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GENERALIZED MINTY PREVARIATIONAL INEQUALITY, INVEX–INCREASE–ALONG–RAYS PROPERTY AND INVEX–STAR–SHAPED OPTIMIZATION PROBLEM

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Abstract. The purpose of this paper is to study some relations between generalized Minty prevariational inequalities, invex-increase-along-rays properties, and invex-star-shaped optimization problems. We introduce the concepts of invex-star-shaped sets and invex-increase-along-rays functions, and establish the relations between invex-increase-along-rays properties and invex-star-shaped optimization problems. Further, under certain conditions, we investigate the relations between invex-increase-along-rays properties and generalized Minty prevariational inequalities. As consequences, we obtain the equivalence of generalized Minty prevariational inequalities and invex-star-shaped optimization problems under suitable conditions. Finally, we prove the equivalence of generalized Minty prevariational inequalities and perturbed generalized Minty prevariational inequalities.

1. Introduction

Variational inequalities have proved to be important mathematical models in the study of many practical problems, especially equilibrium problems [1, 2, 3], complementarity problems [3, 4], arising in economics, mechanics, and engineering science, etc. Because of their wide applications, variational inequalities have been studied by many authors [1, 2, 3, 5, 6, 7, 8, 9, 10]. A Minty variational inequality has close relation with a Stampacchia variational inequality by the classical Minty lemma [1, 2]. A Minty type lemma has shown to be an important tool in the field of variational inequalities when the underlying operator has certain monotonicity and the domain is convex [1, 2, 5, 6, 7]. Further, when the underlying operator is a gradient, the variational principle can be established between a variational inequality and a convex differentiable optimization problem [6, 7]. For more results concerning relations between variational inequalities and optimization problems, we can refer the readers to [8, 9, 10] and the references therein.

The prevariational inequality (also named the variational-like inequality) was first introduced and studied by Parida et al. [11] and Yang and Chen [12], which is one of

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important generalizations of the classical variational inequality. An example of problem appearing as a prevariational inequality can be found in Theorem 3.1 of Yang [13] where it has been shown that a pseudoinvex extremum problem is equivalent to a prevariational inequality problem under suitable conditions. In fact, the prevariational inequality has shown to be an important tool to study an optimization problem involving invexity [13, 14, 15] and it has been studied intensively by many authors [11, 12, 16, 17, 18, 19, 20]. It worths mentioning that the concept of invexity was first introduced by Hanson [14] where a Kuhn-Tucher type condition was shown to be sufficient for optimality of nonlinear programming problems with invexity condition instead of convexity. Invexity can be regarded as one generalization of convexity. Following the development of the concept of generalized invexity, various concepts of generalized invexity as well as generalized invex functions were introduced and studied. For more details on invexity and its applications, we refer the reader to the monograph [15] and [13, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30].

In recent years, more and more authors are focusing on nonmonotone variational inequalities and nondifferentiable optimization problems. Crespi et al [8, 9] introduced and studied a generalized Minty variational inequality with a star-shaped domain, and a nondifferentiable optimization problem. They also introduced the class of increase-along-rays functions (for short, IAR functions), which can be regarded as a generalization of quasiconvex functions. They established some relations between a generalized Minty variational inequality, increasing-along-rays property, and a star-shaped and nondifferentiable optimization problem. Motivated and inspired by the works [8, 9, 14], we introduce the concept of invex-star-shaped sets which can be regarded as a mixture of star-shaped sets and invex sets. We also attempt to generalize the concept of IAR functions to the invexity case by introducing the class of invex-increase-along-rays functions. In terms of lower Dini directional derivative, we introduce the generalized Minty prevariational inequality, which is a generalization of the classical Minty prevariational inequality and the generalized Minty variational inequality in the sense of Crespi et al [8, 9]. We establish some relations between a generalized Minty prevariational inequality, invex-increase-along-rays property, and an invex-star-shaped optimization problem. At last, we introduce perturbed generalized Minty prevariational inequalities and prove the equivalence of generalized Minty prevariational inequalities and perturbed generalized Minty prevariational inequalities. Our results generalize the works of Crespi et al [8, 9] to the invex case.

2. Preliminaries

In what follows, unless otherwise specified, we always suppose that $\bar{R} = R \cup \{+\infty\}$ is the extended real line, K is a nonempty subset of a real linear space E. Let $f : K \to R$ be a given function and $\eta : K \times K \to E$ be a given map. Denote by $\bar{f} : E \to \bar{R}$ the extension of f on E defined by

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } x \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

The lower Dini directional derivative (for short, Dini derivative) of f at $x \in K$ in the direction $d \in E$ is

$$f'_-(x,d) := \liminf_{t \to +0} (1/t)[\bar{f}(x+td) - f(x)].$$

In terms of Dini derivative we introduce the following generalized Minty prevariational inequality problem:

$$GMPVI(f',K): \text{ find } x^* \in K \text{ suth that } \quad f'_-(x, \eta(x^*,x)) \leqslant 0, \ \forall x \in K.$$

REMARK 2.1.

(1) If $\eta(x,y) = x - y$, then $GMPVI(f',K)$ reduces to the generalized Minty variational inequality considered by Crespi et al [8, 9].

(2) If f is differentiable on an open set containing K, then $GMPVI(f',K)$ reduces to the classical Minty prevariational inequality:

$$MPVI(f',K): \text{ find } x^* \in K \text{ such that } \quad \langle f'(x), \eta(x^*,x) \rangle \leqslant 0, \ \forall x \in K.$$

The generalized Stampacchia prevariational inequality problem is formulated as follows:

$$GSPVI(f',K): \text{ find } x^* \in K \text{ suth that } \quad f'_-(x^*, \eta(x,x^*)) \geqslant 0, \ \forall x \in K.$$

When f is differentiable on an open set containing K, then $GSPVI(f',K)$ reduces the classical prevariational inequality studied in [11, 12, 16, 17, 18, 19, 20]. The generalized Minty and Stampacchia prevariational inequality problems have close relations with the following optimization problem:

$$OP(f,K): \quad \min f(x), \quad s.t. \quad x \in K.$$

A point $x^* \in K$ is called a (global) solution of $OP(f,K)$ if $f(x) \geqslant f(x^*)$ for all $x \in K$. A point $x^* \in K$ is a strict solution of $OP(f,K)$ if $f(x) > f(x^*)$ for all $x \in K \backslash \{x^*\}$. A point $x^* \in K$ is called a local solution of $OP(f,K)$ if there exists a neighbourhood U of x^* such that x^* is a solution of $OP(f, K \cap U)$, and x^* is a strict local solution if there exists a neighbourhood U of x^* such that x^* is a strict solution of $OP(f, K \cap U)$.

In the sequel, we give some definitions.

DEFINITION 2.1. [14] For a given set $K \subset E$ and a given map $\eta : K \times K \to E$, K is said to be invex with respect to η if

$$\forall x,y \in K, \ \forall \lambda \in [0,1] \Rightarrow y + \lambda \eta(x,y) \in K.$$

In this case, we say also that K is an invex set with respect to η. Clearly, invexity is a generalization of convexity.

DEFINITION 2.2. [23] Let $K \subset E$ be an invex set with respect to $\eta : K \times K \to E$. $f : K \to R$ is said to be prequasiinvex with respect to η if

$$f(y + \lambda \eta(x,y)) \leqslant \max\{f(x), f(y)\}, \quad \forall x,y \in K, \lambda \in [0,1].$$

Now we introduce the concept of invex-star-shaped sets, which can be regarded as a mixture of invex sets and star-shaped sets.

DEFINITION 2.3. Let K be a nonempty subset of E and $\eta : K \times K \to E$. The set $ker^1_\eta K$ consisting of all $x \in K$ such that $x + t\eta(y,x) \in K$, $\forall y \in K$, $\forall t \in [0,1]$ is called the 1-invex kernel of K with respect to η. K is said to be invex-star-shaped of type 1 (for short, ISS_1) if $ker^1_\eta K \neq \emptyset$. The set $ker^2_\eta K$ consisting of all $x \in K$ such that $y + (1-t)\eta(x,y) \in K$, $\forall y \in K$, $\forall t \in [0,1]$ is called the 2-invex kernel of K with respect to η. K is said to be invex-star-shaped of type 2 (for short, ISS_2) if $ker^2_\eta K \neq \emptyset$. K is invex-star-shaped (for short, ISS) if it is both invex-star-shaped of types 1 and 2. For convenience, we suppose, by definition, that empty set is invex-star-shaped.

REMARK 2.2. (1) Every invex set is invex-star-shaped. (2) $ker^1_\eta K$ and $ker^2_\eta K$ are different in general. This shall be shown in Example 2.1. (3) If $\eta(x,y) = x - y$, then the concept of invex-star-shaped sets reduces to that of star-shaped (for short, SS) sets. In this case, $ker^i_\eta K$ is abbreviated to $ker K$.

EXAMPLE 2.1. Let $K = [-1,1]$ and $\eta(x,y) = |x-y|$. It is easy to verify that $0 \in ker^1_\eta K$ and $0 \notin ker^2_\eta K$.

DEFINITION 2.4. Let K be a nonempty subset of E and $\eta : K \times K \to E$. We say that η satisfies Condition C if, $\forall x, y \in K$, $\lambda \in [0,1]$ with $y + \lambda\eta(x,y) \in K$,

$$\eta(y, y+\lambda\eta(x,y)) = -\lambda\eta(x,y) \quad \text{and} \quad \eta(x, y+\lambda\eta(x,y)) = (1-\lambda)\eta(x,y).$$

REMARK 2.3. Condition C was first introduced by Mohan and Neogy [24] for invex sets and it plays an important role in the study of generalized invex functions. Mohan and Neogy [24] illustrated that Condition C holds for a large class of functions, rather than just for the trivial case $\eta(x,y) = x - y$. Recently, Chudziak and Tabor [30] also give an interesting characterization of the class of functions satisfying Condition C.

The following property associated with Condition C is interesting. The initial proof was given for invex sets in the proof of Theorem 2.1 in [26] and Remark 2.2 of [29]. For the completeness, we here give its proof.

PROPOSITION 2.1. *Let K be a nonempty subset of E and $\eta : K \times K \to E$. Assume that η satisfies Condition C and $x \in ker^2_\eta K$. Then, the following property holds:* $\forall y \in K$, λ_1, $\lambda_2 \in [0,1]$,

$$\eta(y+\lambda_1\eta(x,y), y+\lambda_2\eta(x,y)) = (\lambda_1 - \lambda_2)\eta(x,y).$$

Proof. By Condition C, one has $\eta(y,y) = 0$ for all $y \in K$. So the conclusion holds trivially when $\lambda_1 = \lambda_2$. Without loss of generality, we suppose that $0 \leqslant \lambda_2 < \lambda_1 \leqslant 1$. It follows from Condition C and $x \in ker^2_\eta K$ that

$$\begin{aligned}
&\eta(y+\lambda_2\eta(x,y), y+\lambda_1\eta(x,y)) \\
&= \eta(y+\lambda_2\eta(x,y), y+\lambda_2\eta(x,y) + (\lambda_1-\lambda_2)\eta(x,y))
\end{aligned}$$

$$
\begin{aligned}
&= \eta\left(y+\lambda_2\eta(x,y), y+\lambda_2\eta(x,y)+\frac{\lambda_1-\lambda_2}{1-\lambda_2}\eta(x,y+\lambda_2\eta(x,y))\right) \\
&= -\frac{\lambda_1-\lambda_2}{1-\lambda_2}\eta(x,y+\lambda_2\eta(x,y)) \\
&= (\lambda_2-\lambda_1)\eta(x,y)
\end{aligned}
$$

and

$$
\begin{aligned}
&\eta(y+\lambda_1\eta(x,y), y+\lambda_2\eta(x,y)) \\
&= \eta(y+\lambda_2\eta(x,y)+(\lambda_1-\lambda_2)\eta(x,y), y+\lambda_2\eta(x,y)) \\
&= \eta\left(y+\lambda_2\eta(x,y)+\frac{\lambda_1-\lambda_2}{1-\lambda_2}\eta(x,y+\lambda_2\eta(x,y)), y+\lambda_2\eta(x,y)\right) \\
&= \eta\left(y'+\frac{\lambda_1-\lambda_2}{1-\lambda_2}\eta(x,y'), y'+\frac{\lambda_1-\lambda_2}{1-\lambda_2}\eta(x,y')+\eta\left(x,y'+\frac{\lambda_1-\lambda_2}{1-\lambda_2}\eta(x,y')\right)\right) \\
&\quad (\text{where } y'=y+\lambda_2\eta(x,y)\,) \\
&= -\eta\left(x,y'+\frac{\lambda_1-\lambda_2}{1-\lambda_2}\eta(x,y')\right) \\
&= \frac{\lambda_1-\lambda_2}{1-\lambda_2}\eta(x,y') \\
&= (\lambda_1-\lambda_2)\eta(x,y). \quad \square
\end{aligned}
$$

DEFINITION 2.5. Let $K \subset E$ be a nonempty set, $\eta : K \times K \to E$ a given map, and $f : K \to R$ a given function. We say that f satisfies Condition D_1 if

$$\bar{f}(y+\eta(x,y)) \leqslant \bar{f}(x), \quad \forall x,y \in K,$$

and f satisfies Condition D_2 if

$$\bar{f}(y+\eta(x,y)) \geqslant \bar{f}(x), \quad \forall x,y \in K \text{ with } y+\eta(x,y) \in K.$$

REMARK 2.4. The above definition of Condition D_1 is similar to Condition D in Yang et al [25, 26].

3. Invex-increase-along-rays functions and optimization problems

In this section, we introduce the class of invex-increase-along-rays functions, which can be regarded as generalizations of prequasiinvex functions [23] and increase-along-rays functions studied in Crespi et al [8, 9]. We derive the equivalence of invex-increase-along-rays property of functions and existence of solutions to optimization problems.

DEFINITION 3.1. Let $K \subset E$ be a nonempty set, $\eta : K \times K \to E$, and $x^* \in K$ a given point. A function $f : K \to R$ is said to be invex-increase-along-rays of type 1 starting at x^* (for short, $f \in IIAR(K,x^*)_1$) if $0 \leqslant t_1 < t_2$ implies that $\bar{f}(x^* +$

$t_1\eta(x,x^*)) \leqslant \bar{f}(x^*+t_2\eta(x,x^*))$, $\forall x \in K$. f is said to be invex-increase-along-rays of type 2 starting at x^* (for short, $f \in IIAR(K,x^*)_2$) if $0 \leqslant t_1 < t_2$ implies that $\bar{f}(x+(1-t_1)\eta(x^*,x)) \leqslant \bar{f}(x+(1-t_2)\eta(x^*,x))$, $\forall x \in K$. We say f is invex-increase-along-rays starting at x^* (for short, $f \in IIAR(K,x^*)$) if $f \in IIAR(K,x^*)_1$ and $f \in IIAR(K,x^*)_2$.

PROPOSITION 3.1. *The following conclusions hold.*

(i) If $f \in IIAR(K,x^*)_2$, then K is ISS_2 and $x^* \in ker_\eta^2 K$.

(ii) If $f \in IIAR(K,x^*)_1$ and f satisfies Condition D_1, then K is ISS_1 and $x^* \in ker_\eta^1 K$.

Proof. (i) Let $x \in K$. Since $f \in IIAR(K,x^*)_2$,

$$\bar{f}(x+(1-t)\eta(x^*,x)) \leqslant \bar{f}(x) = f(x) < +\infty, \quad \forall t \in [0,1].$$

This implies that $x+(1-t)\eta(x^*,x) \in K$, $\forall x \in K$, $\forall t \in [0,1]$. So K is ISS_2 and $x^* \in ker_\eta^2 K$.

(ii) Let $x \in K$. Since f satisfies Condition D_1 and $f \in IIAR(K,x^*)_1$,

$$\bar{f}(x^*+t\eta(x,x^*)) \leqslant \bar{f}(x^*+\eta(x,x^*)) \leqslant f(x) < +\infty, \forall t \in [0,1].$$

This implies that $x^*+t\eta(x,x^*) \in K$, $\forall x \in K$, $\forall t \in [0,1]$. So K is ISS_1 and $x^* \in ker_\eta^1 K$. $\square$

REMARK 3.1. Condition D_1 appears in (ii) of Proposition 3.1, but Condition D_2 does not appear in (i) of Proposition 3.1. A natural problem is: whether or not Condition D_1 can be dropped off at all? The following example answers it in the negative.

EXAMPLE 3.1. Let $K = [-1,+1]$, $\eta(x,y) = 2|x-y|$, and $f : K \to R$ defined by $f(x) = |x|$. It is easy to see that $f \in IIAR(K,0)_1$ and $ker_\eta^1 K = \emptyset$.

REMARK 3.2. If $\eta(x,y) = x-y$, then the concepts of both $IIAR(K,x^*)_1$ and $IIAR(K,x^*)_2$ coincide with the concept of $IAR(K,x^*)$ in Crespi et al [8, 9].

PROPOSITION 3.2. *Let $K \subset E$ be a nonempty set and $\eta : K \times K \to E$ satisfy Condition C. Then the following conclusions hold.*

(a) Let f satisfy Condition D_1, $f \in IIAR(K,x^*)_1$, $x \in K$ a given point, and $x_1(t) := x^*+t\eta(x,x^*)$, $\forall t \in [0,+\infty)$. If $x_1(t)$ has left K, then it will not return back.

(b) Let $f \in IIAR(K,x^*)_2$, $x \in K$ a given point, and $x_2(t) := x+(1-t)\eta(x^*,x)$, $\forall t \in [0,+\infty)$. If $x_2(t)$ has left K, then it will not return back.

Proof. (a) Assume by contradiction that, there exist t_1,t_2 with $0 < t_1 < t_2$ such that $x_1(t_1) \notin K$ and $x_1(t_2) \in K$. Condition C implies that

$$x_1(t_1) = x^*+(t_1/t_2)\eta(x_1(t_2),x^*).$$

Since $f \in IIAR(K,x^*)_1$ and f satisfies Condition D_1,

$$\bar{f}(x_1(t_1)) = \bar{f}(x^* + (t_1/t_2)\eta(x_1(t_2),x^*)) \leqslant \bar{f}(x^* + \eta(x_1(t_2),x^*) \leqslant f(x_1(t_2)) < +\infty,$$

which contradicts $x_1(t_1) \notin K$. Thus, if $x_1(t)$ has left K, then it will not return back.

(b) Assume by contradiction that, there exist t_1,t_2 with $0 < t_1 < t_2$ such that $x_2(t_1) \notin K$ and $x_2(t_2) \in K$. Condition C implies that

$$x_2(t_1) = x_2(t_2) + (1 - t_1/t_2)\eta(x^*,x_2(t_2)).$$

Since $f \in IIAR(K,x^*)_2$,

$$\bar{f}(x_2(t_1)) = \bar{f}(x_2(t_2) + (1 - t_1/t_2)\eta(x^*,x_2(t_2))) \leqslant \bar{f}(x_2(t_2)) < +\infty,$$

which contradicts $x_2(t_1) \notin K$. Thus, if $x_2(t)$ has left K, then it will not return back. $\square$

Now we derive some characterizations of $IIAR_i$ functions.

THEOREM 3.1. *Let $K \subset E$ be an ISS_1 set, $x^* \in ker_\eta^1 K$, $\eta : K \times K \to E$, and $f : K \to R$. If f satisfies Condition D_1 and $f \in IIAR(K,x^*)_1$, then x^* is a solution of $OP(f,K)$, and for each $c \in R$ with $c \geqslant f(x^*)$, $x^* \in ker_\eta^1 lev_{\leqslant c} f$, where $lev_{\leqslant c} f := \{x \in K : f(x) \leqslant c\}$ is the lower level set of f. Conversely, if Condition C is satisfied, x^* is a solution of $OP(f,K)$, and for each $c \in R$ with $c \geqslant f(x^*)$, $x^* \in ker_\eta^1 lev_{\leqslant c} f$, then $f \in IIAR(K,x^*)_1$.*

Proof. Let $f \in IIAR(K,x^*)_1$. Since f satisfies Condition D_1,

$$f(x^*) \leqslant f(x^* + t\eta(x,x^*)) \leqslant f(x^* + \eta(x,x^*)) \leqslant f(x),\ \forall t \in [0,1],\ \forall x \in K.$$

This implies that x^* is a solution of $OP(f,K)$. Suppose, on the contrary, that there exists $c \in R$ with $f(x^*) \leqslant c$ such that $x^* \notin ker_\eta^1 lev_{\leqslant c} f$. This means that there exist $x \in K$ with $f(x) \leqslant c$, and $t \in [0,1]$ such that $f(x^* + t\eta(x,x^*)) > c$. But we have

$$f(x^* + t\eta(x,x^*)) \leqslant f(x^* + \eta(x,x^*)) \leqslant f(x) \leqslant c,$$

since $f \in IIAR(K,x^*)_1$ and Condition D_1 holds, a contradiction. Thus, for each $c \in R$ with $c \geqslant f(x^*)$, $x^* \in ker_\eta^1 lev_{\leqslant c} f$.

Conversely, suppose, on the contrary, that $f \notin IIAR(K,x^*)_1$. Then there exist $x \in K$ and t_1,t_2 with $0 \leqslant t_1 < t_2$ such that

$$\bar{f}(x^* + t_1\eta(x,x^*)) > \bar{f}(x^* + t_2\eta(x,x^*)).$$

By (a) of Proposition 3.2, we can assume that $x^* + t_1\eta(x,x^*), x^* + t_2\eta(x,x^*) \in K$. Set $c := f(x^* + t_2\eta(x,x^*))$. Clearly, $x^*, x^* + t_2\eta(x,x^*) \in lev_{\leqslant c} f$ since x^* is a solution of $OP(f,K)$. Condition C implies that

$$x^* + t_1\eta(x,x^*) = x^* + (t_1/t_2)\eta(x^* + t_2\eta(x,x^*),x^*).$$

Since $x^* \in ker_\eta^1 lev_{\leqslant c} f$, we have $x^* + t_1\eta(x,x^*) \in lev_{\leqslant c} f$, a contradiction. Thus, $f \in IIAR(K,x^*)_1$. $\square$

THEOREM 3.2. *Let $K \subset E$ be an ISS_2 set, $x^* \in ker_\eta^2 K$, $\eta : K \times K \to E$, and $f : K \to R$. If f satisfies Condition D_2 and $f \in IIAR(K,x^*)_2$, then x^* is a solution of $OP(f,K)$, and for each $c \in R$ with $c \geqslant f(x^*)$, $x^* \in ker_\eta^2 lev_{\leqslant c} f$. Conversely, if Condition C is satisfied, x^* is a solution of $OP(f,K)$, and for each $c \in R$ with $c \geqslant f(x^*)$, $x^* \in ker_\eta^2 lev_{\leqslant c} f$, then $f \in IIAR(K,x^*)_2$.*

Proof. Let $f \in IIAR(K,x^*)_2$. Since f satisfies Condition D_2,

$$f(x^*) \leqslant f(x+\eta(x^*,x)) \leqslant f(x+(1-t)\eta(x^*,x)) \leqslant f(x),\ \forall t \in [0,1],\ \forall x \in K.$$

This implies that x^* is a solution of $OP(f,K)$. Suppose, on the contrary, that there exists $c \in R$ with $f(x^*) \leqslant c$ such that $x^* \notin ker_\eta^2 lev_{\leqslant c} f$. This means that there exist $x \in K$ with $f(x) \leqslant c$, and $t \in [0,1]$ such that $f(x+(1-t)\eta(x^*,x)) > c$. But we have

$$f(x+(1-t)\eta(x^*,x)) \leqslant f(x) \leqslant c$$

since $f \in IIAR(K,x^*)_2$, a contradiction. Thus, for each $c \in R$ with $c \geqslant f(x^*)$, $x^* \in ker_\eta^2 lev_{\leqslant c} f$.

Conversely, suppose, on the contrary, that $f \notin IIAR(K,x^*)_2$. Then there exist $x \in K$ and t_1,t_2 with $0 \leqslant t_1 < t_2$ such that

$$\bar{f}(x+(1-t_1)\eta(x^*,x)) > \bar{f}(x+(1-t_2)\eta(x^*,x)).$$

By (b) of Proposition 3.2, we can assume that $x+(1-t_1)\eta(x^*,x), x+(1-t_2)\eta(x^*,x) \in K$. Set $c := f(x+(1-t_2)\eta(x^*,x))$. Clearly, $x^*, x+(1-t_2)\eta(x^*,x) \in lev_{\leqslant c} f$ since x^* is a solution of $OP(f,K)$. Condition C implies that

$$x+(1-t_1)\eta(x^*,x) = [x+(1-t_2)\eta(x^*,x)] + (1-t_1/t_2)\eta(x^*, x+(1-t_2)\eta(x^*,x)).$$

Since $x^* \in ker_\eta^2 lev_{\leqslant c} f$, we have $x+(1-t_1)\eta(x^*,x) \in lev_{\leqslant c} f$, a contradiction. Thus, $f \in IIAR(K,x^*)_2$. $\square$

When $\eta(x,y) = x-y$, we obtain the following corollary.

COROLLARY 3.1. *Let K be a star-shaped set and $x^* \in ker K$, and $f : K \to R$ a given function. Then $f \in IAR(K,x^*)$ if and only if x^* is a solution of $OP(f,K)$ and for each $c \in R$ with $c \geqslant f(x^*)$, $x^* \in ker\, lev_{\leqslant c} f$.*

REMARK 3.3. Proposition 2 of Crespi et al [9] also gave a characterization of increase-along-rays property of f without the condition: x^* is a solution of $OP(f,K)$. In the following example, we show that it can not be dropped off.

EXAMPLE 3.2. Let $K = [0,2]$, $\eta(x,y) = x-y$, $f(x) = (x-1)^2$, and $x^* = 0$. Obviously, K is star-shaped and $x^* \in ker K$. For any $c \in R$ with $c \geqslant f(0) = 1$, we have $0 \in ker\, lev_{\leqslant c} f = [0,2]$. But $f \notin IAR(K,0)$.

REMARK 3.4. In Theorems 3.1 and 3.2, the condition that all the nonempty lower level sets of f contain x^* in their i-invex kernels with respect to η is essential to prove $f \in IIAR(K,x^*)_i, i = 1,2$.

EXAMPLE 3.3. Let $K = [-\pi,+\pi]$, $f(x) = \sin x$, $\eta(x,y) = x-y$, and $x^* = -\pi/2$. It is easy to verify that x^* is a solution of $OP(f,K)$, $x^* \notin ker\, lev_{\leqslant 0} f$, and $f \notin IAR(K, -\pi/2)$.

THEOREM 3.3. *Let* $K \subset E$ *be an* ISS_1 *set,* $x^* \in ker^1_\eta K$*,* $\eta : K \times K \to E$*, and* $f : K \to R$*. If* f *satisfies Condition* D_1 *and* $f \in IIAR(K,x^*)_1$*, then* x^* *is a solution of* $OP(f,K)$*, and for each* $c \in R$ *with* $c > f(x^*)$*,* $x^* \in ker^1_\eta lev_{<c} f$*, where* $lev_{<c} f := \{x \in K : f(x) < c\}$*. Conversely, if Condition* C *is satisfied,* x^* *is a solution of* $OP(f,K)$*, and for each* $c \in R$ *with* $c > f(x^*)$*,* $x^* \in ker^1_\eta lev_{<c} f$*, then* $f \in IIAR(K,x^*)_1$*.*

Proof. Let f satisfy Condition D_1 and $f \in IIAR(K,x^*)_1$. By Theorem 3.1, x^* is a solution of $OP(f,K)$. For any $c \in R$ with $c > f(x^*)$ and $x \in lev_{<c} f$, it follows from Condition D_1 that

$$f(x^* + t\eta(x,x^*)) \leqslant f(x^* + \eta(x,x^*)) \leqslant f(x) < c, \ \forall t \in [0,1].$$

Hence, $x^* \in ker^1_\eta lev_{<c} f$ for all $c \in R$ with $c > f(x^*)$.

Conversely, suppose that Condition C is satisfied, x^* is a solution of $OP(f,K)$, and for each $c \in R$ with $c > f(x^*)$, $x^* \in ker^1_\eta lev_{<c} f$. Let $0 \leqslant t_1 < t_2$ and $x \in K$ with $x^* + t_1\eta(x,x^*), x^* + t_2\eta(x,x^*) \in K$. Then

$$x^*, x^* + t_2\eta(x,x^*) \in lev_{<c+\varepsilon} f, \quad \forall \varepsilon \in (0,+\infty),$$

where $c := f(x^* + t_2\eta(x,x^*))$. Since $x^* \in \mathrm{ker}^1_\eta\, lev_{<c+\varepsilon} f$ and Condition C holds,

$$\begin{aligned} f(x^* + t_1\eta(x,x^*)) &= f(x^* + (t_1/t_2)\eta(x^* + t_2\eta(x,x^*),x^*)) < c + \varepsilon \\ &= f(x^* + t_2\eta(x,x^*)) + \varepsilon, \ \forall \varepsilon > 0. \end{aligned}$$

Letting $\varepsilon \to 0$, we have

$$f(x^* + t_1\eta(x,x^*)) \leqslant f(x^* + t_2\eta(x,x^*)),$$

which together with (a) of Proposition 3.2 implies that $f \in IIAR(K,x^*)_1$. □

By similar arguments as in Theorem 3.3, we can obtain the following theorem.

THEOREM 3.4. *Let* $K \subset E$ *be an* ISS_2 *set,* $x^* \in ker^2_\eta K$*,* $\eta : K \times K \to E$*, and* $f : K \to R$*. If* f *satisfies Condition* D_2 *and* $f \in IIAR(K,x^*)_2$*, then* x^* *is a solution of* $OP(f,K)$*, and for each* $c \in R$ *with* $c > f(x^*)$*,* $x^* \in ker^2_\eta lev_{<c} f$*. Conversely, if Condition* C *is satisfied,* x^* *is a solution of* $OP(f,K)$*, and for each* $c \in R$ *with* $c > f(x^*)$*,* $x^* \in ker^2_\eta lev_{<c} f$*, then* $f \in IIAR(K,x^*)_2$*.*

REMARK 3.5. Condition D_1 (res. D_2) appears only in one part of Theorems 3.1 and 3.3 (res. Theorems 3.2 and 3.4). A natural problem is : whether or not it can be dropped off at all? The following examples answer it in the negative.

EXAMPLE 3.4. Let $K = R = (-\infty,+\infty)$, $\eta(x,y) = (x-y)^2$ and $f : K \to R$ defined by $f(x) = x^3$. Choose $x = 0$ and $y = 2$. It follows that

$$f(x + \eta(y,x)) = 4^6 > f(y) = 4^3.$$

Thus, Condition D_1 does not hold. It is easy to verify that $0 \in ker^1_\eta K$, $f \in IIAR(K,0)_1$, and 0 is not a solution of $OP(f,K)$.

EXAMPLE 3.5. Let $K=R=(-\infty,+\infty)$, $\eta(x,y)=2|x-y|$, and $f:K\to R$ defined by $f(x)=-2x^3$. Choose $x=1$ and $y=0$. It follows that

$$f(x+\eta(y,x))=-54<f(y)=0.$$

This implies that Condition D_2 does not hold. It is easy to see that $0\in ker_\eta^2 K, f\in IIAR(K,0)_2$ and 0 is not a solution of $OP(f,K)$.

THEOREM 3.5. *Let $K\subset E$ be an ISS_2 set, $x^*\in ker_\eta^2 K$, $\eta:K\times K\to E$ and $f:K\to R$ a given function such that $f\in IIAR(K,x^*)_2$. Then there is no point $x\in K, x\neq x^*$ which is a strict local solution of $OP(f,K)$.*

Proof. For any $x\in K$ and $t\in[0,1]$, $x+(1-t)\eta(x^*,x)\in K$ since $x^*\in ker_\eta^2 K$. Since $f\in IIAR(K,x^*)_2$,

$$f(x+(1-t)\eta(x^*,x))\leqslant f(x),\quad \forall t\in[0,1].$$

If U is an arbitrary neighborhood of x, then for $t\in[0,1]$ 'near enough' to '1', we have $x+(1-t)\eta(x^*,x)\in U\cap K$. So $x\in K$ can not be a strict local solution of $OP(f,K)$. $\square$

REMARK 3.6. Theorem 3.5 shows that, under suitable conditions, only x^* can be a strict local solution of $OP(f,K)$ with $f\in IIAR(K,x^*)_2$. A natural problem is: whether or not a similar conclusion holds for $f\in IIAR(K,x^*)_1$. The next example answers it in the negative.

EXAMPLE 3.6. Let $K=[-1,1], \eta(x,y)=|x-y|$, and $f:K\to R$ defined by $f(x)=x$. It is easy to verify $0\in ker_\eta^1 K$ and $f\in IIAR(K,0)_1$. However, x^* is not a solution of $OP(f,K)$ and $x=-1$ is a strict local (even global) solution of $OP(f,K)$.

4. Invex-increase-along-rays functions and generalized Minty prevariational inequalities

In this section, we discuss the relations between invex-increase-along-rays functions and generalized Minty prevariational inequalities. First, we introduce the concept of invex radially lower semicontinuous functions.

DEFINITION 4.1. Let $K\subset E$ be a nonempty set, $\eta:K\times K\to E$, and $f:K\to R$. f is said to be 1-invex radially lower semicontinuous along rays starting at $x^*\in K$ (for short, $f\in IRLSC(K,x^*)_1$) if, for any $x\in K$, the restriction of f on $K_{x^*x}^{\eta,1}:=\{z\in K: z=x^*+t\eta(x,x^*), t\in[0,+\infty)\}$ is lower semicontinuous. In another word, $f\in IRLSC(K,x^*)_1$ iff, $\forall x\in K$, the function $\varphi:\mathscr{D}_1\to R, \varphi(t):=f(x^*+t\eta(x,x^*))$ is lower semicontinuous, where $\mathscr{D}_1:=\{t\in[0,+\infty): x^*+t\eta(x,x^*)\in K\}$. We say that f has $IRLSC_1$ property if $f\in IRLSC(K,x^*)_1$ for all $x^*\in K$. f is said to be 2-invex radially lower semicontinuous along rays starting at $x^*\in K$ (for short, $f\in IRLSC(K,x^*)_2$) if, for any $x\in K$, the restriction of f on $K_{x^*x}^{\eta,2}:=\{z\in K: z=x+(1-t)\eta(x^*,x), t\in[0,+\infty)\}$ is lower semicontinuous. In another word, $f\in IRLSC(K,x^*)_2$ iff, $\forall x\in K$, the function $\varphi:\mathscr{D}_2\to R, \varphi(t):=f(x+(1-t)\eta(x^*,x))$ is lower semicontinuous, where

$\mathscr{D}_2 := \{t \in [0,+\infty) : x + (1-t)\eta(x^*,x) \in K\}$. We say that f has $IRLSC_2$ property if $f \in IRLSC(K,x^*)_2$ for all $x^* \in K$.

REMARK 4.1. When $\eta(x,y) = x - y$ and the concepts of $f \in IRLSC(K,x^*)_1$ and $f \in IRLSC(K,x^*)_2$ coincide. In this case we say that $f : K \to R$ is radially lower semicontinuous along rays starting at $x^* \in K$ (for short, $f \in RLSC(K,x^*)$). Note that the concept of $RLSC$ functions here is slightly different from one in Crespi et al [8, 9]. Star-shaped property of K and $x^* \in ker K$ are supposed in their definitions.

THEOREM 4.1. *Let $K \subset E$ be a nonempty set, $x^* \in K$ a point, $\eta : K \times K \to E$, and $f : K \to R$. Then the following conclusions hold:*

(i) If $x^* \in ker^1_\eta K$, $f \in IRLSC(K,x^*)_1$, x^* is a solution of $GMPVI(f',K)$, and Condition C holds, then $f \in IIAR(K,x^*)_1$.

(ii) If $f \in IIAR(K,x^*)_1$, then x^* is a solution of $GSPVI(f',K)$.

Proof. (i) Fix $x \in K$ and define

$$x_1(t) := x^* + t\eta(x,x^*),\ \forall t \in [0,+\infty).$$

It follows from Condition C that

$$f'_-(x_1(t),\eta(x^*,x_1(t))) = f'_-(x_1(t),-t\eta(x,x^*)) = tf'_-(x_1(t),-\eta(x,x^*)) \leqslant 0,\ \forall t \in [0,+\infty)$$

whenever $x_1(t) \in K$. Therefore,

$$f'_-(x_1(t),-\eta(x,x^*)) \leqslant 0,\ \forall t \in [0,+\infty)$$

whenever $x_1(t) \in K$. To prove $f \in IIAR(K,x^*)_1$, we first show that if $x_1(t)$ has left K, then it will not return back. Assume that this is not the case. Then there exist λ_1,λ_2 with $0 < \lambda_1 < \lambda_2$ such that $x_1(\lambda_1) \notin K$ and $x_1(\lambda_2) \in K$. Since $x^* \in ker^1_\eta K$ and Condition C is satisfied,

$$x_1(\lambda_1) = x^* + (\lambda_1/\lambda_2)\eta(x_1(\lambda_2),x^*) \in K,$$

a contradiction. Thus, $0 \leqslant t_1 < t_2$ and $x_1(t_1), x_1(t_2) \in K$ imply that $x_1(t) \in K$, $\forall t \in [t_1,t_2]$. Let $x \in K$ and $0 \leqslant t_1 < t_2$ with $x_1(t_1), x_1(t_2) \in K$. Define the function $\varphi : [t_1,t_2] \to R$ by

$$\varphi(t) = f(x_1(t)) - [(t_2-t)/(t_2-t_1)]f(x_1(t_1)) - [(t-t_1)/(t_2-t_1)]f(x_1(t_2)).$$

Then φ is lower semicontinuous since $f \in IRLSC(K,x^*)_1$. By the Weierstrass theorem, φ attains its global minimum at some $\hat{t} \in [t_1,t_2]$. We may assume that $\hat{t} \neq t_1$. Indeed, if $\hat{t} = t_1$, then the global minimum is attained at $t = t_2$ because $\varphi(t_1) = \varphi(t_2) = 0$. From the definition of $\hat{t}$, we have $\varphi'_-(\hat{t},-1) \geqslant 0$. It follows that

$$\begin{aligned}\varphi'_-(\hat{t},-1) &= \liminf_{s\to+0} \frac{\varphi(\hat{t}-s)-\varphi(\hat{t})}{s}\\ &= \liminf_{s\to+0} \frac{f(x^*+\hat{t}\eta(x,x^*)-s\eta(x,x^*))-f(x^*+\hat{t}\eta(x,x^*))}{s}\end{aligned}$$

$$-\frac{f(x_1(t_1))-f(x_1(t_2))}{t_2-t_1}$$
$$=f'_-(x_1(\hat{t}),-\eta(x,x^*))-\frac{f(x_1(t_1))-f(x_1(t_2))}{t_2-t_1}\geqslant 0,$$

which together with $f'_-(x_1(\hat{t}),-\eta(x,x^*))\leqslant 0$ implies that $f(x_1(t_1))\leqslant f(x_1(t_2))$. Thus $f\in IIAR(K,x^*)_1$.

(ii) Since $f\in IIAR(K,x^*)_1$, it follows that

$$f'_-(x^*,\eta(x,x^*))=\liminf_{t\to+0}\frac{\overline{f}(x^*+t\eta(x,x^*))-f(x^*)}{t}$$
$$=\liminf_{t\to+0}\frac{\overline{f}(x_1(t))-f(x_1(0))}{t}\geqslant 0,\ \forall x\in K.$$

This implies that x^* is a solution of $GSPVI(f',K)$. □

The following example illustrates the conclusion of Theorem 4.1.

EXAMPLE 4.1. Let $K=[-7,-2]\cup[2,10]$ and let $\eta:K\times K\to R$ and $f:K\to R$ be defined by

$$\eta(x,y)=\begin{cases}x-y, & x,y\in[2,10],\\ x-y, & x,y\in[-7,-2],\\ -7-y, & x\in[2,10],\ y\in[-7,-2],\\ 2-y, & x\in[-7,-2],\ y\in[2,10]\end{cases}\qquad\text{and}\qquad f(x)=|x|,\ \forall x\in K.$$

It has been shown in Mohan and Neogy [24] that K is invex with respect to η and Condition C holds. Take $x^*=-2$. It is easy to verify that $f\in IRLSC(K,x^*)_1$, x^* is a solution of $GMPVI(f',K)$ as well as a solution of $GSPVI(f',K)$, and $f\in IIAR(K,x^*)_1$.

REMARK 4.2. Under the assumptions of Theorem 4.1, $f\in IIAR(K,x^*)_1$ does not imply that x^* is a solution of $GMPVI(f',K)$.

EXAMPLE 4.2. Let $K=[-1,1],\eta(x,y)=-|x-y|$, and $f:K\to R$ defined by $f(x)=-2x$. It is easy to see that $0\in ker_\eta^1 K$ and $f\in IIAR(K,0)_1$. It follows that

$$f'_-(0,\eta(x,0))=2|x|\geqslant 0,\quad \forall x\in K$$

and

$$f'_-(x,\eta(0,x))=2|x|>0,\ \forall x\in K \text{ with } x\neq 0.$$

This implies that 0 is a solution of $GSPVI(f',K)$, but not a solution of $GMPVI(f',K)$.

THEOREM 4.2. *Let $K\subset E$ be an ISS_2 set, $x^*\in ker_\eta^2 K$, $\eta:K\times K\to E$, and $f:K\to R$. Then the following conclusions hold:*

(i) If $f\in IRLSC(K,x^*)_2$, x^* is a solution of $GMPVI(f',K)$, and Condition C holds, then $f\in IIAR(K,x^*)_2$.

(ii) If $f \in IIAR(K,x^*)_2$, then x^* is a solution of $GMPVI(f',K)$.

Proof. (i) Fix $x \in K$ and define

$$x_2(t) := x + (1-t)\eta(x^*,x), \ \forall t \in [0,+\infty).$$

It follows from Condition C that

$$f'_-(x_2(t),\eta(x^*,x_2(t))) = f'_-(x_2(t),t\eta(x^*,x)) = tf'_-(x_2(t),\eta(x^*,x)) \leqslant 0, \ \forall t \in [0,+\infty)$$

whenever $x_2(t) \in K$. Therefore,

$$f'_-(x_2(t),\eta(x^*,x)) \leqslant 0, \ \forall t \in [0,+\infty)$$

whenever $x_2(t) \in K$. To prove $f \in IIAR(K,x^*)_2$, we first show that if $x_2(t)$ has left K, then it will not return back. Assume that this is not the case. Then there exist λ_1,λ_2 with $0 < \lambda_1 < \lambda_2$ such that $x_2(\lambda_1) \notin K$ and $x_2(\lambda_2) \in K$. Since $x^* \in ker^2_\eta K$ and Condition C is satisfied,

$$x_2(\lambda_1) = x_2(\lambda_2) + (1-\lambda_1/\lambda_2)\eta(x^*,x_2(\lambda_2)) \in K,$$

a contradiction. Thus, $0 \leqslant t_1 < t_2$ and $x_2(t_1),x_2(t_2) \in K$ imply that $x_2(t) \in K$, $\forall t \in [t_1,t_2]$. Let $x \in K$ and $0 \leqslant t_1 < t_2$ with $x_2(t_1),x_2(t_2) \in K$. Define the function $\varphi : [t_1,t_2] \to R$ by

$$\varphi(t) = f(x_2(t)) - [(t_2-t)/(t_2-t_1)]f(x_2(t_1)) - [(t-t_1)/(t_2-t_1)]f(x_2(t_2)).$$

Then φ is lower semicontinuous since $f \in IRLSC(K,x^*)_2$. By the Weierstrass theorem, φ attains its global minimum at some $\hat{t} \in [t_1,t_2]$. We may assume that $\hat{t} \neq t_1$. Indeed, if $\hat{t} = t_1$, then the global minimum is attained at $t = t_2$ because $\varphi(t_1) = \varphi(t_2) = 0$. From the definition of $\hat{t}$, we have $\varphi'_-(\hat{t},-1) \geqslant 0$. It follows that

$$\begin{aligned}
\varphi'_-(\hat{t},-1) &= \liminf_{s\to+0} \frac{\varphi(\hat{t}-s)-\varphi(\hat{t})}{s} \\
&= \liminf_{s\to+0} \frac{f(x+(1-\hat{t})\eta(x^*,x)+s\eta(x^*,x)) - f(x+(1-\hat{t})\eta(x^*,x))}{s} \\
&\quad - \frac{f(x_2(t_1))-f(x_2(t_2))}{t_2-t_1} \\
&= f'_-(x_2(\hat{t}),\eta(x^*,x)) - \frac{f(x_2(t_1))-f(x_2(t_2))}{t_2-t_1} \geqslant 0,
\end{aligned}$$

which together with $f'_-(x_2(\hat{t}),\eta(x^*,x)) \leqslant 0$ implies that $f(x_2(t_1)) \leqslant f(x_2(t_2))$. Thus $f \in IIAR(K,x^*)_2$.

(ii) Since $f \in IIAR(K,x^*)_2$, it follows that

$$\begin{aligned}
f'_-(x,\eta(x^*,x)) &= \liminf_{t\to+0} \frac{f(x+t\eta(x^*,x))-f(x)}{t} \\
&= \liminf_{t\to+0} \frac{f(x_2(1-t))-f(x_2(1))}{t} \leqslant 0, \ \forall x \in K.
\end{aligned}$$

This implies that x^* is a solution of $GMPVI(f',K)$. $\square$

REMARK 4.3. Under the assumptions of Theorem 4.2, $f \in IIAR(K,x^*)_2$ does not imply that x^* is a solution of $GSPVI(f',K)$. See the following example.

EXAMPLE 4.3. $K = R = (-\infty,+\infty)$, $\eta(x,y) = |x-y|$, and $f(x) = -2x$. It is easy to verify that $0 \in ker^2_\eta K$, and $f \in IIAR(K,0)_2$. It follows that

$$f'_(x,\eta(0,x)) = -2|x| \leqslant 0, \ \forall x \in K$$

and

$$f'_-(0,\eta(x,0)) = -2|x| < 0, \ \forall x \neq 0.$$

Thus 0 is a solution of $GMPVI(f',K)$, but not a solution of $GSPVI(f',K)$.

DEFINITION 4.2. Let $K \subset E$ and $x^* \in E$. We say that K is 1-invex radially closed along rays starting at x^* (for short, $IRCAR^1_{x^*}$) if $K^{\eta,1}_{x^*x}$ (which is defined as in Definition 4.1) is closed in $R^{\eta,1}_{x^*x} := \{z \in E : z = x^* + t\eta(x,x^*), t \in [0,+\infty)\}$ (in view of this parametrization, the topological structure on $R^{\eta,1}_{x^*x}$ is determined by the topological structure on R) for all $x \in E$. We say that K is 1-invex radially closed (for short, $IRCAR^1$) if K is $IRCAR^1_{x^*}$ for all $x^* \in K$. We say that K is 2-invex radially closed along rays starting at x^* (for short, $IRCAR^2_{x^*}$) if $K^{\eta,2}_{x^*x}$ is closed in $R^{\eta,2}_{x^*x} := \{z \in E : z = x + (1-t)\eta(x^*,x), t \in [0,+\infty)\}$ for all $x \in E$. We say that K is 2-invex radially closed (for short, $IRCAR^2$) if K is $IRCAR^2_{x^*}$ for all $x^* \in K$.

REMARK 4.4. The concepts of $IRCAR^1_{x^*}$ and $IRCAR^2_{x^*}$ coincide when $\eta(x,y) = x-y$. In this case, we say that K is radially closed along rays starting at x^* (for short, $RCAR_{x^*}$). Note that the concept of $RCAR_{x^*}$ is slightly different from one in [8, 9].

THEOREM 4.3. *Let $K \subset E$ be a nonempty $IRCAR^1$ set, $\eta : K \times K \to E$, and $f : K \to R$. If $f \in IRLSC(K,x^*)_1$, x^* is a solution of $GMPVI(f',K)$, and Condition C holds, then $f \in IIAR(K,x^*)_1$.*

Proof. For given $x \in K$, define $x_1(t)$ as in the proof of Theorem 4.1. Also in the same way as in the proof of Theorem 4.1, we can obtain

$$f'_-(x_1(t),\eta(x^*,x_1(t))) \leqslant 0, \quad \forall t \in [0,+\infty)$$

whenever $x_1(t) \in K$. To prove $f \in IIAR(K,x^*)_1$, we first show that if $x_1(t)$ has left K, then it will not return back. Suppose that it is not the case. Since K is $IRCAR^1$, then there exist $\delta > 0$ and $t_0 > 0$ such that

$$x_1(t_0) \in K, x_1(t) \notin K, \ \forall t \in (t_0 - \delta, t_0).$$

It follows from Condition C that

$$\begin{aligned} & f'_-(x_1(t_0),\eta(x^*,x_1(t_0))) \\ = & f'_-(x_1(t_0),-t_0\eta(x,x^*)) \end{aligned}$$

$$
\begin{aligned}
&= t_0 f'_-(x_1(t_0), -\eta(x,x^*)) \\
&= t_0 \liminf_{s\to+0} \frac{\overline{f}(x_1(t_0-s) - f(x_1(t_0))}{s} = +\infty,
\end{aligned}
$$

a contradiction. Next, we can show that $0 \leqslant t_1 < t_1$ and $x_1(t) \in K$ for $t_1 \leqslant t \leqslant t_2$ imply that $f(x_1(t_1)) \leqslant f(x_1(t_2))$ in the same way as in the proof of Theorem 4.1. Thus, $f \in IIAR(K,x^*)_1$. $\square$

THEOREM 4.4. *Let $K \subset E$ be a nonempty $IRCAR^2$ set, $\eta : K\times K \to E$, and $f : K \to R$. If $f \in IRLSC(K,x^*)_2$, $x^* \in K$ is a solution of $GMPVI(f',K)$, and Condition C holds, then $f \in IIAR(K,x^*)_2$.*

Proof. For given $x \in K$, define $x_2(t)$ as in the proof of Theorem 4.2. Also in the same way as in the proof of Theorem 4.2, we can obtain

$$f'_-(x_2(t), \eta(x^*, x_2(t))) \leqslant 0, \ \forall t \in [0,+\infty)$$

whenever $x_1(t) \in K$. To prove $f \in IIAR(K,x^*)_2$, we first show that if $x_2(t)$ has left K, then it will not return back. Suppose that it is not the case. Since K is $IRCAR^2$, then there exist $\delta > 0$ and $t_0 > 0$ such that

$$x_2(t_0) \in K, x_2(t) \notin K, \ \forall t \in (t_0-\delta, t_0).$$

It follows from Condition C that

$$
\begin{aligned}
&f'_-(x_2(t_0), \eta(x^*, x_2(t_0))) \\
&= f'_-(x_2(t_0), t_0\eta(x^*,x)) \\
&= t_0 f'_-(x_2(t_0), \eta(x^*,x)) \\
&= t_0 \liminf_{s\to+0} \frac{\overline{f}(x_2(t_0-s) - f(x_2(t_0))}{s} = +\infty,
\end{aligned}
$$

a contradiction. Next, we can show that $0 \leqslant t_1 < t_1$ and $x_2(t) \in K$ for $t_1 \leqslant t \leqslant t_2$ imply that $f(x_2(t_1)) \leqslant f(x_2(t_2))$ in the same way as in the proof of Theorem 4.2. Thus, $f \in IIAR(K,x^*)_2$. $\square$

5. Generalized Minty prevariational inequalities and optimization problems

In this section, we study the relations between generalized Minty prevariational inequalities and optimization problems.

THEOREM 5.1. *Let K be an ISS_2 set, $x^* \in ker^2_\eta K$, $\eta : K\times K \to E$, $f : K \to R$, and Condition C is satisfied. Then the following conclusions hold:*

(i) If $f \in IRLSC(K,x^*)_2$, Condition D_2 holds, and x^* is a solution of $GMPVI(f',K)$, then x^* is a solution of $OP(f,K)$ and $x^* \in ker^2_\eta lev_{\leqslant c} f$ for all $c \in R$ with $c \geqslant f(x^*)$.

(ii) *If x^* is a solution of $OP(f,K)$ and $x^* \in ker_\eta^2 \, lev_{\leqslant c} f$ for all $c \in R$ with $c \geqslant f(x^*)$, then x^* is a solution of $GMPVI(f',K)$.*

Proof. The conclusions follow directly from Theorems 3.2 and 4.2. □

THEOREM 5.2. *Let K be an $IRCAR^2$ set, $x^* \in K$, $\eta : K \times K \to E$, $f : K \to R$, and Condition C is satisfied. Then the following conclusions hold:*

(i) *If $f \in IRLSC(K,x^*)_2$, Condition D_2 holds, and x^* is a solution of $GMPVI(f',K)$, then x^* is a solution of $OP(f,K)$ and $x^* \in ker_\eta^2 \, lev_{\leqslant c} f$ for all $c \in R$ with $c \geqslant f(x^*)$.*

(ii) *If x^* is a solution of $OP(f,K)$ and $x^* \in ker_\eta^2 \, lev_{\leqslant c} f$ for all $c \in R$ with $c \geqslant f(x^*)$, then x^* is a solution of $GMPVI(f',K)$.*

Proof. The conclusions follow directly from Theorems 3.2 and 4.4, and Proposition 3.1. □

THEOREM 5.3. *Let K be an ISS_1 set, $x^* \in ker_\eta^1 K$, $\eta : K \times K \to E$, $f : K \to R$, and Conditions C is satisfied. Then the following conclusions hold:*

(i) *If $f \in IRLSC(K,x^*)_1$, Condition D_1 holds, and x^* is a solution of $GMPVI(f',K)$, then x^* is a solution of $OP(f,K)$ and $x^* \in ker_\eta^1 \, lev_{\leqslant c} f$ for all $c \in R$ with $c \geqslant f(x^*)$.*

(ii) *If x^* is a solution of $OP(f,K)$ and $x^* \in ker_\eta^1 \, lev_{\leqslant c} f$ for all $c \in R$ with $c \geqslant f(x^*)$, then x^* is a solution of $GSPVI(f',K)$.*

Proof. The conclusions follow directly from Theorems 3.1 and 4.1. □

THEOREM 5.4. *Let K be an $IRCAR^1$ set, $x^* \in K$, $\eta : K \times K \to E$, $f : K \to R$, and Condition C is satisfied. Then the following conclusions hold:*

(i) *If $f \in IRLSC(K,x^*)_1$, Condition D_1 holds, and x^* is a solution of $GMPVI(f',K)$, then x^* is a solution of $OP(f,K)$ and $x^* \in ker_\eta^1 \, lev_{\leqslant c} f$ for all $c \in R$ with $c \geqslant f(x^*)$.*

(ii) *If x^* is a solution of $OP(f,K)$ and $x^* \in ker_\eta^1 \, lev_{\leqslant c} f$ for all $c \in R$ with $c \geqslant f(x^*)$, then x^* is a solution of $GSPVI(f',K)$.*

Proof. The conclusions follow directly from Theorems 3.1 and 4.3. □

When $\eta(x,y) = x - y$, we obtain the following corollary from Theorems 5.1 and 5.3.

COROLLARY 5.1. *Let K be a star-shaped set, $x^* \in kerK$, and $f : K \to R$. Then the following conclusions hold:*

(i) If $f \in RLSC(K,x^*)$ and x^* is a solution of $GMPVI(f',K)$, then x^* is a solution of $OP(f,K)$ and $x^* \in ker\, lev_{\leqslant c} f$ for all $c \in R$ with $c \geqslant f(x^*)$.

(ii) If x^* is a solution of $OP(f,K)$ and $x^* \in ker\, lev_{\leqslant c} f$ for all $c \in R$ with $c \geqslant f(x^*)$, then x^* is a solution of both $GMPVI(f',K)$ and $GSPVI(f',K)$.

REMARK 5.1. Denote by SOL and GM the solution sets of $GMPVI(f',K)$ and $OP(f,K)$ respectively. When $\eta(x,y) = x-y$, Crespi et al [9] established the following inclusions:

$$SOL \subset ker\, GM \subset GM$$

and pointed out these conclusions can be strict. From Theorems 5.1 and 5.2, we know $SOL = ker^2_\eta\, GM = GM$ under the condition: $x^* \in ker^2_\eta\, lev_{\leqslant c} f$ for all $c \in R$ with $c \geqslant f(x^*)$.

COROLLARY 5.2. *Let $K \subset E$ be an invex set with respect to $\eta : K \times K \to E$ and $f : K \to R$ be a prequasiinvex function satisfying $IRLSC^2$ property. Suppose that Conditions C and D_2 are satisfied. Then the solution set SOL of $GMPVI(f',K)$ and the solution set GM of $OP(f,K)$ coincide.*

Proof. Let $x^* \in GM$. In view of Theorem 5.1, it remains to show that for each $c \in R$ with $c \geqslant f(x^*)$, $x^* \in ker^2_\eta\, lev_{\leqslant c} f$. For given $x \in ker^2_\eta\, lev_{\leqslant c} f$, it follows that

$$f(x+(1-t)\eta(x^*,x)) \leqslant \max\{f(x), f(x^*)\} = f(x) \leqslant c, \ \forall t \in [0,1].$$

Thus $x+(1-t)\eta(x^*,x) \in lev_{\leqslant c} f$, $\forall t \in [0,1]$ and so $x^* \in ker^2_\eta\, lev_{\leqslant c} f$. The proof is complete. $\square$

REMARK 5.2. Corollary 5.2 generalizes Theorem 5.1 of [8] to the prequasiinvex case.

REMARK 5.3. The condition: $x^* \in ker^2_\eta\, lev_{\leqslant c} f$ for all $c \in R$ with $c \geqslant f(x^*)$ can not be dropped off in (ii) of Theorem 5.1. See the following example.

EXAMPLE 5.1. Let $K = [-1,1], \eta(x,y) = x-y$ and $f(x) = \sqrt{1-x^2}$. It is easy to see that K is star-shaped, $1 \in ker K$ is a solution of $OP(f,K)$, and $1 \notin ker\, lev_{\leqslant c} f$ with $0 < c < 1$. But 1 is not a solution of $GMPVI(f',K)$.

6. Perturbed generalized Minty prevariational inequality

The perturbed Minty variational inequality was introduced by Giannessi [6] and the equivalence of the Minty variational inequality and the perturbed Minty variational inequality was studied in [6, 7] in the case of convexity. In this section, we introduce the perturbed generalized Minty prevariational inequality with an invex star-shaped domain and study the relations between generalized Minty prevariational inequalities and perturbed generalized Minty variational inequalities.

DEFINITION 6.1. Let $K \subset E$, $\eta : K \times K \to E$, and $f : K \to R$. f is said to be invex pseudomonotone if for any $x, y \in K$,

$$f'_-(x, \eta(y,x)) \geqslant 0 \Rightarrow f'_-(y, \eta(x,y)) \leqslant 0.$$

REMARK 6.1. (1) Definition 6.1 is a generalization of invariant pseudomonotonicity in the sense of Yang et al [26] to the nondifferentiable case. (2) We say that f is pseudomonotone if f is invex pseudomotone with $\eta(x,y) = x - y$.

DEFINITION 6.2. [31] A function $s : E \to \bar{R}$ is said to be subodd if

$$s(d) + s(-d) \geqslant 0, \quad \forall d \in E.$$

Now consider the following perturbed generalized Minty prevariational inequalities: find $x^* \in K$ for which $\exists \bar{\varepsilon} \in (0,1]$ such that

$$PGMPVI(f',K)_1 \quad f'_-(x^* + \varepsilon\eta(x,x^*), \eta(x,x^*)) \geqslant 0, \quad \forall x \in K,\ \forall \varepsilon \in (0,\bar{\varepsilon}],$$

and find $x^* \in K$ for which $\exists \bar{\varepsilon} \in (0,1]$ such that

$$PGMPVI(f',K)_2 \quad f'_-(x + (1-\varepsilon)\eta(x^*,x), \eta(x^*,x)) \leqslant 0, \quad \forall x \in K,\ \forall \varepsilon \in (0,\bar{\varepsilon}].$$

THEOREM 6.1. *Let $K \subset E$ be an ISS_1 set, $x^* \in ker^1_\eta K$, $\eta : K \times K \to E$, and $f : K \to R$. Assume that*

(i) Condition C holds.

(ii) f is invex pseudomonotone.

(iii) for each $u \in E$, $f'_-(u,\cdot)$ is subodd and $f'_-(u, \eta(\cdot,u))$ is lower semicontinuous.

Then x^* is a solution of $GMPVI(f',K)$ if and only if x^* is a solution of $PGMPVI(f',K)_1$.

Proof. Let x^* be a solution of $GMPVI(f',K)$. Since $x^* \in ker^1_\eta K$, we have $x^* + \varepsilon\eta(x,x^*) \in K$, $\forall x \in K$, $\forall \varepsilon \in (0,1]$. It follows that that

$$\begin{aligned} 0 &\geqslant f'_-(x^* + \varepsilon\eta(x,x^*), \eta(x^*, x^* + \varepsilon\eta(x,x^*))) \\ &= f'_-(x^* + \varepsilon\eta(x,x^*), -\varepsilon\eta(x,x^*)) \\ &= \varepsilon f'_-(x^* + \varepsilon\eta(x,x^*), -\eta(x,x^*)) \\ &\geqslant -\varepsilon f'_-(x^* + \varepsilon\eta(x,x^*), \eta(x,x^*)),\ \forall x \in K,\ \forall \varepsilon \in (0,\bar{\varepsilon}]. \end{aligned}$$

This implies that

$$f'_-(x^* + \varepsilon\eta(x,x^*), \eta(x,x^*)) \geqslant 0,\ \forall x \in K,\ \forall \varepsilon \in (0,\bar{\varepsilon}]$$

and so x^* is a solution of $PGMPVI(f',K)_1$.

Conversely, let x^* be a solution of $PGMPVI(f',K)_1$. For given $x \in K$, we have $x^* + \varepsilon\eta(x,x^*) \in K$, $\forall \varepsilon \in [0,1]$ since $x^* \in ker^1_\eta K$. It follows that for any $\varepsilon \in (0,\bar{\varepsilon}]$, that

$$\begin{aligned} &f'_-(x^* + \varepsilon\eta(x,x^*), \eta(x, x^* + \varepsilon\eta(x,x^*))) \\ &= f'_-(x^* + \varepsilon\eta(x,x^*), (1-\varepsilon)\eta(x,x^*)) \\ &= (1-\varepsilon) f'_-(x^* + \varepsilon\eta(x,x^*), \eta(x,x^*)) \geqslant 0. \end{aligned}$$

Since f is invex pseudomonotonicity, we have

$$f'_{-}(x,\eta(x^*+\varepsilon\eta(x,x^*),x)) \leqslant 0, \quad \forall x \in K,\ \forall \varepsilon \in (0,\bar{\varepsilon}].$$

Letting $\varepsilon \to 0$ in the above inequality, we have

$$f'_{-}(x,\eta(x^*,x)) \leqslant 0, \quad \forall x \in K.$$

Thus x^* is a solution of $GMPVI(f',K)$. $\square$

THEOREM 6.2. *Let $K \subset E$ be an ISS_2 set, $x^* \in ker^2_\eta K$, $\eta : K \times K \to E$, and $f : K \to R$. Assume that Condition C is satisfied and f is invex pseudomonotone, and for each $u \in E$, $f'_{-}(u,\cdot) : E \to \bar{R}$ is subodd. Then x^* is a solution of $GMPVI(f',K)$ if and only if x^* is a solution of $PGMPVI(f',K)_2$.*

Proof. Let x^* be a solution of $GMPVI(f',K)$. Since $x^* \in ker^2_\eta K$, we have $x + (1-\varepsilon)\eta(x^*,x) \in K$, $\forall x \in K$, $\forall \varepsilon \in (0,1]$. It follows that

$$\begin{aligned} 0 &\geqslant f'_{-}(x+(1-\varepsilon)\eta(x^*,x),\eta(x^*,x+(1-\varepsilon)\eta(x^*,x))) \\ &= f'_{-}(x+(1-\varepsilon)\eta(x^*,x),\varepsilon\eta(x^*,x)) \\ &= \varepsilon f'_{-}(x+(1-\varepsilon)\eta(x^*,x),\eta(x^*,x)),\ \forall x \in K,\ \forall \varepsilon \in (0,\bar{\varepsilon}]. \end{aligned}$$

This implies that x^* is a solution of $PGMPVI(f',K)_2$.

Conversely, let x^* be a solution of $PGMPVI(f',K)_2$. For given $x \in K$, we have $x + (1-\varepsilon)\eta(x^*,x) \in K$, $\forall \varepsilon \in [0,1]$ since $x^* \in ker^2_\eta K$. It follows that

$$\begin{aligned} &f'_{-}(x+(1-\varepsilon)\eta(x^*,x),\eta(x,x+(1-\varepsilon)\eta(x^*,x))) \\ &= f'_{-}(x+(1-\varepsilon)\eta(x^*,x),-(1-\varepsilon)\eta(x^*,x)) \\ &= (1-\varepsilon)f'_{-}(x+(1-\varepsilon)\eta(x^*,x),-\eta(x^*,x)) \\ &\geqslant -(1-\varepsilon)f'_{-}(x+(1-\varepsilon)\eta(x^*,x),\eta(x^*,x)) \geqslant 0,\ \forall \varepsilon \in (0,\bar{\varepsilon}]. \end{aligned}$$

The above inequality together with the invex pseudomonotonicity of f implies that

$$\begin{aligned} f'_{-}(x,\eta(x+(1-\varepsilon)\eta(x^*,x),x)) &= f'_{-}(x,(1-\varepsilon)\eta(x^*,x)) \\ &= (1-\varepsilon)f'_{-}(x,\eta(x^*,x)) \leqslant 0, \quad \forall x \in K,\ \forall \varepsilon \in (0,\bar{\varepsilon}]. \end{aligned}$$

This implies that x^* is a solution of $GMPVI(f',K)$. $\square$

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