Journal of Mathematical Inequalities
Volume 8, Number 3 (2014), 661–667
doi:10.7153/jmi-08-48

SCHUR M–POWER CONVEXITY OF GENERALIZED HAMY SYMMETRIC FUNCTION

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(*Communicated by J. Pečarić*)

Abstract. In this paper, we investigate the Schur m-power convexity of the generalized hamy symmetric function

$$F_n^*(\mathbf{x},r)=\sum_{i_1+i_2+\cdots+i_n=r}(x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n})^{\frac{1}{r}}$$

for $x\in\mathbb{R}_+^n$ and $r\in\mathbb{N}$ with $1\leqslant r\leqslant n$, which generalizes some known results.

1. Introduction

Let $\mathbb{R}_+ := (0,\infty)$. For $\mathbf{x}=(x_1,x_2,\cdots,x_n)\in\mathbb{R}_+^n$, let

$$A_n(\mathbf{x})=\frac{1}{n}\sum_{i=1}^{n}x_i,\quad\text{and}\quad G_n(\mathbf{x})=\left(\prod_{i=1}^{n}x_i\right)^{\frac{1}{n}}. \tag{1.1}$$

denote the classical arithmetic and geometric means, respectively.

The classical arithmetic and geometric mean inequality states that:

$$G_n(\mathbf{x})\leqslant A_n(\mathbf{x}). \tag{1.2}$$

A large number of proofs, generalizations and refinements of inequality (1.2) were given in [2, 3, 11, 14, 15].

The Hamy symmetric function [2, 7, 11] is defined by

$$F_n(\mathbf{x},r)=F_n(x_1,x_2,\cdots,x_n;r)=\sum_{1\leqslant i_1<i_2<\cdots<i_n\leqslant n}\left(\prod_{j=1}^{r}x_{i_j}\right)^{\frac{1}{r}},\ r=1,2,\cdots,n \tag{1.3}$$

Its properties and applications can be found in [2].

Mathematics subject classification (2010): 05E05, 26B25.
Keywords and phrases: Symmetric function, Schur m-power convexity, hamy symmetric function.
This work is supported by the Doctoral Programs Foundation of Education Ministry of China (20113401110009), Natural Science Research Project of Hefei Normal University (2012kj11), Universities Natural Science Foundation of Anhui Province (KJ2013A220).

In 2006, Guan [7] defined the following generalized Hamy symmetric function

$$F_n^*(\mathbf{x},r)=\sum_{i_1+i_2+\cdots+i_n=r}(x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n})^{\frac{1}{r}}, \tag{1.4}$$

where $i_1,i_2,\cdots,i_n$ are non-negative integers, $r\in\mathbb{N}$.

In 1923, the Schur convex function was introduced in [17] by Schur, and it plays an important role in the study of analytic inequalities [2-35]. Hereafter, Hard, Littlewood, Polya, and Wang also investigated the Schur convexity of several functions, and obtained some important inequalities [12, 23].

Guan [7] proved Hamy symmetric function $F_n(\mathbf{x},r)$ and its generalized Hamy one $F_n^*(\mathbf{x},r)$ are Schur concave in $\mathbb{R}_+^n$, and obtained some interesting analytic inequalities.

The notions of Schur geometrical convexity and Schur harmonic convexity were introduced by Zhang [35] and Anderson et al. [1], respectively. Guan [8] proved the Hamy symmetric function $F_n(\mathbf{x},r)$ and its generalized one $F_n^*(\mathbf{x},r)$ are geometrically convex on $\mathbb{R}_+^n$. Chu and Sun [4] showed that the generalized Hamy symmetric function $F_n^*(\mathbf{x},r)$ is Schur harmonic convex on $\mathbb{R}_+^n$.

Recently, Yang [29-31] generalized the notion of Schur convexity to Schur f-convexity, which contains the Schur geometrical convexity, Schur harmonic convexity and so on. Moreover, he discussed Schur m-power convexity of Stolarsky means [29], Gini means [30] and Daróczy means [31].

The purpose of this paper is to discuss the Schur m-power convexity of the generalized Hamy symmetric function $F_n^*(\mathbf{x},r)$.

Our main results are stated as follows.

THEOREM 1.1. *For fixed $r\in\mathbb{N}$ with $1\leqslant r\leqslant n$, $F_n^*(\mathbf{x},r)$ is Schur m-power concave on $\mathbb{R}_+^n$ when $m\geqslant 1$ and Schur m-power convex on $\mathbb{R}_+^n$ when $m\leqslant\frac{1}{r}$.*

Taking $r=1$, we get

COROLLARY 1.2. *For fixed $r\in\mathbb{N}$ with $1\leqslant r\leqslant n$, $F_n^*(\mathbf{x},1)=\sum_{i=1}^n x_i$ is Schur m-power concave on $\mathbb{R}_+^n$ when $m\geqslant 1$ and Schur m-power convex on $\mathbb{R}_+^n$ when $m\leqslant 1$.*

COROLLARY 1.3. *For fixed $r\in\mathbb{N}$ with $1\leqslant r\leqslant n$, the function $\frac{F_n^*(\mathbf{x},r)}{F_n^*(1-\mathbf{x},1)}$ is Schur m-power concave on $(0,\frac{1}{2})^n$ when $m\geqslant 1$ and Schur m-power convex on $(0,\frac{1}{2})^n$ when $m\leqslant\frac{1}{r}$.*

REMARK. Since the Schur m-power convexity contains the Schur convexity, Schur geometrical convexity and Schur harmonic convexity, respectively, our result generalizes some known results.

2. Definitions and Lemmas

We first introduce the following definitions and lemmas.

LEMMA 2.1. ([17,23]) *Let* $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ *and* $\mathbf{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ *be two* n*-tuples real numbers.* $\mathbf{y}$ *majorizes* $\mathbf{x}$ *(in symbols* $\mathbf{x} \prec \mathbf{y}$*), if* $\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]}$, $(k = 1, 2, \cdots, n-1)$ *and* $\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$, *where* $x_{[1]} \geqslant x_{[2]} \geqslant \cdots \geqslant x_{[n]}$, $y_{[1]} \geqslant y_{[2]} \geqslant \cdots \geqslant y_{[n]}$ *are rearrangements of* $\mathbf{x}$ *and* $\mathbf{y}$ *in a descending order.*

LEMMA 2.2. ([17,23]) *A real-valued function* $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ *is said to be Schur convex on* Ω *if*

$$\mathbf{x} \prec \mathbf{y} \quad on \ \Omega \Rightarrow \qquad f(\mathbf{x}) \leqslant f(\mathbf{y}).$$

f *is a Schur concave function on* Ω *if and only if* $-f$ *is a Schur convex function.*

Recently, Yang present the Schur f-convexity in [29] as follows.

LEMMA 2.3. ([29 − 31]) *Let* $\Omega \subseteq \mathbb{R}^n$ *be a set with nonempty interior and* f *be a strictly monotone function defined on* Ω. *Suppose that*

$$f(\mathbf{x}) = (f(x_1), f(x_2), \cdots, f(x_n)) \text{ and } f(\mathbf{y}) = (f(y_1), f(y_2), \cdots, f(y_n)).$$

Then function $\varphi : \Omega \to \mathbb{R}$ *is said to be Schur f-convex on* Ω *if* $f(\mathbf{x}) \prec f(\mathbf{y})$ *on* Ω *implies* $\psi(\mathbf{x}) \leqslant \psi(\mathbf{y})$.

ψ *is said to be Schur m-power concave if* $-\psi$ *is Schur m-power convex.*

LEMMA 2.4. ([29 − 31]) *Let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be defined by* $f(x) = (x^m - 1)/m$ *if* $m \neq 0$ *and* $f(x) = lnx$ *if* $m = 0$. *Then function* $\psi : \Omega \subseteq \mathbb{R}^n_+ \to \mathbb{R}$ *is said to be Schur m-power convex on* Ω *if* $f(\mathbf{x}) \prec f(\mathbf{y})$ *on* Ω *implies* $\psi(\mathbf{x}) \leqslant \psi(\mathbf{y})$.

LEMMA 2.5. ([29 − 31]) *Let* $\psi : \Omega \subseteq \mathbb{R}^n_+ \to \mathbb{R}$ *be continuous on* Ω *and differentiable in* Ω^0. *Then* ψ *is schur m-power convex (Schur m-power concave) on* Ω *if and only if* ψ *is symmetric on* Ω *and*

$$\frac{x_1^m - x_2^m}{m}\left(x_1^{1-m}\frac{\partial \varphi}{\partial x_1} - x_2^{1-m}\frac{\partial \varphi}{\partial x_2}\right) \geqslant (\leqslant) 0 \quad if\, m \neq 0, \tag{2.1}$$

$$(\ln x_1 - \ln x_2)\left(x_1\frac{\partial \varphi}{\partial x_1} - x_2\frac{\partial \varphi}{\partial x_2}\right) \geqslant (\leqslant) 0 \quad if\, m = 0, \tag{2.2}$$

hold for any $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \Omega^0$ *with* $x_1 \neq x_2$, *where* $\Omega \subseteq \mathbb{R}^n_+$ *is a symmetric set with nonempty interior* Ω^0.

The following lemma is clearly due to the monotonicity property of the function x^p on $(0, \infty)$, which will be used to prove our main result.

LEMMA 2.6. *For $x_1, x_2 > 0$ with $x_1 \neq x_2$, let U be defined by*

$$U(p;x_1,x_2) := \frac{x_1^p - x_2^p}{p(x_1 - x_2)}, \quad \text{if } p \neq 0 \text{ and } U(0;x_1,x_2) := \frac{\ln x_1 - \ln x_2}{x_1 - x_2}. \tag{2.3}$$

Then $\operatorname{sgn}\left(\frac{x_1^p - x_2^p}{p(x_1 - x_2)}\right) = 1$.

REMARK 2.7. *Lemma 2.6, we see that*

$$\operatorname{sgn}\left(\frac{x_1^p - x_2^p}{p}\right) = \operatorname{sgn}(x_1 - x_2) \quad \text{if } p \neq 0 \text{ and } \operatorname{sgn}(\ln x_1 - \ln x_2) = \operatorname{sgn}(x_1 - x_2).$$

Then the two discrimination inequalities in Lemma 2.5 are equivalent to

$$(x_1 - x_2)\left(x_1^{1-m}\frac{\partial \varphi}{\partial x_1} - x_2^{1-m}\frac{\partial \varphi}{\partial x_3}\right) \geqslant (\leqslant) 0. \tag{2.4}$$

The valuable suggestion is due to Ming Li.

Next, recall that the complete symmetric function in [9] defined by

$$C_r(\mathbf{x}) = C_r(x_1, x_2, \cdots, x_n) = \sum_{i_1 + i_2 + \cdots + i_n = r} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n},$$

where $i_1, i_2, \cdots, i_n$ are non-negative integers, $r \in \mathbb{N}$ with $1 \leqslant r \leqslant n$, and $C_0(\mathbf{x}) = 1$. Guan [34] obtained its property as follows.

LEMMA 2.8. ([9]) *Suppose that $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n$. If*

$$\overline{\mathbf{x}_i} = (x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n),$$

then

$$C_r(\mathbf{x}) = x_i C_{r-1}(\mathbf{x}) + C_r(\overline{\mathbf{x}_i}). \tag{2.5}$$

3. Proof of main results

Proof of Theorem 1.1. For fixed r, let $\mathbf{u} = (u_1, u_2, \cdots, u_n)$, $u_i = \sqrt[r]{x_i}$, $i = 1, 2, \cdots, n$. Then $F_n^*(x, r) = C_r(\mathbf{u})$. From Lemma 2.8, it follows that

$$\frac{\partial C_r(\mathbf{u})}{\partial u_k} = C_{r-1}(\mathbf{u}) + u_k \frac{\partial C_{r-1}(\mathbf{u})}{\partial u_k}, \; k = 1, 2, \cdots, n. \tag{3.1}$$

Hence, we get

$$\frac{\partial C_r(\mathbf{u})}{\partial u_k} = C_{r-1}(\mathbf{u}) + u_k C_{r-2}(\mathbf{u}) + u_k^2 C_{r-2}(\mathbf{u}) + \cdots + u_k^{r-2} C_{r-2}(\mathbf{u}) + u_k^{r-1}. \tag{3.2}$$

Differentiating $F_n^*(x,r)$ with respect to x_1 and using (3.2), we obtain

$$\begin{aligned}\frac{\partial F_n^*(\mathbf{x},r)}{\partial x_1} &= \sum_{k=1}^{n} \frac{\partial C_r(\mathbf{u})}{\partial u_k} \cdot \frac{\partial u_k}{\partial x_1} = \frac{\partial C_r(\mathbf{u})}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} \\ &= \left(C_{r-1}(\mathbf{u}) + u_1 C_{r-2}(\mathbf{u}) + u_1^2 C_{r-2}(\mathbf{u}) + \cdots + u_1^{r-2} C_{r-2}(\mathbf{u}) + u_1^{r-1}\right) \frac{\sqrt[r]{x_1}}{r x_1} \\ &= \frac{1}{r} \sum_{j=1}^{r} C_{r-j}(\mathbf{u}) x_1^{\frac{j-r}{r}}.\end{aligned} \tag{3.3}$$

Similarly, we have

$$\frac{\partial F_n^*(\mathbf{x},r)}{\partial x_2} = \frac{1}{r} \sum_{j=1}^{r} C_{r-j}(\mathbf{u}) x_2^{\frac{j-r}{r}}. \tag{3.4}$$

By Lemma 2.5 and Remark 2.7, and combining (3.3) with (3.4), we find that

$$\begin{aligned}\Delta_1 =&(x_1 - x_2)\left(x_1^{1-m} \frac{\partial F_n^*(\mathbf{x},r)}{\partial x_1} - x_2^{1-m} \frac{\partial F_n^*(\mathbf{x},r)}{\partial x_2}\right) \\ =&\frac{(x_1 - x_2)}{r} \sum_{j=1}^{r} \left[C_{r-j}(\mathbf{u}) \left(x_1^{\frac{j}{r}-m} - x_2^{\frac{j}{r}-m}\right)\right] \\ =&\frac{(x_1 - x_2)^2}{r} \sum_{j=1}^{r} \left[\left(\frac{j}{r} - m\right) C_{r-j}(\mathbf{u}) \frac{\left(x_1^{\frac{j}{r}-m} - x_2^{\frac{j}{r}-m}\right)}{(\frac{j}{r} - m)(x_1 - x_2)}\right] \\ =&\frac{(x_1 - x_2)^2}{r} \sum_{j=1}^{r} \left[\left(\frac{j}{r} - m\right) C_{r-j}(\mathbf{u}) U\left(\frac{j}{r} - m; x_1, x_2\right)\right],\end{aligned} \tag{3.5}$$

where $U(p;x_1,x_2)$ is defined by (2.3).

On the other hand, Lemma 2.6 implies that for $x_1, x_2 > 0$, $U(p;x_1,x_2) > 0$. So, to guarantee $\Delta_1 \geqslant (\leqslant) 0$, it suffices to

$$\frac{j}{r} - m \geqslant (\leqslant) 0, \quad j = 1, 2, \cdots, r.$$

Solving the inequalities for m yield $m \leqslant \frac{1}{r}$ or $m \geqslant 1$. Thus, the proof of Theorem 1.1 is complete. □

Proof of Corollary 1.3. For fixed $r \in \mathbb{N}$ with $1 \leqslant r \leqslant n$, let $\phi(x,r) = \frac{F_n^*(\mathbf{x},r)}{F_n^*(1-\mathbf{x},1)}$. Differentiating ϕ with respect to x_i shows that

$$\frac{\partial \phi(\mathbf{x},r)}{\partial x_i} = \frac{1}{(F_n^*(1-\mathbf{x},1))^2} \left(F_n^*(1-\mathbf{x},1) \cdot \frac{\partial F_n^*(\mathbf{x},r)}{\partial x_i} + F_n^*(x,r)\right), \quad i = 1,2. \tag{3.7}$$

From (3.7) it follows that

$$\begin{aligned}\Delta_2 &= (x_1-x_2)\left(x_1^{1-m}\frac{\partial\phi(\mathbf{x},r)}{\partial x_1}-x_2^{1-m}\frac{\partial\phi(\mathbf{x},r)}{\partial x_2}\right)\\ &=\frac{x_1-x_2}{F_n^*(1-\mathbf{x},1)}\left(x_1^{1-m}\frac{\partial F_n^*(\mathbf{x},r)}{\partial x_1}-x_2^{1-m}\frac{\partial F_n^*(\mathbf{x},r)}{\partial x_2}\right)+\frac{\phi(\mathbf{x},r)}{F_n^*(1-\mathbf{x},1)}(x_1-x_2)(x_1^{1-m}-x_2^{1-m})\\ &=\frac{1}{F_n^*(1-\mathbf{x},1)}\cdot\Delta_1+\frac{\phi(\mathbf{x},r)}{F_n^*(1-\mathbf{x},1)}\frac{x_1^{1-m}-x_2^{1-m}}{(1-m)(x_1-x_2)}\cdot(1-m)(x_1-x_2)^2\\ &=\frac{1}{F_n^*(1-\mathbf{x},1)}\cdot\Delta_1+\frac{\phi(\mathbf{x},r)\cdot(1-m)(x_1-x_2)^2}{F_n^*(1-\mathbf{x},1)}U(1-m;x_1,x_2),\end{aligned}$$

where $U(p;x_1,x_2)$ is defined by (2.3), and $U(p;x_1,x_2)>0$.

Clearly, for $m\geqslant 1$, it is seen that $\Delta_1\leqslant 0$ and $1-m\leqslant 0$, which yield $\Delta_2\leqslant 0$, that is, $\frac{F_n^*(\mathbf{x},r)}{F_n^*(1-\mathbf{x},1)}$ is Schur m-power concave on $(0,\frac{1}{2})^n$. In the same way, for $m\leqslant\frac{1}{r}$, $\frac{F_n^*(\mathbf{x},r)}{F_n^*(1-\mathbf{x},1)}$ is Schur m-power convex on $(0,\frac{1}{2})^n$.

This completes the proof. □

Acknowledgements. The authors wish to thank the anonymous referees for their careful reading of the manuscript and their fruitful comments and suggestions.

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(Received May 28, 2013)

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