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FEKETE–SZEGÖ INEQUALITY FOR GENERALIZED SUBCLASSES OF UNIVALENT FUNCTIONS

LIANGPENG XIONG, XIANGDONG FENG AND JIANLIANG ZHANG

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Abstract. Let $\mathscr{P}_{\varphi}(n,b,\lambda)$ denote the class of normalized univalent functions $f(z)=z+a_2z^2+\ldots$, which are defined in the unit disk Δ and satisfying $1+[(\lambda D^{n+2}f(z)+(1-\lambda)D^{n+1}f(z))/(\lambda D^{n+1}f(z)+(1-\lambda)D^{n}f(z))-1]/b \prec \varphi(z)$, where $\varphi(z)$ is the function with positive real part, $D^n f$ denotes the sălăgean operator, $n \geqslant 0$, $0 \leqslant \lambda \leqslant 1$, $b \in \mathbb{C}$. In this paper, for the class $\mathscr{P}_{\varphi}(n,b,\lambda)$, the Fekete-Szegö inequalities are completely solved. A more general class $\mathscr{K}(\beta,n,\lambda,g(z))$ related $\mathscr{P}_{\varphi}(n,b,\lambda)$ is also considered with same subject, which extends the earlier corresponding results for the class of strongly close-to-convex functions of order β.

1. Introduction

Let $\mathscr{A}$ be the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k, \tag{1.1}$$

which are analytic univalent in the open unit disk $\triangle=\{z:|z|<1\}$ and normalized by the conditions $f(0)=0$, $f'(0)=1$. It is well-known that $f(z)\in\mathscr{A}$, $|a_3-a_2^2|\leqslant 1$. If f and g are analytic in $\triangle$, we say that f is subordinate to g, written $f(z)\prec g(z)$, provided there exists a analytic function $\omega(z)$ defined on $\triangle$ with $\omega(0)=0$ and $|\omega(z)|<1$ satisfying $f(z)=g(\omega(z))$.

For $f(z)\in\mathscr{A}$, Sălăgean [20] defined the following operator:

$$D^0f(z)=f(z),\ D^1f(z)=Df(z)=zf'(z),\ldots,D^nf(z)=D(D^{n-1}f(z)),$$

where $n\in N=\{1,2,\ldots\}$. We note that

$$D^nf(z)=z+\sum_{k=2}^{\infty}k^na_kz^k,\quad n\in N_0=\{0\}\cup N. \tag{1.2}$$

Let S^*, C and $\mathscr{K}$ denote the usual starlike function, convex function and close-to-convex function, respectively. Ma and Minda [12] unified various subclasses of

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starlike and convex functions for which either one of the quantities $zf'(z)/f(z)$ or $1+zf''(z)/f'(z)$ is subordinate to a more general superordinate function. The classes $S^*(\varphi)$ and $C(\varphi)$ of Ma-Minda starlike and Ma-Minda convex functions, are respectively characterized by $zf'(z)/f(z) \prec \varphi(z)$ and $1+zf''(z)/f'(z) \prec \varphi(z)$, where function φ with positive real part in $\triangle$, $\varphi(0)=0$, $\varphi'(0)>1$. The coefficient functional $\rho_\mu(f)=a_3-\mu a_2^2$ on the normalized analytic functions f in $\triangle$ plays an important role in function theorem. The problem of maximizing the absolute value of the functional $\rho_\mu(f)$ is called the Fekete-Szegö problem. A classical theorem of Fekete-Szegö (see [5]) states that for $f \in \mathscr{A}$ given by (1.1),

$$|a_3-\mu a_2^2| \leqslant \begin{cases} 3-4\mu, & if \quad \mu \leqslant 0, \\ 1+2\exp\left(\frac{-2\mu}{1-\mu}\right), & if \quad 0 \leqslant \mu < 1, \\ 4\mu-3, & if \quad \mu \geqslant 1. \end{cases}$$

Later, Pfluger [17] considered the problem when μ is complex. In the case of C, S^* and $\mathscr{K}$, the above inequalities can be improved [9, 10]. Actually, many authors have considered the Fekete-Szegö problem for various subclasses of $\mathscr{A}$, the upper bound for $|a_3-\mu a_2^2|$ was investigated by many different authors (see [3, 6, 19, 22]). Recently, some results on this subject were improved (see [2, 4, 8, 13, 14, 15, 16, 21, 23]).

We denote by $\mathscr{P}$ a class of analytic function in $\triangle$ with $p(0)=0$ and $\Re p(z)>0$. Here we assume that $\varphi \in \mathscr{P}$, satisfying $\varphi(0)=0$, $\varphi'(0)>0$, and $\varphi(\triangle)$ is symmetric with respect to the real axis. Also, $\varphi(z)$ has a series expansion of the form

$$\varphi(z)=1+A_1z+A_2z^2+A_3z^3+..., (A_1>0). \tag{1.3}$$

With the aid of salagean operator, we introduce the class $\mathscr{P}_\varphi(n,b,\lambda)$ as follows:

A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{P}_\varphi(n,b,\lambda)$ if and only if

$$1+\frac{1}{b}\left(\frac{\lambda D^{n+2}f(z)+(1-\lambda)D^{n+1}f(z)}{\lambda D^{n+1}f(z)+(1-\lambda)D^nf(z)}-1\right) \prec \varphi(z),\ z\in\Delta, \tag{1.4}$$

where b is nonzero complex number, φ is defined as (1.3), $n \geqslant 0$, $0 \leqslant \lambda \leqslant 1$.

By giving specific values to the parameters b, λ and φ, we obtain the following important subclasses studied by various authors in earlier works, for instance,

$$\mathscr{P}_\varphi(0,1,0) \equiv S^*(\varphi), \quad \mathscr{P}_\varphi(0,1,1) \equiv C(\varphi),$$

and

$$\mathscr{P}_{\frac{1+Az}{1+Bz}}(0,1,0) \equiv S^*[A,B], \quad \mathscr{P}_{\frac{1+Az}{1+Bz}}(0,1,1) \equiv C[A,B],$$

where $-1 \leqslant A < B \leqslant 1$. The $S^*(\varphi)$, $C(\varphi)$ were introduced by Ma-Minda [12]. The $S^*[A,B]$, $C[A,B]$ were defined by Janowski [7].

By taking $b=1$, $\varphi(z)=(\frac{1+z}{1-z})^\beta$ $(0 \leqslant \beta \leqslant 1)$, we want to extend the $\mathscr{P}_\varphi(n,b,\lambda)$ to a more general class $\mathscr{K}(\beta,n,\lambda,g(z))$.

A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{K}(\beta, n, \lambda, g(z))$ if and only if

$$\left|\arg\left(\frac{\lambda D^{n+2} f(z) + (1-\lambda) D^{n+1} f(z)}{\lambda D^{n+1} g(z) + (1-\lambda) D^{n} g(z)}\right)\right| \leqslant \frac{\pi}{2}\beta,\ z \in \Delta, \tag{1.5}$$

where $n \geqslant 0$, $0 \leqslant \lambda \leqslant 1$, $0 \leqslant \beta \leqslant 1$, $g(z) = z + b_2 z^2 + b_3 z^3 + \ldots \in S^*$.

We note that $\mathscr{K}(\beta, 0, 0, g(z)) \equiv \mathscr{K}(\beta)$, where $\mathscr{K}(\beta)$ is the class of strongly close-to-convex functions of order β defined by Koepf [11] and Abdel-Gawad [1]. Koepf [11] considered the Fekete-Szegö problem for $\mathscr{K}(\beta)$ with some particular values of μ. Later, Abdel-Gawad [1] improved the results for any $\mu \in \mathbb{R}$ without $\mu = 1$.

In this paper, we concentrate on the Fekete-Szegö problem for the subclass $\mathscr{P}_{\varphi}(n, b, \lambda)$, which is discussed with four different cases as: (i) μ is special number, $b \in \mathbb{C}$. (ii) $\mu \in \mathbb{C}$, $b \in \mathbb{C}$. (iii) $b > 0$, $\mu \in \mathbb{R}$. (IV) $\mu \in \mathbb{R}$, $b \in \mathbb{C}$. Also, a more general class $\mathscr{K}(\beta, n, \lambda, g(z))$ is considered on the same subject, which extends the corresponding earlier results for the class $\mathscr{K}(\beta)$ of strongly close-to-convex functions of order β.

2. Main results

In order to derive our main results, we have to recall here the following Lemmas.

LEMMA 1. ([10]) *Let $g \in S^*$ with $g(z) = z + b_2 z^2 + b_3 z^3 + \ldots$, then for real μ,*

$$|b_3 - \mu b_2^2| \leqslant \max\{1, |3 - 4\mu|\}.$$

The result is sharp.

LEMMA 2. ([18]) *Let $p \in \mathscr{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$, then $|c_n| \leqslant 2$ for $n \geqslant 1$. If $|c_1| = 2$ then $p(z) \equiv p_1(z) = (1 + \gamma_1 z)/(1 - \gamma_1 z)$ with $\gamma_1 = c_1/2$. Conversely, if $p(z) \equiv p_1(z)$ for some $|\gamma_1| = 1$, then $c_1 = 2\gamma_1$ and $|c_1| = 2$. Furthermore we have*

$$\left|c_2 - \frac{c_1^2}{2}\right| \leqslant 2 - \frac{|c_1|^2}{2}.$$

If $|c_1| < 2$ and $|c_2 - \frac{c_1^2}{2}| = 2 - \frac{|c_1|^2}{2}$, then $p(z) \equiv p_2(z)$, where

$$p_2(z) = \frac{1 + z\frac{\gamma_2 z + \gamma_1}{1 + \overline{\gamma_1}\gamma_2 z}}{1 - z\frac{\gamma_2 z + \gamma_1}{1 + \overline{\gamma_1}\gamma_2 z}}$$

and $\gamma_1 = c_1/2$, $\gamma_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$. Conversely if $p(z) = p_2(z)$ for some $|\gamma_1| < 1$ and $|\gamma_2| = 1$, then $\gamma_1 = c_1/2$, $\gamma_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$ and $|c_2 - \frac{c_1^2}{2}| = 2 - \frac{|c_1|^2}{2}$.

THEOREM 1. *Let $n \geqslant 0$, $0 \leqslant \lambda \leqslant 1$, and Let b be nonzero complex number. If $f(z) \in \mathscr{P}_{\varphi}(n,b,\lambda)$, where $\varphi(z) = 1 + A_1 z + A_2 z^2 + \ldots + A_n z^n + \ldots, (A_1 > 0)$, then*

$$|a_2| \leqslant \frac{1}{2^n}\frac{1}{1+\lambda}|b|A_1, \tag{2.1}$$

$$|a_3| \leqslant \frac{1}{2}.\frac{1}{1+2\lambda}\frac{1}{3^n}|b|A_1 \max\left\{1, \left|bA_1 + \frac{A_2}{A_1}\right|\right\}, \tag{2.2}$$

$$\left|a_3 - \frac{2.4^{n-1}.(1+\lambda)^2.(bA_1 + \frac{A_2}{A_1} - 1)}{3^n(1+2\lambda)bA_1}a_2^2\right| \leqslant \frac{A_1|b|}{(2+4\lambda)3^n}. \tag{2.3}$$

These results are sharp.

Proof. Let $f(z) \in \mathscr{P}_{\varphi}(n,b,\lambda)$. Then there is a function $w(z)$, such that

$$1 + \frac{1}{b}\left(\frac{\lambda D^{n+2}f(z) + (1-\lambda)D^{n+1}f(z)}{\lambda D^{n+1}f(z) + (1-\lambda)D^n f(z)} - 1\right) = \varphi(w(z)),\ z \in \Delta.$$

Define the function $p(z)$ by

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + r_1 z + r_2 z^2 + \ldots \prec \frac{1+z}{1-z},\ z \in \Delta. \tag{2.4}$$

We can note that $p(0) = 1$ and $p(z)$ is a function with positive real part. In fact, using the (2.4), it is easy to know that

$$w(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left(r_1 z + \left(r_2 - \frac{r_1^2}{2}\right)z^2 + \ldots\right),$$

So

$$1 + \frac{1}{b}\left(\frac{\lambda D^{n+2}f(z) + (1-\lambda)D^{n+1}f(z)}{\lambda D^{n+1}f(z) + (1-\lambda)D^n f(z)} - 1\right) = \varphi(w(z))$$

$$= 1 + \frac{1}{2}A_1 r_1 z + \left(\frac{1}{2}A_1\left(r_2 - \frac{r_1^2}{2}\right) + \frac{1}{4}A_2 r_1^2\right)z^2 + \ldots \tag{2.5}$$

Actually, a computation shows that

$$1 + \frac{1}{b}\left(\frac{\lambda D^{n+2}f(z) + (1-\lambda)D^{n+1}f(z)}{\lambda D^{n+1}f(z) + (1-\lambda)D^n f(z)} - 1\right)$$

$$= 1 + \frac{1}{b}2^n(1+\lambda)a_2 z + \frac{1}{b}[(2+4\lambda).3^n a_3 - (1+\lambda)^2 4^n a_2^2]z^2 + \ldots \tag{2.6}$$

The equations (2.5) and (2.6) yield

$$\frac{1}{b}2^n(1+\lambda)a_2 = \frac{1}{2}A_1 r_1,\ \frac{1}{b}[(2+4\lambda).3^n a_3 - (1+\lambda)^2 4^n a_2^2] = \frac{1}{2}A_1\left(r_2 - \frac{r_1^2}{2}\right) + \frac{1}{4}A_2 r_1^2. \tag{2.7}$$

Taking into account (2.7) and Lemma 2, we obtain

$$|a_2| = \left|\frac{1}{2^{n+1}}\frac{1}{1+\lambda}bA_1r_1\right| \leqslant \frac{1}{2^n}\frac{1}{1+\lambda}|b|A_1, \tag{2.8}$$

and

$$\begin{aligned}
|a_3| &= \left|\frac{1}{4}.\frac{b}{3^n}.\frac{1}{1+2\lambda}A_1\left[r_2+\left(\frac{1}{2}bA_1+\frac{1}{2}\frac{A_2}{A_1}-\frac{1}{2}\right)r_1^2\right]\right| \\
&= \left|\frac{1}{4}.\frac{b}{3^n}.\frac{1}{1+2\lambda}A_1\left[r_2-\frac{1}{2}r_1^2+\frac{r_1^2}{2}(bA_1+\frac{A_2}{A_1})\right]\right| \\
&\leqslant \frac{1}{4}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1\left[|r_2-\frac{1}{2}r_1^2|+\frac{|r_1^2|}{2}|bA_1+\frac{A_2}{A_1}|\right] \\
&\leqslant \frac{1}{4}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1\left[2-\frac{1}{2}|r_1|^2+\frac{|r_1|^2}{2}|bA_1+\frac{A_2}{A_1}|\right] \\
&= \frac{1}{4}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1\left[2+\frac{1}{2}|r_1|^2[|bA_1+\frac{A_2}{A_1}|-1]\right] \\
&\leqslant \frac{1}{2}.\frac{1}{1+2\lambda}\frac{1}{3^n}|b|A_1\max\left\{1,|bA_1+\frac{A_2}{A_1}|\right\}.
\end{aligned}$$

Furthermore,

$$\left|a_3-\frac{2.4^{n-1}.(1+\lambda)^2.(bA_1+\frac{A_2}{A_1}-1)}{3^n(1+2\lambda)bA_1}a_2^2\right| = \left|\frac{1}{4}.\frac{b}{3^n}.\frac{1}{1+2\lambda}A_1r_2\right| \leqslant \frac{A_1|b|}{(2+4\lambda)3^n}. \tag{2.9}$$

An examination of the proof shows the first equality holds if $c_1 = 2$. Equivalently, we have $p(z) = p_1(z) = (1+z)/(1-z)$. Therefore, the extremal function in $\mathscr{P}_\varphi(n,b,\lambda)$ is defined by

$$1+\frac{1}{b}\left(\frac{\lambda D^{n+2}f(z)+(1-\lambda)D^{n+1}f(z)}{\lambda D^{n+1}f(z)+(1-\lambda)D^nf(z)}-1\right) = \varphi\Big(\frac{p_1(z)-1}{p_1(z)+1}\Big). \tag{2.10}$$

Next, in (2.2), for first case, the equality holds if $c_1 = 0$, $c_2 = 2$. Equivalently, we have $p(z) = p_2(z) = \frac{1+z^2}{1-z^2}$. Therefore, the extremal functions in $\mathscr{P}_\varphi(n,b,\lambda)$ is given by

$$1+\frac{1}{b}\left(\frac{\lambda D^{n+2}f(z)+(1-\lambda)D^{n+1}f(z)}{\lambda D^{n+1}f(z)+(1-\lambda)D^nf(z)}-1\right) = \varphi\Big(\frac{p_2(z)-1}{p_2(z)+1}\Big). \tag{2.11}$$

In (2.2), for the second case, the equality holds if $c_1 = 2$, $c_2 = 2$. Therefore, the extremal function in $\mathscr{P}_\varphi(n,b,\lambda)$ is given by (2.10).

Finally, in (2.3), the equality holds. Obtained extremal function for (2.1) is also valid for (2.3).

In fact, Theorem 1 gives a special case of Fekete-Szegö problem with

$$\mu = \frac{2.4^{n-1}.(1+\lambda)^2.(bA_1+\frac{A_2}{A_1}-1)}{3^n(1+2\lambda)bA_1},$$

which obtain the naturally and simple estimate. Thus the proof is completed. □

Now, we consider the Fekete-Szegö problem with complex μ.

THEOREM 2. *Let $n \geqslant 0,\ 0 \leqslant \lambda \leqslant 1$ and Let b be a nonzero complex number. If $f(z) \in \mathscr{P}_{\varphi}(n,b,\lambda)$, then for any complex μ,*

$$|a_3 - \mu a_2^2| \leqslant \begin{cases} \frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1, & \mathscr{U} \leqslant 1, \\ \frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1\mathscr{U}, & \mathscr{U} > 1. \end{cases}$$

where $\varphi(z) = 1 + A_1 z + A_2 z^2 + \ldots, (A_1 > 0)$, $\mathscr{U} = \left|bA_1 + \frac{A_2}{A_1} - 2\left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2}\mu b A_1\right|$. The results are sharp.

Proof. Following (2.7), we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{4}\cdot\frac{b}{3^n}\cdot\frac{1}{1+2\lambda}A_1\left[r_2 + \left(\frac{1}{2}bA_1 + \frac{1}{2}\frac{A_2}{A_1} - \frac{1}{2}\right)r_1^2\right] - \mu\frac{1}{(1+\lambda)^2}\frac{b^2A_1^2r_1^2}{4^{n+1}} \\ &= \frac{1}{4}\cdot\frac{b}{3^n}\cdot\frac{1}{1+2\lambda}A_1\left[r_2 + \left(\frac{1}{2}bA_1 + \frac{1}{2}\frac{A_2}{A_1} - \frac{1}{2}\right)r_1^2 - \left(\frac{3}{4}\right)^n\frac{1+2\lambda}{(1+\lambda)^2}\mu bA_1 r_1^2\right] \\ &= \frac{1}{4}\cdot\frac{b}{3^n}\cdot\frac{1}{1+2\lambda}A_1\left[r_2 - \frac{1}{2}r_1^2 + \frac{1}{2}r_1^2\left(bA_1 + \frac{A_2}{A_1} - 2\left(\frac{3}{4}\right)^n\frac{1+2\lambda}{(1+\lambda)^2}\mu bA_1\right)\right]. \end{aligned}$$

Again, using the Lemma 2, it has

$$\begin{aligned} |a_3 - \mu a_2^2| &\leqslant \frac{1}{4}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1\left|r_2 - \frac{1}{2}r_1^2 + \frac{1}{2}r_1^2\left(bA_1 + \frac{A_2}{A_1} - 2\left(\frac{3}{4}\right)^n\frac{1+2\lambda}{(1+\lambda)^2}\mu bA_1\right)\right| \\ &\leqslant \frac{1}{4}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1\left[2 + \frac{1}{2}|r_1|^2\left[\left|bA_1 + \frac{A_2}{A_1} - 2\left(\frac{3}{4}\right)^n\frac{1+2\lambda}{(1+\lambda)^2}\mu bA_1\right| - 1\right]\right] \\ &\leqslant \frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1\max\left\{1, \left|bA_1 + \frac{A_2}{A_1} - 2\left(\frac{3}{4}\right)^n\frac{1+2\lambda}{(1+\lambda)^2}\mu bA_1\right|\right\}. \end{aligned}$$

Equality holds for each μ with the first case if fonctions in (2.11) and the second case if functions in (2.10). Thus the proof is completed. □

Next, we want to consider the Fekete-Szegö problem with real μ and real b.

THEOREM 3. *Let $n \geqslant 0,\ 0 \leqslant \lambda \leqslant 1$ and Let $b > 0$. If $f(z) \in \mathscr{P}_{\varphi}(n,b,\lambda)$, then for any real μ,* *(1) If* $bA_1 \geqslant 1$, *we have* $|a_3 - \mu a_2^2|$

$$\leqslant \begin{cases} \frac{1}{2}\frac{b}{3^n}\frac{1}{1+2\lambda}A_1\left[\mathscr{N}_1 - 2\left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2}\mu b A_1\right], & \mu < \frac{A_2}{A_1}\mathscr{N}_1, \\ \frac{1}{2}\frac{b^2}{3^n}\frac{1}{1+2\lambda}A_1^2, & \frac{A_2}{A_1}\mathscr{N}_1 \leqslant \mu < (\mathscr{N}_1 - 1)\mathscr{N}_2, \\ \frac{1}{2}\frac{b}{3^n}\frac{1}{1+2\lambda}A_1, & (\mathscr{N}_1 - 1)\mathscr{N}_2 \leqslant \mu < (\mathscr{N}_1 + 1)\mathscr{N}_2, \\ \frac{1}{2}\frac{b}{3^n}\frac{1}{1+2\lambda}A_1\left[2\left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2}\mu b A_1 - \mathscr{N}_1\right], & \mu \geqslant (\mathscr{N}_1 + 1)\mathscr{N}_2. \end{cases}$$

(2) If $bA_1 < 1$, *we have* $|a_3 - \mu a_2^2|$

$$\leqslant \begin{cases} \frac{1}{2}\frac{b}{3^n}\frac{1}{1+2\lambda}A_1\left[\mathscr{N}_1 - 2\left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2}\mu b A_1\right], & \mu < (\mathscr{N}_1 - 1)\mathscr{N}_2, \\ \frac{1}{2}\frac{b}{3^n}\frac{1}{1+2\lambda}A_1, & (\mathscr{N}_1 - 1)\mathscr{N}_2 \leqslant \mu < (\mathscr{N}_1 + 1)\mathscr{N}_2, \\ \frac{1}{2}\frac{b}{3^n}\frac{1}{1+2\lambda}A_1\left[2\left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2}\mu b A_1 - \mathscr{N}_1\right], & \mu \geqslant (\mathscr{N}_1 + 1)\mathscr{N}_2. \end{cases}$$

where $\varphi(z) = 1 + A_1 z + A_2 z^2 + \ldots, (A_1 > 0, A_2 > 0)$,

$$\mathscr{N}_1 = bA_1 + \frac{A_2}{A_1}, \qquad \mathscr{N}_2 = \frac{4^n(1+\lambda)^2}{2A_1 b 3^n (1+2\lambda)}.$$

For each μ *there is a function* $f \in \mathscr{P}_\varphi(n,b,\lambda)$ *such that equality holds.*

Proof. It follows from (2.7) that

$$a_3 - \mu a_2^2 = \frac{1}{4}\cdot\frac{b}{3^n}\cdot\frac{1}{1+2\lambda}A_1\left[r_2 - \frac{1}{2}r_1^2 + \frac{1}{2}r_1^2\left(bA_1 + \frac{A_2}{A_1} - 2\left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2}\mu b A_1\right)\right], \tag{2.12}$$

As the Lemma 2, we have

$$|a_3 - \mu a_2^2| \leqslant \frac{1}{4}\cdot\frac{b}{3^n}\cdot\frac{1}{1+2\lambda}A_1\left[2 + \frac{1}{2}|r_1|^2\left[\left|bA_1 + \frac{A_2}{A_1} - 2\left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2}\mu b A_1\right| - 1\right]\right]. \tag{2.13}$$

Firstly, we want to consider the case with $bA_1 \geqslant 1$. Several possible cases need to further analyze.

Case 1. If $\mu \leqslant \frac{4^n A_2 (1+\lambda)^2}{2A_1^2 b 3^n (1+2\lambda)}$, using (2.13), we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\leqslant \frac{1}{4}\cdot\frac{b}{3^n}\cdot\frac{1}{1+2\lambda}A_1\left[2 + \frac{1}{2}|r_1|^2\left[bA_1 + \frac{A_2}{A_1} - 2\left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2}\mu b A_1 - 1\right]\right] \\ &\leqslant \frac{1}{2}\frac{b}{3^n}\frac{1}{1+2\lambda}A_1\left[bA_1 + \frac{A_2}{A_1} - 2\left(\frac{3}{4}\right)^n \frac{1+2\lambda}{(1+\lambda)^2}\mu b A_1\right]. \end{aligned}$$

Case 2. If $\frac{4^nA_2(1+\lambda)^2}{2A_1^2b3^n(1+2\lambda)} \leqslant \mu \leqslant \left(bA_1+\frac{A_2}{A_1}-1\right)\frac{4^n(1+\lambda)^2}{2A_1b3^n(1+2\lambda)}$, using (2.13), we have

$$\begin{aligned}|a_3-\mu a_2^2| &\leqslant \frac{1}{4}\cdot\frac{b}{3^n}\cdot\frac{1}{1+2\lambda}A_1\left[2+\frac{1}{2}|r_1|^2\left[bA_1+\frac{A_2}{A_1}-2\left(\frac{3}{4}\right)^n\frac{1+2\lambda}{(1+\lambda)^2}\mu bA_1-1\right]\right]\\ &\leqslant \frac{1}{2}\frac{b}{3^n}\frac{1}{1+2\lambda}A_1\left[bA_1+\frac{A_2}{A_1}-2\left(\frac{3}{4}\right)^n\frac{1+2\lambda}{(1+\lambda)^2}\mu bA_1\right] \leqslant \frac{1}{2}\frac{b^2}{3^n}\frac{1}{1+2\lambda}A_1^2.\end{aligned}$$

Case 3. If $\left(bA_1+\frac{A_2}{A_1}-1\right)\frac{4^n(1+\lambda)^2}{2A_1b3^n(1+2\lambda)} \leqslant \mu \leqslant \left(bA_1+\frac{A_2}{A_1}\right)\frac{4^n(1+\lambda)^2}{2A_1b3^n(1+2\lambda)}$, we have

$$\begin{aligned}|a_3-\mu a_2^2| &\leqslant \frac{1}{4}\cdot\frac{b}{3^n}\cdot\frac{1}{1+2\lambda}A_1\left[2+\frac{1}{2}|r_1|^2\left[bA_1+\frac{A_2}{A_1}-2\left(\frac{3}{4}\right)^n\frac{1+2\lambda}{(1+\lambda)^2}\mu bA_1-1\right]\right]\\ &\leqslant \frac{1}{2}\frac{b}{3^n}\frac{1}{1+2\lambda}A_1.\end{aligned}$$

Case 4. If $\left(bA_1+\frac{A_2}{A_1}\right)\frac{4^n(1+\lambda)^2}{2A_1b3^n(1+2\lambda)} \leqslant \mu \leqslant \left(bA_1+\frac{A_2}{A_1}+1\right)\frac{4^n(1+\lambda)^2}{2A_1b3^n(1+2\lambda)}$, we have

$$\begin{aligned}|a_3-\mu a_2^2| &\leqslant \frac{1}{4}\cdot\frac{b}{3^n}\cdot\frac{1}{1+2\lambda}A_1\left[2+\frac{1}{2}|r_1|^2\left[2\left(\frac{3}{4}\right)^n\frac{1+2\lambda}{(1+\lambda)^2}\mu bA_1-bA_1-\frac{A_2}{A_1}-1\right]\right]\\ &\leqslant \frac{1}{2}\frac{b}{3^n}\frac{1}{1+2\lambda}A_1.\end{aligned}$$

Case 5. If $\mu \geqslant \left(bA_1+\frac{A_2}{A_1}+1\right)\frac{4^n(1+\lambda)^2}{2A_1b3^n(1+2\lambda)}$, we have

$$\begin{aligned}|a_3-\mu a_2^2| &\leqslant \frac{1}{4}\cdot\frac{b}{3^n}\cdot\frac{1}{1+2\lambda}A_1\left[2+\frac{1}{2}|r_1|^2\left[2\left(\frac{3}{4}\right)^n\frac{1+2\lambda}{(1+\lambda)^2}\mu bA_1-bA_1-\frac{A_2}{A_1}-1\right]\right]\\ &\leqslant \frac{1}{2}\frac{b}{3^n}\frac{1}{1+2\lambda}A_1\left[2\left(\frac{3}{4}\right)^n\frac{1+2\lambda}{(1+\lambda)^2}\mu bA_1-bA_1-\frac{A_2}{A_1}\right].\end{aligned}$$

Finally, if $bA_1<1$, the similar discussions can readily yield the desired results.

If $bA_1 \geqslant 1$, equality is attained for the second and third case on choosing $c_1=0$, $c_2=2$ in (2.11). Also, equality is attained for the first and fourth case on choosing $c_1=2$, $c_2=2$ and $c_1=2i$, $c_2=-2$ in (2.10), respectively.

If $bA_1 \leqslant 1$, equality is attained for the second case on choosing $c_1=0$, $c_2=2$ in (2.11). Also, equality is attained for the first and third case on choosing $c_1=2$, $c_2=2$ and $c_1=2i$, $c_2=-2$ in (2.10), respectively. Thus the proof is completed. □

Here, we discuss the Fekete-Szegö problem with complex b and real μ.

THEOREM 4. *Let $n \geqslant 0$, $0 \leqslant \lambda \leqslant 1$ and Let b be a nonzero complex number. If $f(z) \in \mathscr{P}_\varphi(n,b,\lambda)$, then for any real μ,* *(1) If* $|\frac{A_2}{A_1}\sin\theta| < 1$*, we have*

$$|a_3-\mu a_2^2| \leqslant \begin{cases} \frac{|b|^2A_1^2}{4^n(1+\lambda)^2}(\Re(\mathscr{X}_1)-\mu)+\frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}|A_2||\sin\theta|, & \mu \leqslant \mathscr{M}_1, \\ \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1, & \mathscr{M}_1 < \mu \leqslant \mathscr{M}_2, \\ \frac{|b|^2A_1^2}{4^n(1+\lambda)^2}(\mu-\Re(\mathscr{X}_1))+\frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}|A_2||\sin\theta|, & \mu > \mathscr{M}_2. \end{cases}$$

(2) If $|\frac{A_2}{A_1}\sin\theta| > 1$*, we have*

$$|a_3-\mu a_2^2| \leqslant \begin{cases} \frac{|b|^2A_1^2}{4^n(1+\lambda)^2}(\Re(\mathscr{X}_1)-\mu)+\frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}|A_2||\sin\theta|, & \mu \leqslant \Re(\mathscr{X}_1), \\ \frac{|b|^2A_1^2}{4^n(1+\lambda)^2}(\mu-\Re(\mathscr{X}_1))+\frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}|A_2||\sin\theta|, & \mu > \Re(\mathscr{X}_1). \end{cases}$$

where $\varphi(z) = 1+A_1z+A_2z^2+\ldots$, $b=|b|e^{-i\theta}$, $\mathscr{X}_1 = \frac{4^n(1+\lambda)^2}{2.3^n.(1+2\lambda)}+\frac{4^{n+1}(1+\lambda)^2A_2e^{i\theta}}{8.3^n(1+2\lambda)A_1^2|b|}$, $\mathscr{J}_1 = \frac{4^{n+1}(1+\lambda)^2}{8.3^n.(1+2\lambda)|b|A_1}$, $\mathscr{M}_1 = \Re(\mathscr{X}_1)-\mathscr{J}_1(1-|\frac{A_2}{A_1}||\sin\theta|)$, $\mathscr{M}_2 = \Re(\mathscr{X}_1)+\mathscr{J}_1(1-|\frac{A_2}{A_1}||\sin\theta|)$*. For each* μ *there is a function in* $\mathscr{P}_\varphi(n,b,\lambda)$*, such that the equality holds.*

Proof. Suppose $f(z) = z+\sum_{k=2}^{\infty}a_kz^k \in \mathscr{P}_\varphi(n,b,\lambda)$, using (2.7), then we have

$$\begin{aligned} |a_3-\mu a_2^2| &= \frac{1}{4}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1\left|r_2-\frac{1}{2}r_1^2+\frac{1}{2}r_1^2\left(bA_1+\frac{A_2}{A_1}-2\left(\frac{3}{4}\right)^n\frac{1+2\lambda}{(1+\lambda)^2}\mu bA_1\right)\right| \\ &\leqslant \frac{1}{4}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1\left[2+\frac{1}{2}|r_1|^2\left[\left|bA_1+\frac{A_2}{A_1}-2\left(\frac{3}{4}\right)^n\frac{1+2\lambda}{(1+\lambda)^2}\mu bA_1\right|-1\right]\right] \\ &= \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1+\frac{1}{8}\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1\left[\left|bA_1+\frac{A_2}{A_1}-2\left(\frac{3}{4}\right)^n\frac{1+2\lambda}{(1+\lambda)^2}\mu bA_1\right|-1\right]|r_1|^2 \\ &= \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1+\frac{1}{8}\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1\left[\left|2\left(\frac{3}{4}\right)^n\frac{1+2\lambda}{(1+\lambda)^2}\mu bA_1-bA_1-\frac{A_2}{A_1}\right|-1\right]|r_1|^2 \\ &= \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1+\frac{1}{8}\frac{|b|^2}{3^n}.\frac{1}{1+2\lambda}A_1\left[\left|2\left(\frac{3}{4}\right)^n\frac{1+2\lambda}{(1+\lambda)^2}\mu A_1-A_1-\frac{A_2}{bA_1}\right|-\frac{1}{|b|}\right]|r_1|^2 \\ &= \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1+\frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}\left[\left|\mu-\frac{4^n(1+\lambda)^2}{2.3^n.(1+2\lambda)}-\frac{4^{n+1}(1+\lambda)^2A_2}{8.3^n(1+2\lambda)A_1^2}\frac{1}{b}\right|\right. \\ &\qquad \left. -\frac{4^{n+1}(1+\lambda)^2}{8.3^n.(1+2\lambda)|b|A_1}\right]|r_1|^2. \end{aligned} \tag{2.14}$$

Taking $b=|b|e^{-i\theta}$, $\frac{4^n(1+\lambda)^2}{2.3^n.(1+2\lambda)}+\frac{4^{n+1}(1+\lambda)^2A_2e^{i\theta}}{8.3^n(1+2\lambda)A_1^2|b|} = \mathscr{X}_1$, $\frac{4^{n+1}(1+\lambda)^2}{8.3^n.(1+2\lambda)|b|A_1} = \mathscr{J}_1$, a direct calculation with (2.14) shows that

$$
\begin{aligned}
&|a_3-\mu a_2^2|\\
\leqslant&\frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1+\frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}\Big[|\mu-\mathscr{X}_1|-\mathscr{J}_1\Big]|r_1|^2\\
\leqslant&\frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1+\frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}\Big[|\mu-\Re(\mathscr{X}_1)-i(Im(\mathscr{X}_1))|-\mathscr{J}_1\Big]|r_1|^2\\
\leqslant&\frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1+\frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}\Big[|\mu-\Re(\mathscr{X}_1)|+\mathscr{J}_1\Big|\frac{A_2}{A_1}\Big||\sin\theta|-\mathscr{J}_1\Big]|r_1|^2\\
=&\frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1+\frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}\Big[|\mu-\Re(\mathscr{X}_1)|-\mathscr{J}_1\Big(1-\Big|\frac{A_2}{A_1}\Big||\sin\theta|\Big)\Big]|r_1|^2.
\end{aligned}
\tag{2.15}
$$

Here, for later convenience as well, we set $\Re(\mathscr{X}_1)-\mathscr{J}_1(1-|\frac{A_2}{A_1}||\sin\theta|)=\mathscr{M}_1$, $\Re(\mathscr{X}_1)+\mathscr{J}_1(1-|\frac{A_2}{A_1}||\sin\theta|)=\mathscr{M}_2$. Now we make some discussions for several different cases.

Firstly, if $|\frac{A_2}{A_1}||\sin\theta|\leqslant 1$, we can note that $\mathscr{M}_1\leqslant\Re(\mathscr{X}_1)\leqslant\mathscr{M}_2$. Thus, it gives

(i) Let $\mu\leqslant\mathscr{M}_1$. Then from (2.15) we have

$$
\begin{aligned}
&|a_3-\mu a_2^2|\\
\leqslant&\ \frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1+\frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}\Big[\Re(\mathscr{X}_1)-\mathscr{J}_1\Big(1-\Big|\frac{A_2}{A_1}\Big||\sin\theta|\Big)-\mu\Big]|r_1|^2\\
=&\ \frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1+\frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}[\mathscr{M}_1-\mu]|r_1|^2\\
\leqslant&\ \frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1+\frac{|b|^2A_1^2}{4^{n}(1+\lambda)^2}[\mathscr{M}_1-\mu]\\
=&\ \frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1+\frac{|b|^2A_1^2}{4^{n}(1+\lambda)^2}[\Re(\mathscr{X}_1)-\frac{4^{n+1}(1+\lambda)^2}{8.3^n.(1+2\lambda)|b|A_1}\Big(1-\Big|\frac{A_2}{A_1}\Big||\sin\theta|\Big)-\mu]\\
=&\ \frac{|b|^2A_1^2}{4^{n}(1+\lambda)^2}(\Re(\mathscr{X}_1)-\mu)+\frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}|A_2||\sin\theta|.
\end{aligned}
$$

(ii) Let $\mathscr{M}_1<\mu\leqslant\Re(\mathscr{X}_1)$, we have

$$
\begin{aligned}
|a_3-\mu a_2^2|\leqslant&\ \frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1+\frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}\Big[\Re(\mathscr{X}_1)-\mathscr{J}_1\Big(1-\Big|\frac{A_2}{A_1}\Big||\sin\theta|\Big)-\mu\Big]|r_1|^2\\
=&\ \frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1+\frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}[\mathscr{M}_1-\mu]|r_1|^2\leqslant\frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1.
\end{aligned}
$$

(iii) Let $\Re(\mathscr{X}_1)<\mu\leqslant\mathscr{M}_2$. Then from (2.15) we have

$$
|a_3-\mu a_2^2|\leqslant\frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1+\frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}\Big[\mu-\Re(\mathscr{X}_1)-\mathscr{J}_1\Big(1-\Big|\frac{A_2}{A_1}\Big||\sin\theta|\Big)\Big]|r_1|^2
$$

$$= \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1 + \frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}[\mu - \mathscr{M}_2]|r_1|^2 \leqslant \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1.$$

(iiii) Let $\mu > \mathscr{M}_2$, we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\leqslant \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1 + \frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}[\mu - \mathscr{M}_2]|r_1|^2 \\ &\leqslant \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1 + \frac{|b|^2A_1^2}{4^{n}(1+\lambda)^2}[\mu - \mathscr{M}_2] \end{aligned}$$

$$= \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1 + \frac{|b|^2A_1^2}{4^{n}(1+\lambda)^2}\left[\mu - \Re(\mathscr{X}_1) - \frac{4^{n+1}(1+\lambda)^2}{8.3^n.(1+2\lambda)|b|A_1}\left(1 - \left|\frac{A_2}{A_1}\right||\sin\theta|\right)\right]$$

$$= \frac{|b|^2A_1^2}{4^{n}(1+\lambda)^2}(\mu - \Re(\mathscr{X}_1)) + \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}|A_2||\sin\theta|.$$

Moreover, if $|\frac{A_2}{A_1}||\sin\theta| > 1$, we can note that $\mathscr{M}_2 \leqslant \Re(\mathscr{X}_1) \leqslant \mathscr{M}_1$. Thus, it gives

(i) Let $\mu \leqslant \mathscr{M}_2$. Then from (2.15), we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\leqslant \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1 + \frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}\left[\Re(\mathscr{X}_1) - \mathscr{J}_1\left(1 - \left|\frac{A_2}{A_1}\right||\sin\theta|\right) - \mu\right]|r_1|^2 \\ &= \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1 + \frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}[\mathscr{M}_1 - \mu]|r_1|^2 \end{aligned}$$

$$\leqslant \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1 + \frac{|b|^2A_1^2}{4^{n}(1+\lambda)^2}\left[\Re(\mathscr{X}_1) - \frac{4^{n+1}(1+\lambda)^2}{8.3^n.(1+2\lambda)|b|A_1}\left(1 - \left|\frac{A_2}{A_1}\right||\sin\theta|\right) - \mu\right]$$

$$= \frac{|b|^2A_1^2}{4^{n}(1+\lambda)^2}(\Re(\mathscr{X}_1) - \mu) + \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}|A_2||\sin\theta|.$$

(ii) Let $\mathscr{M}_2 < \mu \leqslant \Re(\mathscr{X}_1)$. Then

$$\begin{aligned} |a_3 - \mu a_2^2| &\leqslant \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1 + \frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}\left[\Re(\mathscr{X}_1) - \mathscr{J}_1\left(1 - \left|\frac{A_2}{A_1}\right||\sin\theta|\right) - \mu\right]|r_1|^2 \\ &\leqslant \frac{|b|^2A_1^2}{4^{n}(1+\lambda)^2}(\Re(\mathscr{X}_1) - \mu) + \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}|A_2||\sin\theta|. \end{aligned}$$

(iii) Let $\Re(\mathscr{X}_1) < \mu \leqslant \mathscr{M}_1$. Then we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\leqslant \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1 + \frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}\left[\mu - \Re(\mathscr{X}_1) - \mathscr{J}_1\left(1 - \left|\frac{A_2}{A_1}\right||\sin\theta|\right)\right]|r_1|^2 \\ &= \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}A_1 + \frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}[\mu - \mathscr{M}_2]|r_1|^2 \\ &\leqslant \frac{|b|^2A_1^2}{4^{n}(1+\lambda)^2}(\mu - \Re(\mathscr{X}_1)) + \frac{1}{2}.\frac{|b|}{3^n}.\frac{1}{1+2\lambda}|A_2||\sin\theta|. \end{aligned}$$

(iiii) Let $\mu > \mathscr{M}_1$. Then we have

$$\begin{aligned}
&|a_3-\mu a_2^2|\\
&\leqslant \frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1+\frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}\left[\mu-\Re(\mathscr{X}_1)-\mathscr{J}_1\left(1-\left|\frac{A_2}{A_1}\right||\sin\theta|\right)\right]|r_1|^2\\
&=\frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}A_1+\frac{|b|^2A_1^2}{4^{n+1}(1+\lambda)^2}[\mu-\mathscr{M}_2]|r_1|^2\\
&\leqslant \frac{|b|^2A_1^2}{4^n(1+\lambda)^2}(\mu-\Re(\mathscr{X}_1))+\frac{1}{2}\cdot\frac{|b|}{3^n}\cdot\frac{1}{1+2\lambda}|A_2||\sin\theta|.
\end{aligned}$$

Thus the proof is completed. □

Motivated essentially by the earlier works of Abdel-Gawad [1], Koepf [11], we extend the corresponding results by investigating the class $\mathscr{K}(\beta,n,\lambda,g(z))$ in the next Theorem.

THEOREM 5. *If* $f(z)=z+\sum_{k=2}^{\infty}a_kz^k\in\mathscr{K}(\beta,n,\lambda,g(z))$, *then for* $0\leqslant\beta\leqslant 1$, $0\leqslant\lambda\leqslant 1$, $\mu\in\mathbb{R}$, *we have*

$$|a_3-\mu a_2^2|\leqslant$$

$$\begin{cases}
\Lambda(x_0)|_{\mu=\mathscr{W}_1}+(\mathscr{W}_1-\mu)\left[\frac{\beta}{2^n(1+\lambda)}+1\right]^2, & if \quad \mu\leqslant\mathscr{W}_1,\\
\Lambda(x_0), & if \quad \mathscr{W}_1\leqslant\mu\leqslant\mathscr{W}_2\\
\frac{(\beta+1)(\mu-\mathscr{W}_2)}{\mathscr{W}_2}\Theta(1,1)+\frac{\beta+1}{\mathscr{W}_2}(\mathscr{W}_2\frac{\beta+2}{\beta+1}-\mu).\Lambda(x_0)|_{\mu=\mathscr{W}_2}, & if \quad \mathscr{W}_2\leqslant\mu\leqslant\mathscr{W}_2\frac{\beta+2}{\beta+1},\\
\Theta(1,1)+\left(\mu-\mathscr{W}_2\frac{\beta+2}{\beta+1}\right)\left[\frac{\beta}{2^n(1+\lambda)}+1\right]^2, & if \quad \mu\geqslant\mathscr{W}_2\frac{\beta+2}{\beta+1}.
\end{cases}$$

where

$$\mathscr{W}_1=\frac{8^{n+1}(1+\lambda)^3-2.4^{n+1}(1+\lambda)^2(1-\beta)}{4\beta.3^{n+1}(1+2\lambda)+2.6^{n+1}(1+\lambda)(1+2\lambda)},\quad \mathscr{W}_2=\frac{1}{2}\left(\frac{4}{3}\right)^{n+1}.\frac{(1+\lambda)^2}{1+2\lambda},$$

$$\begin{aligned}
\Lambda(x_0)&=1-\mu+\frac{\beta}{3^{n+1}(1+2\lambda)}\left(2-\frac{1}{2}x_0^2\right)+\frac{\beta^2[(1+\lambda)^2 4^{n+1}-2.3^{n+1}(1+2\lambda)\mu]}{2.12^{n+1}(1+\lambda)^2(1+2\lambda)}x_0^2\\
&\quad+\beta\frac{4^{n+1}(1+\lambda)^2-2.3^{n+1}(1+2\lambda)\mu}{6^{n+1}(1+\lambda)(1+2\lambda)}x_0,
\end{aligned}$$

$$\begin{aligned}
\Theta(1,1)&=\frac{2^{n+3}\beta(1+\lambda)-4\beta-8.3^n(1+2\lambda)(\beta+1)+4^{n+1}(1+\lambda)^2(\beta+2)}{2.3^{n+1}(1+2\lambda)(\beta+1)}\\
&\quad+\frac{1}{3}+\frac{2\beta}{3^{n+1}(1+2\lambda)}.
\end{aligned}$$

$$x_0 = \frac{2^{n+1}(1+\lambda)[4^{n+1}(1+\lambda)^2 - 2.3^{n+1}(1+2\lambda)\mu]}{4^{n+1}(1+\lambda)^2(1-\beta)+2\beta.3^{n+1}(1+2\lambda)\mu}.$$

For each μ there are functions in $\mathscr{K}(\beta,n,\lambda,g(z))$ such that equality holds in all cases.

Proof. If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathscr{K}(\beta,n,\lambda,g(z))$, then there are analytic functions $g(z)$ and $h(z)$, such that

$$\frac{\lambda D^{n+2}f(z)+(1-\lambda)D^{n+1}f(z)}{\lambda D^{n+1}g(z)+(1-\lambda)D^n g(z)} = (h(z))^{\beta}, \tag{2.16}$$

where $g(z) = z + b_2z^2 + b_3z^3 + ... \in S^*$, $h(z) = 1 + c_1z + c_2z^2 + ... \in \mathscr{P}$. Equating coefficients of the power series in the relation with (2.16) we find that

$$2^{n+1}(1+\lambda)a_2 = c_1\beta + 2^n(1+\lambda)b_2, \tag{2.17}$$

and

$$3^{n+1}(2\lambda+1)a_3 = \frac{\beta(\beta-1)}{2}c_1^2 + 2^n(1+\lambda)b_2c_1\beta + \beta c_2 + 3^n(2\lambda+1)b_3. \tag{2.18}$$

From (2.17) and (2.18), it follows that

$$a_3 - \mu a_2^2 = \frac{1}{3}(b_3 - \frac{3}{4}\mu b_2^2) + \beta\Big(\frac{2^n(1+\lambda)}{3^{n+1}(1+2\lambda)} - \frac{\mu}{2^{n+1}(1+\lambda)}\Big)b_2c_1$$
$$+\frac{\beta}{3^{n+1}(1+2\lambda)}\Big\{c_2 + \Big[\frac{\beta[(1+\lambda)^2 4^{n+1} - 2.3^{n+1}(1+2\lambda)\mu]}{2(1+\lambda)^2.4^{n+1}} - \frac{1}{2}\Big]c_1^2\Big\}. \tag{2.19}$$

Suppose

$$\mathscr{W}_1 = \frac{8^{n+1}(1+\lambda)^3 - 2.4^{n+1}(1+\lambda)^2(1-\beta)}{4\beta.3^{n+1}(1+2\lambda)+2.6^{n+1}(1+\lambda)(1+2\lambda)}, \quad \mathscr{W}_2 = \frac{1}{2}\Big(\frac{4}{3}\Big)^{n+1}.\frac{(1+\lambda)^2}{1+2\lambda},$$

where $0 \leqslant \beta \leqslant 1$, $0 \leqslant \lambda \leqslant 1$. We next consider the different cases for μ.

Firstly, let $\mathscr{W}_1 \leqslant \mu \leqslant \mathscr{W}_2$, with the aid of Lemma 1 and Lemma 2, we obtain

$$\begin{aligned}
&|a_3 - \mu a_2^2| \\
\leqslant &\frac{1}{3}\Big|b_3 - \frac{3}{4}\mu b_2^2\Big| + \frac{\beta}{3^{n+1}(1+2\lambda)}\Big|c_2 - \frac{1}{2}c_1^2\Big| \\
&+ \frac{\beta^2[(1+\lambda)^2 4^{n+1} - 2.3^{n+1}(1+2\lambda)\mu]}{2.12^{n+1}(1+\lambda)^2(1+2\lambda)}|c_1^2| + \beta\frac{4^{n+1}(1+\lambda)^2 - 2.3^{n+1}(1+2\lambda)\mu}{6^{n+1}(1+\lambda)(1+2\lambda)}|c_1| \\
\leqslant &1 - \mu + \frac{\beta}{3^{n+1}(1+2\lambda)}\Big(2 - \frac{1}{2}|c_1|^2\Big) + \frac{\beta^2[(1+\lambda)^2 4^{n+1} - 2.3^{n+1}(1+2\lambda)\mu]}{2.12^{n+1}(1+\lambda)^2(1+2\lambda)}|c_1|^2 \\
&+\beta\frac{4^{n+1}(1+\lambda)^2 - 2.3^{n+1}(1+2\lambda)\mu}{6^{n+1}(1+\lambda)(1+2\lambda)}|c_1| = \Lambda(x) \text{ say, with } \quad x = |c_1|.
\end{aligned} \tag{2.20}$$

Now, we take $\Lambda'(x)=0,$ it gives the stable point

$$x_0=\frac{2^{n+1}(1+\lambda)[4^{n+1}(1+\lambda)^2-2.3^{n+1}(1+2\lambda)\mu]}{4^{n+1}(1+\lambda)^2(1-\beta)+2\beta.3^{n+1}(1+2\lambda)\mu},$$

moreover,

$$\Lambda''(x)=\frac{\beta^2[(1+\lambda)^2.4^{n+1}-2.3^{n+1}(1+2\lambda)\mu]-4^{n+1}(1+\lambda)^2\beta}{12^{n+1}(1+2\lambda)(1+\lambda)^2}<0,$$

it implies that $\max\{\Lambda(x):x=|c_1|\}=\Lambda(x_0)$. So (2.20) gives the desired estimate on $|a_3-\mu a_2^2|$.

In fact, since $x=|c_1|\leqslant 2$, it follows that $\mu\geqslant\mathscr{W}_1$. Furthermore, equality is attained for this case by choosing $c_1=x_0, c_2=2, b_2=2, b_3=3$ in (2.19).

Let, now $\mu\leqslant\mathscr{W}_1$, then

$$|a_3-\mu a_2^2|\leqslant|a_3-\mathscr{W}_1a_2^2|+(\mathscr{W}_1-\mu)|a_2|^2. \tag{2.21}$$

Using the result already proved in first case with $\mu=\mathscr{W}_1$, we have $|a_3-\mathscr{W}_1a_2^2|\leqslant\Lambda(x_0)|_{\mu=\mathscr{W}_1}$. Also, applying (2.17) we get $|a_2|\leqslant\frac{\beta}{2^n(1+\lambda)}+1$. Thus (2.21) shows that

$$|a_3-\mu a_2^2|\leqslant\Lambda(x_0)|_{\mu=\mathscr{W}_1}+(\mathscr{W}_1-\mu)\Big[\frac{\beta}{2^n(1+\lambda)}+1\Big]^2. \tag{2.22}$$

The equality for (2.22) is attained when $c_1=x_0|_{\mu=\mathscr{W}_1}$, $c_2=2$, $b_2=2$, $b_3=3$ in (2.19).

Let, now $\mathscr{W}_2\leqslant\mu\leqslant\mathscr{W}_2\frac{\beta+2}{\beta+1}$. Then a computation shows that

$$a_3-\mu a_2^2=\Big(\frac{\beta+1}{\mathscr{W}_2}\mu-\beta-1\Big)\Big(a_3-\mathscr{W}_2\frac{\beta+2}{\beta+1}a_2^2\Big)+\frac{\beta+1}{\mathscr{W}_2}\Big(\mathscr{W}_2\frac{\beta+2}{\beta+1}-\mu\Big)\Big(a_3-\mathscr{W}_2a_2^2\Big).$$

It yields

$$|a_3-\mu a_2^2|=\Big(\frac{\beta+1}{\mathscr{W}_2}\mu-\beta-1\Big)\Big|a_3-\mathscr{W}_2\frac{\beta+2}{\beta+1}a_2^2\Big|+\frac{\beta+1}{\mathscr{W}_2}\Big(\mathscr{W}_2\frac{\beta+2}{\beta+1}-\mu\Big)|a_3-\mathscr{W}_2a_2^2|. \tag{2.23}$$

We deal first with the case $\mu=\mathscr{W}_2\frac{\beta+2}{\beta+1}$. Since $g\in S^*$, so there is a function $p(z)=1+p_1z+p_2z^2+\ldots\in\mathscr{P}$, satisfying $zg'(z)=g(z)p(z)$, where $b_2=p_1$, $2b_3=p_1^2+p_2$. Thus we have

$$\begin{aligned}&a_3-\mathscr{W}_2\frac{\beta+2}{\beta+1}a_2^2\\&=\frac{1}{6}\Big(p_2-\frac{1}{2}p_1^2\Big)+\Big[\frac{1}{4}-\frac{1}{6}\Big(\frac{4}{3}\Big)^n\frac{(1+\lambda)^2}{1+2\lambda}\frac{\beta+2}{\beta+1}\Big]p_1^2+\frac{\beta}{3^{n+1}(1+2\lambda)}(c_2-\frac{1}{2}c_1^2)\\&\quad+\frac{-\beta^2}{2.3^{n+1}(\beta+1)(1+2\lambda)}c_1^2+\beta\Big[\frac{2^n(1+\lambda)}{3^{n+1}(1+2\lambda)}-\frac{1}{2}\Big(\frac{2}{3}\Big)^{n+1}\frac{1+\lambda}{1+2\lambda}\frac{\beta+2}{\beta+1}\Big]p_1c_1,\end{aligned} \tag{2.24}$$

moreover,

$$\begin{aligned}
&\left|a_3-\mathscr{W}_2\frac{\beta+2}{\beta+1}a_2^2\right| \\
\leqslant&\frac{1}{6}\left|p_2-\frac{1}{2}p_1^2\right|+\left[\frac{1}{6}\left(\frac{4}{3}\right)^n\frac{(1+\lambda)^2}{1+2\lambda)}\frac{\beta+2}{\beta+1}-\frac{1}{4}\right]|p_1^2|+\frac{\beta}{3^{n+1}(1+2\lambda)}\left|c_2-\frac{1}{2}c_1^2\right| \\
&+\frac{\beta^2}{2.3^{n+1}(\beta+1)(1+2\lambda)}|c_1^2|+\beta\left[\frac{1}{2}\left(\frac{2}{3}\right)^{n+1}\frac{1+\lambda}{1+2\lambda}\frac{\beta+2}{\beta+1}-\frac{2^n(1+\lambda)}{3^{n+1}(1+2\lambda)}\right]|p_1c_1| \\
\leqslant&\frac{1}{6}\left(2-\frac{1}{2}|p_1|^2\right)+\left[\frac{1}{6}\left(\frac{4}{3}\right)^n\frac{(1+\lambda)^2}{1+2\lambda)}\frac{\beta+2}{\beta+1}-\frac{1}{4}\right]|p_1|^2+\frac{\beta}{3^{n+1}(1+2\lambda)}\left(2-\frac{1}{2}|c_1|^2\right) \\
&+\frac{\beta^2}{2.3^{n+1}(\beta+1)(1+2\lambda)}|c_1|^2+\beta\frac{2^{n+1}(1+\lambda)}{2.3^{n+1}(1+2\lambda)(\beta+1)}|p_1c_1|, \\
=&\frac{1}{3}+\frac{2\beta}{3^{n+1}(1+2\lambda)}-\frac{2.3^n(1+2\lambda)(\beta+1)-4^n(1+\lambda)^2(\beta+2)}{6.3^n(1+2\lambda)(\beta+1)}|p_1|^2 \\
&-\frac{\beta}{2.3^{n+1}(\beta+1)(1+2\lambda)}|c_1|^2+\beta\frac{2^{n+1}(1+\lambda)}{2.3^{n+1}(1+2\lambda)(\beta+1)}|p_1c_1|.
\end{aligned}$$

Letting $p_1=2re^{i\theta}$, $c_1=2Re^{i\varphi}$, where $0\leqslant\theta\leqslant 2\pi$, $0\leqslant\varphi\leqslant 2\pi$, $0\leqslant r\leqslant 1$, and $0\leqslant R\leqslant 1$. Following the above inequality, we have

$$\begin{aligned}
&\left|a_3-\mathscr{W}_2\frac{\beta+2}{\beta+1}a_2^2\right|\leqslant\frac{1}{3}+\frac{2\beta}{3^{n+1}(1+2\lambda)}-4r^2\frac{2.3^n(1+2\lambda)(\beta+1)-4^n(1+\lambda)^2(\beta+2)}{6.3^n(1+2\lambda)(\beta+1)} \\
&-4R^2\frac{\beta}{2.3^{n+1}(\beta+1)(1+2\lambda)}+4Rr\beta\frac{2^{n+1}(1+\lambda)}{2.3^{n+1}(1+2\lambda)(\beta+1)}=\Theta(R,r).
\end{aligned}$$

Letting n, β and λ fixed and differentiating $\Theta(R,r)$ partially with $n\geqslant 0$, $0\leqslant\beta\leqslant 1$, and $0\leqslant\lambda\leqslant 1$, we have

$$\Theta_{RR}.\Theta_{rr}-\Theta_{Rr}^2=\frac{128\beta.3^n(1+2\lambda)(\beta+1)-128\beta.4^n(1+\lambda)^2(\beta+1)}{12.3^{2n+1}(1+2\lambda)^2(1+\beta)^2}<0.$$

Therefore, the maximum of $\Theta(R,r)$ occurs on the boundaries, which yields

$$\left|a_3-\mathscr{W}_2\frac{\beta+2}{\beta+1}a_2^2\right|\leqslant\Theta(R,r)\leqslant\Theta(1,1)$$

$$=\frac{1}{3}+\frac{2\beta}{3^{n+1}(1+2\lambda)}+\frac{2^{n+3}\beta(1+\lambda)-4\beta-8.3^n(1+2\lambda)(\beta+1)+4^{n+1}(1+\lambda)^2(\beta+2)}{2.3^{n+1}(1+2\lambda)(\beta+1)}. \tag{2.25}$$

Now, applying the first case with $\mu=\mathscr{W}_2$, we get $|a_3-\mathscr{W}_2a_2^2|\leqslant\Lambda(x_0)|_{\mu=\mathscr{W}_2}$. It follows from the (2.23) and (2.25) that

$$|a_3-\mu a_2^2|\leqslant\left(\frac{\beta+1}{\mathscr{W}_2}\mu-\beta-1\right)\Theta(1,1)+\frac{\beta+1}{\mathscr{W}_2}\left(\mathscr{W}_2\frac{\beta+2}{\beta+1}-\mu\right).\Lambda(x_0)|_{\mu=\mathscr{W}_2}.$$

Equality is attained on choosing $c_1 = b_2 = 2i$, $c_2 = -2$, $b_3 = -3$ in (2.19).

Finally, if $\mu \geqslant \mathscr{W}_2 \frac{\beta+2}{\beta+1}$, then with the aid of the result already proved for $\mu = \mathscr{W}_2$ and $a_2 \leqslant \frac{\beta}{2^n(1+\lambda)} + 1$, we have

$$
\begin{aligned}
|a_3 - \mu a_2^2| &\leqslant \left|a_3 - \mathscr{W}_2 \frac{\beta+2}{\beta+1} a_2^2\right| + \left|\mathscr{W}_2 \frac{\beta+2}{\beta+1} - \mu\right| |a_2|^2 \\
&\leqslant \Theta(1,1) + \left(\mu - \mathscr{W}_2 \frac{\beta+2}{\beta+1}\right) \left[\frac{\beta}{2^n(1+\lambda)} + 1\right]^2.
\end{aligned}
$$

Equality is attained on choosing $c_1 = b_2 = 2i$, $c_2 = -2$, $b_3 = -3$ in (2.19). Thus the proof is completed. □

REMARK.

(1) Setting $n = 0$, $b = 1$, $\lambda = 0$ in Theorem 1, Theorem 2, Theorem 3 and Theorem 4, respectively, we obtain the corresponding results on the classes $S^*(\varphi)$ defined by Ma-Minda [12].

(2) Setting $n = 0$, $b = 1$, $\lambda = 1$ in Theorem 1, Theorem 2, Theorem 3 and Theorem 4, respectively, we obtain the corresponding results on the classes $C(\varphi)$ defined by Ma-Minda [12]

(3) Setting $n = 0$, $b = 1$, $\lambda = 0$, $\varphi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leqslant A < B \leqslant 1)$ in Theorem 1, Theorem 2, Theorem 3 and Theorem 4, respectively, we obtain the corresponding results on the classes $S^*[A,B]$ defined by Janowski [7].

(4) Setting $n = 0$, $b = 1$, $\lambda = 1$, $\varphi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leqslant A < B \leqslant 1)$ in Theorem 1, Theorem 2, Theorem 3 and Theorem 4, respectively, we obtain the corresponding results on the classes $C[A,B]$ defined by Janowski [7].

(5) Setting $n = 0$, $b = 1$, $\lambda = 0$ in Theorem 5, we obtain the results proved by Abdel-Gawad [1].

(6) Setting $n = 0$, $b = 1$, $\lambda = 0$, $\beta = 1$ in Theorem 5, we obtain the results proved by Keogh [9].

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Liangpeng Xiong
Engineering and technical college
of Chengdu university of technology Leshan
Sichuan, 614007, P.R. China e-mail: xlpwxf@163.com

Xiangdong Feng
Engineering and technical college
of Chengdu university of technology Leshan
Sichuan, 614007, P.R. China e-mail: fengxiangdong@sina.com

Jianliang Zhang
Engineering and technical college
of Chengdu university of technology Leshan
Sichuan, 614007, P.R. China e-mail: zhangjianliang48@163.com