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OPERATOR INEQUALITIES AMONG ARITHMETIC MEAN, GEOMETRIC MEAN AND HARMONIC MEAN

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Abstract. We give an upper bound for the weighted geometric mean using the weighted arithmetic mean and the weighted harmonic mean. We also give a lower bound for the weighted geometric mean. These inequalities are proven for two invertible positive operators.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space. We represent the set of all bounded operators on $\mathscr{H}$ by $B(\mathscr{H})$. If $A \in B(\mathscr{H})$ satisfies $A^* = A$, then A is called a self-adjoint operator. If a self-adjoint operator A satisfies $\langle x|A|x\rangle \geqslant 0$ for any $|x\rangle \in \mathscr{H}$, then A is called a positive operator. For two self-adjoint operators A and B, $A \geqslant B$ means $A - B \geqslant 0$. The notation $A > 0$ means A is an invertible positive operator.

It is well-known that we have the following Young inequalities for invertible positive operators A and B:

$$(1-\nu)A + \nu B \geqslant A\#_\nu B \geqslant \left\{(1-\nu)A^{-1} + \nu B^{-1}\right\}^{-1}, \tag{1}$$

where $A\#_\nu B \equiv A^{1/2}(A^{-1/2}BA^{-1/2})^\nu A^{1/2}$ represents the geometric mean for two positive operators A and B and a weighted parameter $\nu \in [0,1]$ [1]. (In this paper, we use the notation $A\#B$ instead of $A\#_{1/2}B$ for the simplicity.) $(1-\nu)A + \nu B$ and $\left\{(1-\nu)A^{-1} + \nu B^{-1}\right\}^{-1}$ are called weighted arithmetic mean and harmonic mean for two positive operators, respectively. The simplified and elegant proof for the inequalities (1) was given in [2]. Recently, refinements of the inequalities (1) were given in our papers [3, 4]. It is also notable that improvements of [4] have been given in the paper [5]. And further improvements have been given in quite recent papers [6] and [7]. In this short note, we consider the relations among operator means for two positive operators.

We start from the following proposition.

PROPOSITION 1.1. *Let A, B be invertible positive operators and r be a real number. Then we have the following inequalities.*

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(i) If $r \geqslant 2$, *then* $rA\#B + (1-r)\frac{A+B}{2} \leqslant \left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}$.

(ii) If $r \leqslant 1$, *then* $rA\#B + (1-r)\frac{A+B}{2} \geqslant \left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}$.

Proof. In general, by using the notion of the representing function $f_m(x) = 1mx$ for operator mean m, it is well-known [1] that $f_m(x) \leqslant f_n(x)$ holds for $x > 0$ if and only if $AmB \leqslant AnB$ holds for all positive operators A and B. Thus we can prove this proposition from the following scalar inequalities for $t > 0$.

(i) $r\sqrt{t} + (1-r)\frac{t+1}{2} \leqslant \frac{2t}{t+1}, \quad (r \geqslant 2)$.

(ii) $r\sqrt{t} + (1-r)\frac{t+1}{2} \geqslant \frac{2t}{t+1}, \quad (r \leqslant 1)$.

Actually (i) above can be proven in the following way. We set $f_r(t) \equiv \frac{2t}{t+1} - r\sqrt{t} - (1-r)\frac{t+1}{2}$, then $\frac{df_r(t)}{dr} = -\sqrt{t} + \frac{t+1}{2} \geqslant 0$ implies $f_r(t) \geqslant f_2(t)$ for $r \geqslant 2$. From the relation $\frac{2t}{t+1} + \frac{t+1}{2} \geqslant 2\sqrt{t}$, we have $f_2(t) \geqslant 0$. We also give the proof for (ii) above. We set $g_r(t) \equiv r\sqrt{t} + (1-r)\frac{t+1}{2} - \frac{2t}{t+1}$, then $\frac{dg_r(t)}{dr} = \sqrt{t} - \frac{t+1}{2} \leqslant 0$ implies $g_r(t) \geqslant g_1(t)$ for $r \leqslant 1$. From the relation $\frac{2t}{t+1} \leqslant \sqrt{t}$, we have $g_1(t) \geqslant 0$. $\square$

REMARK 1.2. *We have counter-examples of both inequalities (i) and (ii) in Proposition 1.1 for* $1 < r < 2$. *For example, we take* $r = 1.5$. *Then we have the following computations.* $\frac{2t}{t+1} - r\sqrt{t} - (1-r)\frac{t+1}{2} \simeq 0.122302$ *when* $t = 0.01$ *and* $\frac{2t}{t+1} - r\sqrt{t} - (1-r)\frac{t+1}{2} \simeq -0.037987$ *when* $t = 2$.

2. Main results

Proposition 1.1 can be generalized by means of weighted parameter $\nu \in [0,1]$, as the second inequality in (2) below.

THEOREM 2.1. *If (i)* $0 \leqslant \nu \leqslant 1/2$ *and* $0 < A \leqslant B$ *or (ii)* $1/2 \leqslant \nu \leqslant 1$ *and* $0 < B \leqslant A$, *then the following inequalities hold*

$$A\#B + \left(\nu - \frac{1}{2}\right)(B-A) \leqslant A\#_\nu B \leqslant \frac{1}{2}\{(1-\nu)A + \nu B\} + \frac{1}{2}\{(1-\nu)A^{-1} + \nu B^{-1}\}^{-1}. \quad (2)$$

REMARK 2.2. *Under the same conditions as in Theorem 2.1, we have* $A\#B \geqslant A\#_\nu B$.

In order to prove Theorem 2.1, we firstly prove the corresponding scalar inequalities, as it was similarly done in Proposition 1.1.

LEMMA 2.3. *If (i)* $0 \leqslant \nu \leqslant 1/2$ *and* $t \geqslant 1$ *or (ii)* $1/2 \leqslant \nu \leqslant 1$ *and* $0 < t \leqslant 1$, *then the following inequalities hold*

$$2\sqrt{t} + (2\nu - 1)(t-1) \leqslant 2t^\nu \leqslant (1-\nu) + \nu t + \left\{(1-\nu) + \frac{\nu}{t}\right\}^{-1}. \quad (3)$$

Proof. It is trivial for the case $t = 1$. For the cases $\nu = 0$, $1/2$ or 1, the inequalities (3) hold. So we assume $t \neq 1$ and $\nu \neq 0, 1/2, 1$. We firstly prove the first inequality of the inequalities (3), under the condition (i) $0 < \nu < 1/2$ and $t > 1$ or (ii) $1/2 < \nu < 1$ and $0 < t < 1$. Here we put $f_\nu(t) \equiv t^\nu - \sqrt{t} - \left(\nu - \frac{1}{2}\right)(t-1)$. Then we have $f'_\nu(t) = \nu t^{\nu-1} - \frac{1}{2}\frac{1}{\sqrt{t}} - \left(\nu - \frac{1}{2}\right)$ and $f'_\nu(1) = 0$. We also have $f''_\nu(t) = -\nu(1-\nu)t^{\nu-2} + \frac{1}{4}t^{-3/2}$. Thus we have $f''_\nu(t) = 0 \Leftrightarrow t = t_\nu \equiv \{4\nu(1-\nu)\}^{\frac{2}{1-2\nu}}$. We find $t_\nu < 1$ in the case $0 < \nu < 1/2$ and $t > 1$. Then we find $f''_\nu(t) \geqslant 0$ for $t > 1 (> t_\nu)$. So $f'_\nu(t)$ is monotone increasing for $t > 1$ and we have $f'_\nu(1) = 0$. Thus we find $f'_\nu(t) \geqslant 0$ for $t > 1$. So $f_\nu(t)$ is monotone increasing for $t > 1$. Therefore we have $f_\nu(t) \geqslant f_\nu(1) = 0$. We also find $t_\nu > 1$ in the case $1/2 < \nu < 1$ and $0 < t < 1$. Then we find $f''_\nu(t) \geqslant 0$ for $0 < t < 1 (< t_\nu)$. So $f'_\nu(t)$ is monotone increasing for $0 < t < 1$ and we have $f'_\nu(1) = 0$. Thus we find $f'_\nu(t) \leqslant 0$ for $0 < t < 1$. So $f_\nu(t)$ is monotone decreasing for $0 < t < 1$. Therefore we have $f_\nu(t) \geqslant f_\nu(1) = 0$. Thus the proof for the first inequality of the inequalities (3) is done.

We prove the second inequality of the inequalities (3). We put

$$g_\nu(t) \equiv (1-\nu) + \nu t + \frac{1}{1-\nu+\frac{\nu}{t}} - 2t^\nu.$$

Then we have

$$g_\nu(t) = (1-\nu) + \nu t + \frac{t}{(1-\nu)t+\nu} - 2t^\nu \geqslant 2\sqrt{\frac{\{(1-\nu)+\nu t\}t}{(1-\nu)t+\nu}} - 2t^\nu.$$

Since $g_\nu(t) \geqslant 0$ is equivalent to $\frac{(1-\nu)+\nu t}{(1-\nu)t+\nu} \geqslant t^{2\nu-1}$, we put again $h_\nu(t) \equiv (1-\nu) + \nu t - \{(1-\nu)t+\nu\}t^{2\nu-1}$. Then we prove $h_\nu(t) > 0$ under the condition (i) $0 < \nu < 1/2$ and $t > 1$ or (ii) $1/2 < \nu < 1$ and $0 < t < 1$. By the elementary calculations, we have $h_\nu{}'(t) = \nu - 2\nu(1-\nu)t^{2\nu-1} - \nu(2\nu-1)t^{2\nu-2}$, $h_\nu{}'(1) = 0$ and $h_\nu{}''(t) = -2\nu(1-\nu)(2\nu-1)t^{2\nu-3}(t-1)$. Then we find $h_\nu{}''(t) = 0 \Leftrightarrow t = 1$. In the case $t > 1$, we have $h_\nu{}''(t) \geqslant 0$. So $h_\nu{}'(t)$ is monotone increasing for $t > 1$ and we have $h_\nu{}'(1) = 0$. Thus we have $h_\nu{}'(t) \geqslant 0$ for $t > 1$. So $h_\nu(t)$ is monotone increasing for $t > 1$. Thus we have $h_\nu(t) \geqslant h_\nu(1) = 0$. In the case $0 < t < 1$, we also have $h_\nu{}''(t) \geqslant 0$. So $h_\nu{}'(t)$ is monotone increasing for $0 < t < 1$ and we have $h_\nu{}'(1) = 0$. Thus we have $h_\nu{}'(t) \leqslant 0$ for $0 < t < 1$. So $h_\nu(t)$ is monotone decreasing for $0 < t < 1$. Thus we have $h_\nu(t) \geqslant h_\nu(1) = 0$. Thus the proof for the second inequality of the inequalities (3) is done. □

LEMMA 2.4. *Let $r \in \mathbb{R}$. Then the function $k_{r,\nu}(t) \equiv rt^\nu + (1-r)\{(1-\nu)+\nu t\}$, $(0 \leqslant \nu \leqslant 1, t > 0)$ is monotone decreasing with respect to r. Therefore, $k_{r,\nu}(t) \leqslant k_{2,\nu}(t)$ for $r \geqslant 2$ and $k_{r,\nu}(t) \geqslant k_{1,\nu}(t)$ for $r \leqslant 1$.*

Proof. The proof is done by $\frac{\partial k_{r,\nu}(t)}{\partial r} = t^\nu - \{(1-\nu)+\nu t\} \leqslant 0$, for $\nu \in [0,1]$ and $t > 0$. □

Lemma 2.4 provides the following results.

LEMMA 2.5. *Let* $r \geqslant 2$. *If (i)* $0 \leqslant \nu \leqslant 1/2$ *and* $t \geqslant 1$ *or (ii)* $1/2 \leqslant \nu \leqslant 1$ *and* $0 < t \leqslant 1$, *then*

$$rt^{\nu} + (1-r)\{(1-\nu)+\nu t\} \leqslant \left\{(1-\nu)+\frac{\nu}{t}\right\}^{-1}.$$

Proof. The proof follows directly from Lemma 2.3 and Lemma 2.4. □

LEMMA 2.6. *Let* $r \leqslant 1$. *For* $0 < \nu \leqslant 1$ *and* $t > 0$, *we have*

$$rt^{\nu} + (1-r)\{(1-\nu)+\nu t\} \geqslant \left\{(1-\nu)+\frac{\nu}{t}\right\}^{-1}.$$

Proof. For $r \leqslant 1$, it follows from Lemma 2.4 that $rt^{\nu} + (1-r)\{(1-\nu)+\nu t\} \geqslant t^{\nu}$. Since we have $t^{\nu} \geqslant \left\{(1-\nu)+\frac{\nu}{t}\right\}^{-1}$, the proof is done. □

Finally we have the following corollary.

COROLLARY 2.7. *Let* $r \geqslant 2$. *If (i)* $0 < \nu \leqslant 1/2$ *and* $0 < A \leqslant B$ *or (ii)* $1/2 \leqslant \nu \leqslant 1$ *and* $0 < B \leqslant A$, *then*

$$rA\#_{\nu}B + (1-r)\{(1-\nu)A+\nu B\} \leqslant \left\{(1-\nu)A^{-1}+\nu B^{-1}\right\}^{-1}.$$

Let $r \leqslant 1$. *For* $0 < \nu < 1$ *and* $t > 0$, *we have*

$$rA\#_{\nu}B + (1-r)\{(1-\nu)A+\nu B\} \geqslant \left\{(1-\nu)A^{-1}+\nu B^{-1}\right\}^{-1}.$$

Proof. The proof can be done applying Lemma 2.5, Lemma 2.6 and Theorem 2.1. □

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