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REFINEMENTS OF SOME INEQUALITIES RELATED TO JENSEN'S INEQUALITY

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Dedicated to Professor Sin-Ei Takahasi on the occasion of his 70th birthday

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Abstract. A finite form of Jensen's inequality for a continuous *convex* function from a topological abelian semigroup to another topological ordered abelian semigroup is given by the author and S.-E. Takahasi. As an application of this abstract Jensen's inequality, two inequalities with respect to geometric mean and arithmetic mean are obtained. The first gives a new refinement of the geometric-arithmetic mean inequality. The second gives a refinement between the arithmetic mean and a certain mean.

1. Introduction

The finite form of Jensen's inequality proved by Jensen [1] in 1906 asserts that if $t_1,\cdots,t_n$ are positive numbers with $\sum_{i=1}^n t_i = 1$ and f is a continuous convex (resp. concave) function on a real interval I, then

$$f\left(\sum_{i=1}^n t_i x_i\right) \leqslant \sum_{i=1}^n t_i f(x_i) \quad \left(\text{resp.}\, f\left(\sum_{i=1}^n t_i x_i\right) \geqslant \sum_{i=1}^n t_i f(x_i)\right)$$

holds for all $x_1,\cdots,x_n \in I$.

In [3], the author and Takahasi have introduced a concept called $(*,\circ)$-*convex* (or *concave*) for a continuous function from a topological abelian semigroup $(I,*)$ to another topological ordered abelian semigroup $(J,\circ)$, and give an abstract Jensen's inequality for such a function. Applying this abstract Jensen's inequality, we give two interesting inequalities related to the geometric mean and the arithmetic mean. The first (Theorem 1) is a new refinement of the geometric-arithmetic mean inequality and the second (Theorem 2) is a refinement between the arithmetic mean and a certain mean related to the geometric mean.

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2. Terminology and main results

Let I be a topological space and $*$ a topological abelian semigroup operation on I. For any $x \in I$ and $n \in \mathbf{N}$, define the n-th power $x^{(n)_*}$ of x recursively by $x^{(1)_*} = x$ and $x^{(n+1)_*} = x^{(n)_*} * x$ for $n \geqslant 1$.

We assume that

($\sharp_1$) any n-th power function: $x \mapsto x^{(n)_*}$ is a bijection of I onto itself.

By the assumption ($\sharp_1$), for each $x \in I$ and $n \in \mathbf{N}$, there exists a unique element a of I such that $a^{(n)_*} = x$. Denote by $x^{(1/n)_*}$ such an element a. Moreover, we define

$$x^{(m/n)_*} = \left(x^{(1/n)_*}\right)^{(m)_*}$$

for each $m,n \in \mathbf{N}$. Then we can easily see that this definition is well-defined. In this case, we can easily show that the following power laws:

$$x^{(p+q)_*} = x^{(p)_*} * y^{(q)_*}, x^{(pq)_*} = \left(x^{(p)_*}\right)^{(q)_*} \quad \text{and} \quad (x*y)^{(p)_*} = x^{(p)_*} * y^{(p)_*} \tag{1}$$

for all $p,q \in \mathbf{Q}_+$ and $x,y \in I$. Here $\mathbf{Q}_+$ denotes the set of all positive rational numbers. Furthermore, we assume that

($\sharp_2$) for each $x \in I$, the function $p \mapsto x^{(p)_*}$ is continuous on $\mathbf{Q}_+$ and it has a continuous extension to $\mathbf{R}_+$, say $t \mapsto x^{(t)_*}$.

Here $\mathbf{R}_+$ denotes the set of all positive real numbers. Therefore power laws (1) hold for all $p,q \in \mathbf{R}_+$. Denote by $\mathscr{A}_+(I)$ the set of all topological abelian semigroup operations on I satisfying both ($\sharp_1$) and ($\sharp_2$). Our assumption ($\sharp_1$) leads to the following important concept called *mean*. For each $x,y \in I$, put

$$M_*(x,y) = (x*y)^{(1/2)_*}.$$

We call $M_*(x,y)$ the mean of x and y with respect to the operation $*$.

Now let J be a topological ordered space with relation $\leqslant$, and denote by $\mathscr{A}_+^0(J) = \mathscr{A}_+^0(J,\leqslant)$ the set of all operations $\circ \in \mathscr{A}_+(J)$ satisfying the following two conditions:
($\flat_1$) $a \leqslant b \Leftrightarrow a \circ c \leqslant b \circ c$ for all $a,b,c \in J$
and
($\flat_2$) $a \leqslant b \Rightarrow a^{(t)_\circ} \leqslant b^{(t)_\circ}$ for all $a,b \in J$ and $t \in \mathbf{R}_+$.

Let $C(I,J)$ be the set of all continuous functions from I to J. Take $* \in \mathscr{A}_+(I)$, $\circ \in \mathscr{A}_+^0(J,\leqslant)$ and $f \in C(I,J)$ arbitrarily. If f satisfies

$$f(M_*(x,y)) \leqslant M_\circ(f(x),f(y)) \quad (\text{resp. } f(M_*(x,y)) \geqslant M_\circ(f(x),f(y)))$$

for all $x,y \in I$, then f is said to be $(*,\circ)$-convex (resp. concave).

In [3], the author and Takahasi have shown the following theorem which states a finite form of Jensen's inequality for a $(*,\circ)$-convex (or concave) function.

THEOREM A. *Let* $* \in \mathscr{A}_+(I)$ *and* $\circ \in \mathscr{A}_+^0(J,\leqslant)$. *If* $f \in C(I,J)$ *is* $(*,\circ)$*-convex, then*

$$f(x_1^{(t_1)_*} * \cdots * x_n^{(t_n)_*}) \leqslant f(x_1)^{(t_1)_\circ} \circ \cdots \circ f(x_n)^{(t_n)_\circ}$$

holds for all $n \in \mathbf{N}$, $x_1,\cdots,x_n \in I$ *and* $t_1,\cdots,t_n \in \mathbf{R}_+$ *with* $t_1+\cdots+t_n = 1$.

If f *is* $(*,\circ)$*-concave, then the inequality above is reversed.*

REMARK 1. The above theorem is the inheritance of the idea of [2, Theorem 1] which gives a new interpretation of Jensen's inequality by φ-mean.

As an application of Theorem A, we have the following

THEOREM 1. *Let* $a_1,\cdots,a_n,t_1,\cdots,t_n > 0$ *with* $t_1+\cdots+t_n = 1$ *and put*

$$(GA)_t = \prod_{i=1}^{n}(a_i+t)^{t_i} - t$$

for each $t \geqslant 0$. *Then* $\{(GA)_t : t \geqslant 0\}$ *is strictly monotone increasing and*

$$\lim_{t\to\infty}(GA)_t = \sum_{i=1}^{n} t_i a_i$$

holds.

REMARK 2. The above theorem gives a strict refinement of the geometric-arithmetic mean inequality:

$$\prod_{i=1}^{n} a_i^{t_i} < (GA)_t < \sum_{i=1}^{n} t_i a_i \quad (t > 0).$$

Furthermore, as another application of Theorem A, we have the following

THEOREM 2. *Let* $0 < a_1,\cdots,a_n,t_1,\cdots,t_n < 1$ *with* $t_1+\cdots+t_n = 1$ *and put*

$$(AP)_t = \frac{1}{t}\frac{\prod_{i=1}^{n}(1+ta_i)^{t_i} - \prod_{i=1}^{n}(1-ta_i)^{t_i}}{\prod_{i=1}^{n}(1+ta_i)^{t_i} + \prod_{i=1}^{n}(1-ta_i)^{t_i}}$$

for each t *with* $0 < t \leqslant 1$. *Then* $\{(AP)_t : 0 < t \leqslant 1\}$ *is strictly monotone increasing and*

$$\lim_{t\downarrow 0}(AP)_t = \sum_{i=1}^{n} t_i a_i.$$

holds.

REMARK 3. The above theorem gives a strict refinement between $\sum_{i=1}^{n} t_i a_i$ and $(AP)_1$:

$$\sum_{i=1}^{n} t_i a_i < (AP)_t < \frac{\prod_{i=1}^{n}(1+a_i)^{t_i} - \prod_{i=1}^{n}(1-a_i)^{t_i}}{\prod_{i=1}^{n}(1+a_i)^{t_i} + \prod_{i=1}^{n}(1-a_i)^{t_i}} \quad (0 < t < 1).$$

3. Proofs of main results

In order to show our main results, we must prepare several lemmas.

LEMMA 1. *Let I and J be two ordered topological spaces. Suppose that there exists a homeomorphism φ of I onto J such that both φ and φ^{-1} are monotone increasing or monotone decreasing. Let $\circ \in \mathscr{A}_+^0(J)$ and put $a \circ_\varphi b = \varphi^{-1}(\varphi(a) \circ \varphi(b))$ for each $a,b \in I$. Then*

(i) $\circ_\varphi \in \mathscr{A}_+^0(I)$.

(ii) $a_1^{(t_1)\circ_\varphi} \circ_\varphi \cdots \circ_\varphi a_n^{(t_n)\circ_\varphi} = \varphi^{-1}\left(\varphi(a_1)^{(t_1)\circ} \circ \cdots \circ \varphi(a_n)^{(t_n)\circ}\right)$ *holds for all* $a_1, \cdots,$ $a_n \in I$ *and* $t_1, \cdots, t_n > 0$.

Proof. (i) It is obvious that $\circ_\varphi$ is a topological abelian semigroup operation on I. Note that $a^{(n)\circ_\varphi} = \varphi^{-1}((\varphi(a)^{(n)\circ})$ for each $a \in I$ and $n \in \mathbf{N}$. Then $\circ_\varphi$ satisfies the condition $(\sharp_1)$. Moreover we have that $a^{(p)\circ_\varphi} = \varphi^{-1}((\varphi(a)^{(p)\circ})$ for each $a \in I$ and $p \in \mathbf{Q}_+$. Then $\circ_\varphi$ satisfies the condition $(\sharp_2)$. Suppose that both φ and φ^{-1} are monotone increasing. Then

$$\begin{aligned} a \leqslant b &\Leftrightarrow \varphi(a) \leqslant \varphi(b) \\ &\Leftrightarrow \varphi(a) \circ \varphi(c) \leqslant \varphi(b) \circ \varphi(c) \\ &\Leftrightarrow \varphi^{-1}(\varphi(a) \circ \varphi(c)) \leqslant \varphi^{-1}(\varphi(b) \circ \varphi(c)) \\ &\Leftrightarrow a \circ_\varphi c \leqslant b \circ_\varphi c \end{aligned}$$

for all $a,b,c \in I$. Then $\circ_\varphi$ satisfies the condition $(\flat_1)$. Furthermore we have

$$\begin{aligned} a \leqslant b &\Rightarrow \varphi(a) \leqslant \varphi(b) \\ &\Rightarrow \varphi(a)^{(t)\circ} \leqslant \varphi(b)^{(t)\circ} \\ &\Rightarrow \varphi^{-1}(\varphi(a)^{(t)\circ}) \leqslant \varphi^{-1}(\varphi(b)^{(t)\circ}) \\ &\Rightarrow a^{(t)\circ_\varphi} \leqslant b^{(t)\circ_\varphi} \end{aligned}$$

for all $a,b \in I$ and $t \in \mathbf{R}_+$. Then $\circ_\varphi$ satisfies the condition $(\flat_2)$. Consequently we obtain that $\circ_\varphi \in \mathscr{A}_+^0(J)$. If both φ and φ^{-1} are monotone decreasing, then we obtain the same result by using the same method above.

(ii) Let $a_1, \cdots, a_n, t_1, \cdots, t_n > 0$. Then we have

$$\begin{aligned} &a_1^{(t_1)\circ_\varphi} \circ_\varphi \cdots \circ_\varphi a_n^{(t_n)\circ_\varphi} \\ &\quad = \varphi^{-1}\left(\varphi(a_1)^{(t_1)\circ}\right) \circ_\varphi \cdots \circ_\varphi \varphi^{-1}\left(\varphi(a_n)^{(t_n)\circ}\right) \\ &\quad = \varphi^{-1}\left(\varphi(a_1)^{(t_1)\circ} \circ \varphi(a_2)^{(t_2)\circ}\right) \circ_\varphi \cdots \circ_\varphi \varphi^{-1}\left(\varphi(a_n)^{(t_n)\circ}\right) \\ &\qquad \vdots \\ &\quad = \varphi^{-1}\left(\varphi(a_1)^{(t_1)\circ} \circ \cdots \circ \varphi(a_n)^{(t_n)\circ}\right), \end{aligned}$$

and hence the desired equality holds. □

LEMMA 2. *Let $f_1,\cdots,f_n$ be differentiable positive-valued functions on a real interval I and $t_1,\cdots,t_n>0$. Then*

$$\frac{d}{dx}\prod_{i=1}^{n} f_i(x)^{t_i} = \prod_{i=1}^{n} f_i(x)^{t_i}\sum_{i=1}^{n}\frac{t_i f_i'(x)}{f_i(x)}$$

holds for all $x\in I$.

Proof. Straightforward. □

Now note that $\mathbf{R}$ is an ordered topological space with the ordinary order and the ordinary topology. Let $+$ be the ordinary additive operation on $\mathbf{R}$. Then it is obvious that $+\in\mathscr{A}_+^0(\mathbf{R})$.

Proof of Theorem 1. Take $t\geqslant 0$ arbitrarily and put $I_t=\{x\in\mathbf{R}:x>-t\}$. Then I_t is an ordered topological spaces with the ordinary order and the ordinary topology. Consider the following function

$$\varphi_t(x)=\log(x+t)\ \ (x\in I_t).$$

Then φ_t is a homeomorphism of I_t onto $\mathbf{R}$. Moreover both φ_t and φ_t^{-1} are strictly monotone increasing, and hence $(+)_{\varphi_t}\in\mathscr{A}_+^0(I_t)$ by Lemma 1-(i). For the sake of simplicity, let $\circ_t=(+)_{\varphi_t}$. Since $\varphi_t^{-1}(y)=e^y-t$ for all $y\in\mathbf{R}$, we have from simple computation that

$$a\circ_t b=(a+t)(b+t)-t$$

for all $a,b\in I_t$.

Now let $a_1,\cdots,a_n\in I_t$ and $t_1,\cdots,t_n>0$. Then we have from Lemma 1-(ii) that

$$\begin{aligned} a_1^{(t_1)_{\circ_t}}\circ\cdots\circ a_n^{(t_n)_{\circ_t}} &= \varphi_t^{-1}(t_1\varphi_t(a_1)+\cdots+t_n\varphi_t(a_n)) \\ &= \prod_{i=1}^{n}(a_i+t)^{t_i}-t=(GA)_t. \end{aligned}$$

Assume that $0\leqslant t<s$, and so $I_t\subset I_s$. Let $\imath:I_t\to I_s$ be the identity mapping. Then $\imath$ is $(\circ_t,\circ_s)$-convex. Indeed,

$$\begin{aligned} \imath M_{\circ_t}(a,b) &= M_{\circ_t}(a,b)=(a\circ_t b)^{(1/2)_{\circ_t}} \\ &= \sqrt{(a+t)(b+t)}-t \\ &\leqslant \sqrt{(a+s)(b+s)}-s \\ &= M_{\circ_s}(a,b)=M_{\circ_s}(\imath a,\imath b) \end{aligned}$$

holds for all $a,b\in I_t$. Therefore if $t_1+\cdots+t_n=1$, then we have from Theorem A that $(GA)_t\leqslant(GA)_s$. However, $(GA)_t<(GA)_s$ holds. Indeed, if $(GA)_t=(GA)_s$, then

$\prod_{i=1}^{n}(a_i+x)^{t_i}=x+c$ must hold for all $x\in\mathbf{R}$ with $t<x<s$ and a constant $c\in\mathbf{R}$. This is a contradiction.

We next show that $\lim_{t\to\infty}(GA)_t=\sum_{i=1}^{n}t_ia_i$. To do this, put

$$p(t)=\prod_{i=1}^{n-1}\left(\frac{a_i+t}{a_n+t}\right)^{t_i}$$

for each $t>0$. Then we have that $(GA)_t=a_np(t)+(p(t)-1)t\ \ (t>0)$. Obviously, $\lim_{t\to\infty}p(t)=1$. Also we have from Lemma 2 that

$$\begin{aligned}\lim_{t\to\infty}(p(t)-1)t&=\lim_{x\downarrow 0}\frac{p(1/x)-1}{x}\\&=\lim_{x\downarrow 0}\frac{d}{dx}\prod_{i=1}^{n-1}\left(\frac{1+a_ix}{1+a_nx}\right)^{t_i}\\&=\lim_{x\downarrow 0}\prod_{i=1}^{n-1}\left(\frac{1+a_ix}{1+a_nx}\right)^{t_i}\sum_{i=1}^{n-1}\frac{t_i(a_i-a_n)}{(1+a_ix)(1+a_nx)}\\&=t_1a_1+\cdots+t_{n-1}a_{n-1}-(t_1+\cdots+t_{n-1})a_n.\end{aligned}$$

Therefore

$$\lim_{t\to\infty}(GA)_t=a_n+t_1a_1+\cdots+t_{n-1}a_{n-1}-(t_1+\cdots+t_{n-1})a_n=\sum_{i=1}^{n}t_ia_i,$$

as required. Thus we obtain the desired result. □

Proof of Theorem 2. Let $0<t\leqslant 1$ and put $I_t=\{x\in\mathbf{R}:0<x<1/t\}$. Then I_t is an ordered topological spaces with the ordinary order and the ordinary topology. Consider the following function

$$\varphi_t(x)=\tanh^{-1}(tx)\ \ (x\in I_t)$$

Then φ_t is a homeomorphism of I_t onto $\mathbf{R}_+$. Moreover both φ_t and φ_t^{-1} are strictly monotone increasing, and hence $(+)_{\varphi_t}\in\mathscr{A}_+^0(I_t)$ by Lemma 1-(i). For the sake of simplicity, let $\circ_t=(+)_{\varphi_t}$.

Now let $0<a_1,\cdots,a_n,t_1,\cdots,t_n<1$ with $t_1+\cdots+t_n=1$. Then $a_1,\cdots,a_n\in I_t$, and hence we have from Lemma 1-(ii) that

$$a_1^{(t_1)\circ_t}\circ_t\cdots\circ_t a_n^{(t_n)\circ_t}=\frac{1}{t}\tanh\left(t_1\tanh^{-1}(ta_1)+\cdots+t_n\tanh^{-1}(ta_n)\right).$$

Since $\tanh^{-1}(x)=\log\sqrt{\frac{1+x}{1-x}}\ \ (0<x<1)$, it follows that

$$t_1\tanh^{-1}(ta_1)+\cdots+t_n\tanh^{-1}(ta_n)=\log\prod_{i=1}^{n}\left(\frac{1+ta_i}{1-ta_i}\right)^{t_i/2}.$$

Therefore we have

$$\begin{aligned} a_1^{(t_1)_{\circ_t}} \circ_t \cdots \circ_t a_n^{(t_n)_{\circ_t}} &= \frac{1}{t} \tanh \log \prod_{i=1}^{n} \left(\frac{1+ta_i}{1-ta_i} \right)^{t_i/2} \\ &= \frac{1}{t} \frac{\prod_{i=1}^n \left(\frac{1+ta_i}{1-ta_i}\right)^{t_i/2} - \prod_{i=1}^n \left(\frac{1+ta_i}{1-ta_i}\right)^{-t_i/2}}{\prod_{i=1}^n \left(\frac{1+ta_i}{1-ta_i}\right)^{t_i/2} + \prod_{i=1}^n \left(\frac{1+ta_i}{1-ta_i}\right)^{-t_i/2}} \\ &= \frac{1}{t} \frac{\prod_{i=1}^n \left(\frac{1+ta_i}{1-ta_i}\right)^{t_i} - 1}{\prod_{i=1}^n \left(\frac{1+ta_i}{1-ta_i}\right)^{t_i} + 1} \\ &= (AP)_t. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} a \circ_t b &= \frac{1}{t} \tanh \left(\tanh^{-1}(ta) + \tanh^{-1}(tb) \right) \\ &= \frac{1}{t} \tanh \log \sqrt{\frac{(1+ta)(1+tb)}{(1-ta)(1-tb)}} \\ &= \frac{1}{t} \frac{\sqrt{\frac{(1+ta)(1+tb)}{(1-ta)(1-tb)}} - \sqrt{\frac{(1+ta)(1+tb)}{(1-ta)(1-tb)}}^{-1}}{\sqrt{\frac{(1+ta)(1+tb)}{(1-ta)(1-tb)}} + \sqrt{\frac{(1+ta)(1+tb)}{(1-ta)(1-tb)}}^{-1}} \\ &= \frac{1}{t} \frac{ta+tb}{1+(ta)(tb)} \\ &= \frac{a+b}{1+t^2ab} \ (t > 0, a, b \in I_t). \end{aligned}$$

Let $a, b \in I_t$. Then the solution of the equation $x \circ_t x = a \circ_t b, x \in I_t$ is given by

$$M_{\circ_t}(a,b) = \frac{1+t^2ab - \sqrt{(1-t^2a^2)(1-t^2b^2)}}{t^2(a+b)}.$$

Put $f(x) = M_{\circ_x}(a,b)$ for each $x \in \mathbf{R}$ with $0 < x < t$. By simple computation, we have

$$f'(x) = \frac{2x^{-2} - a^2 - b^2 - 2\sqrt{(x^{-2}-a^2)(x^{-2}-b^2)}}{(a+b)x^3\sqrt{(x^{-2}-a^2)(x^{-2}-b^2)}} \geqslant 0 \ \ (0 < x < t).$$

Then $M_{\circ_t}(a,b) \geqslant M_{\circ_s}(a,b)$ holds when $0 < s < t$. This means that the identity mapping from I_t to I_s is $(\circ_t, \circ_s)$-concave when $0 < s < t$. Therefore it follows from Theorem A that

$$(AP)_t = a_1^{(t_1)_{\circ_t}} \circ_t \cdots \circ_t a_n^{(t_n)_{\circ_t}} \geqslant a_1^{(t_1)_{\circ_s}} \circ_s \cdots \circ_s a_n^{(t_n)_{\circ_s}} = (AP)_s \ (0 < s < t)$$

holds. However, $(AP)_t < (AP)_s$ $(0 < s < t)$ holds. Indeed, if $(AP)_t = (AP)_s$ and $0 < s < t$, then

$$\frac{\prod_{i=1}^n (1+xa_i)^{t_i} - \prod_{i=1}^n (1-xa_i)^{t_i}}{\prod_{i=1}^n (1+xa_i)^{t_i} + \prod_{i=1}^n (1-xa_i)^{t_i}} = cx$$

must hold for all $x \in \mathbf{R}$ with $s < x < t$ and a constant $c \in \mathbf{R}$. This is a contraction. Also we have from Lemma 2 that

$$\begin{aligned}
\lim_{t \downarrow 0} (AP)_t &= \frac{1}{2} \lim_{t \downarrow 0} \frac{d}{dt} \left(\prod_{i=1}^n (1+a_i t)^{t_i} - \prod_{i=1}^n (1-a_i t)^{t_i} \right) \\
&= \frac{1}{2} \lim_{t \downarrow 0} \left(\prod_{i=1}^n (1+a_i t)^{t_i} \sum_{i=1}^n \frac{t_i a_i}{1+a_i t} + \prod_{i=1}^n (1-a_i t)^{t_i} \sum_{i=1}^n \frac{t_i a_i}{1-a_i t} \right) \\
&= \sum_{i=1}^n t_i a_i.
\end{aligned}$$

Thus we obtain the desired result. □

REMARK 4. We can show that $\sum_{i=1}^n t_i a_i \leqslant (AP)_t$ $(0 < t \leqslant 1)$ in other ways. Indeed,

$$M_{\circ_t}(a,b) - M_+(a,b) = \frac{2 - t^2 a^2 - t^2 b^2 - 2\sqrt{(1-t^2a^2)(1-t^2b^2)}}{2t^2(a+b)} \geqslant 0$$

holds for all $a, b \in I_t$. Then the identity mapping from I_t to $\mathbf{R}_+$ is $(\circ_t, +)$-concave, and hence

$$(AP)_t = a_1^{(t_1)_{\circ_t}} \circ_t \cdots \circ_t a_n^{(t_n)_{\circ_t}} \geqslant a_1^{(t_1)_+} + \cdots + a_n^{(t_n)_+} = \sum_{i=1}^n t_i a_i$$

holds by Theorem A.

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