coordinates,

$$I_1\leqslant\int_{|x|\leqslant R}\frac{1}{\beta(|x|)}n^{p(|x|)}\,(f(|x|))^{p(|x|)}\,|x|^{-a(|x|)}\left(1-\left(\frac{|x|}{R}\right)^{\beta(|x|)}\right)dx+J_0.$$

The proof of the case $0<p(x)\leqslant 1$ follows in the same way. For the case $R=\infty$ the proof follows by making obvious modifications of the arguments above so the proof is complete. □

REMARK 3.2. Theorem 3.1 is very different from other known multidimensional Hardy type inequalities in the literature (see e.g. the books [7], [14], [15] and the references there) e.g. the exponent p can have different values which can even be negative on one part of the ball and $\geqslant 1$ on another, also the power exponent a can take different values in a similar way and the "error term" which appears can be 0, >0 or <0.

The dual version of Theorem 3.1 reads:

THEOREM 3.3. *Let $f(x)$ and $p(x)$ be defined as in Theoren 3.1 with $b=R_0$ and $l=R$, $0\leqslant R<\infty$, where $p_0\geqslant 1$, $p_1\geqslant 1$ or $p_0\geqslant 1$, $p_1<0$ or $p_1\geqslant 1$, $p_0<0$ or p_0, $p_1<0$. Moreover, let*

$$a(r)=\begin{cases} a_0, & 0\leqslant r<R_0 \\ a_1, & r\geqslant R_0,\end{cases}$$

where $a_0,a_1\in\mathbb{R}\setminus\{0\}$ and $\beta(r):=(n-1)(p(r)+1)+a(r)>0$. Then

$$\begin{aligned} &\int_{|x|\geqslant R}\left(V_{R,n}\int_{|t|\geqslant|x|} f(|t|)\frac{dt}{(V_{|t|,n})^2}\right)^{p(x)} |x|^{a(|x|)}\,\frac{dx}{|x|^2} \\ &\leqslant\int_{|x|\geqslant R}\frac{1}{\beta(|x|)}n^{p(|x|)}\,(f(|x|)^{p(|x|)}\,|x|^{a(x)}\left(1-\left(\frac{R}{|x|}\right)^{\beta(|x|)}\right)\frac{dx}{|x|^2}+J_1, \end{aligned} \tag{3.2}$$

where $J_1 = 0$ *if* $R_0 \leqslant R$ *and*

$$J_1 = \frac{1}{\beta_0}\left(R_0^{\beta_0} - R^{\beta_0}\right)\int_{R_0}^{\infty}\left(nf(r)r^{1-n}\right)^{p_0}\frac{dr}{r^2}$$
$$-\frac{1}{\beta_1}\left(R_0^{\beta_1} - R^{\beta_1}\right)\int_{R_0}^{\infty}\left(nf(r)r^{1-n}\right)^{p_1}\frac{dr}{r^2}$$

if $R_0 > R$ *with* $\beta_i := (n-1)(p_i+1)+a_i,\ i=0,1$. *If* $0 < p(x) \leqslant 1$, *then (3.2) holds in the reversed direction (For the case* $R=0$ *we have* $\left(1-\left(\frac{R}{|x|}\right)^{\beta(|x|)}\right) \equiv 1$ *and* $R^{\beta_0} = R^{\beta_1} = 0$).

Proof. Let $R > 0$. By using notations and arguments as in the proof of Theorem 3.1 we obtain that

$$\begin{aligned} I_2 &= \int_{|x|\geqslant R}\left(V_{|x|,n}\int_{|t|\geqslant|x|} f(|t|)\frac{dt}{\left(V_{|t|},r\right)^2}\right)^{p(x)}|x|^{a(|x|)}\frac{dx}{|x|^2} \\ &= \int_R^{\infty}\left(V_n.r^n\int_r^{\infty} f(u)S_{n-1}u^{n-1}\frac{du}{(V_nu^n)^2}\right)^{p(r)} r^{a(r)}S_{n-1}r^{n-1}\frac{dr}{r^2} \\ &= \int_R^{\infty}\left(\frac{S_{n-1}r^n}{V_n}\int_r^{\infty} f(u)u^{-n-1}du\right)^{p(r)} r^{a(r)}S_{n-1}r^{n-1}\frac{dr}{r^2} \\ &= \int_R^{\infty}\left(r\int_r^{\infty} nf(u)u^{-n+1}\frac{du}{u^2}\right)^{p(r)} S_{n-1}r^{(n-1)p(r)+a(r)+n-1}\frac{dr}{r^2}. \end{aligned}$$

By now using Theorem 2.5 with $\beta(r) = (n-1)(p(r)+1)+a(r)$ we find that

$$I_2 \leqslant \int_R^{\infty}\frac{1}{\beta(r)}\left(nf(r)\right)^{p(r)} r^{(1-n)p(r)}S_{n-1}r^{(n-1)p(r)+a(r)+n-1}\left(1-\left(\frac{R}{r}\right)^{\beta(r)}\right)\frac{dr}{r^2}+J_1,$$

so that, by again using spherical coordinates in $\mathbb{R}^n$,

$$I_2 \leqslant \int_{|x|\geqslant R}\frac{1}{\beta(|x|)}n^{p(|x|)}\left(f(|x|)\right)^{p(|x|)}|x|^{a(|x|)}\left(1-\left(\frac{R}{|x|}\right)^{\beta(|x|)}\right)\frac{dx}{|x|^2}+J_1,$$

where $J_1 = 0$ if $R_0 \leqslant R$ and

$$J_1 = \frac{1}{\beta_0}\left(R_0^{\beta_0} - R^{\beta_0}\right)\int_{R_0}^{\infty}\left(nf(r)r^{-n+1}\right)^{p_0}\frac{dr}{r^2}$$
$$-\frac{1}{\beta_1}\left(R_0^{\beta_1} - R^{\beta_1}\right)\int_{R_0}^{\infty}\left(nf(r)r^{-n+1}\right)^{p_1}\frac{dr}{r^2}$$

if $R_0 > R$. The result for the case $0 < p(x) \leqslant 1$ follows in the same way. For the case $R = \infty$ the proof follows by making suitable modifications in the proof above so the proof is complete. $\square$

4. Concluding remarks

REMARK 4.1. It seems to be Godunova (see [2] and [3]) who first discovered the convexity technique to prove classical Hardy's inequality in its original form (1.1). However, this proof was not much noticed and it was only after Kaijser et al. [11] rediscovered this technique of proof which was a great development of the area, see e.g. [10], [11], [13], [16], [17], [18], [19], [20], [21], [22], and even the PhD thesis of K. Krulic [12].

REMARK 4.2. The ideas and results in this paper were initiated in [22] and it seems tempting to develop the theory in the standard way (see e.g. the books [14] and [15]) in this direction. The idea to consider Hardy inequalities with "general" variable exponent $p = p(x)$ is considered before in a more general way (see e.g. [1]) but this approach (for a special case) is new and give us possibility to consider even negative values on $p(x)$ and to have sharp constant (cf. [22]).

REMARK 4.3. Inspired by our investigations in [21], our original aim was to generalize Theorem 2.1 to a multidimensional situation, where the Hardy operator

$$H = H_1 f(x) \to \frac{1}{x}\int_0^x f(t)dt$$

is replaced by a corresponding multidimensional operator

$$H_n : f(x_1,...,x_n) \to \frac{1}{x_1...x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(t_1,...,t_n)dt_1...dt_n.$$

However, as expected from the general theory this leads to many difficulties even in the two-dimensional case (cf. the result of Sawyer [23]). For the moment we leave this as an open (probably difficult) question.

Finally we give the following example of our results in Theorem 3.1 and Theorem 3.3 with $R=\infty$ and $R=0$, respectively, and $p_0 = p_1 = p$.

EXAMPLE 4.4. With the notations in Theorem 3.1 it yields that, for $p \geqslant 1$ and $p<0$,

(a)

$$\int_{\mathbb{R}^n}\left(\frac{1}{V_{r,n}}\int_{|t|\leqslant|x|} f(|t|)dt\right)^p |x|^{-\alpha}\,dx \leqslant \frac{n^p}{\beta_1}\int_{\mathbb{R}^n}(f(|x|))^p\,|x|^{-a}\,dx \tag{4.1}$$

where $\beta_1 := (n-1)(p-1)+a>0$,

(b)

$$\int_{\mathbb{R}^n}\left(\frac{1}{V_{r,n}}\int_{|t|\geqslant|x|} f(|t|)\frac{dt}{\left(V_{|t|,n}\right)^2}\right)^p |x|^{\alpha}\,\frac{dx}{x^2} \leqslant \frac{n^p}{\beta_2}\int_{\mathbb{R}^n}(f(|x|))^p\,|x|^{\alpha}\,\frac{dx}{x^2}, \tag{4.2}$$

where $\beta_2 := (n-1)(p+1)+a>0$. For the case $0<p\leqslant 1$ both (4.1) and (4.2) hold in the reversed direction.

REMARK 4.5. For the one-dimensional case we obtain the inequalities

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^{-\alpha}dx \leqslant \frac{1}{\alpha}\int_0^\infty (f(x))^p x^{-\alpha}dx \tag{4.3}$$

and

$$\int_0^\infty \left(x\int_x^\infty f(t)\frac{dt}{t^2}\right)^p x^{\alpha-2}dx \leqslant \frac{1}{\alpha}\int_0^\infty (f(x))^p x^{\alpha-2}dx$$

yielding for $p \geqslant 1$ and $p < 0$, and $\alpha > 0$.

As we have noted before, (4.3) even with $\alpha = 1$ is equivalent with the first power weighted forms (1.3) and (1.4) of Hardy's inequality via the substitution pointed out in (c) in the introduction.

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