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PACHPATTE'S TYPE INTEGRAL INEQUALITIES WITH INTEGRAL IMPULSES

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Abstract. The purpose of the present paper is to establish some new Pachpatte's type integral inequalities with integral impulses, which provide explicit bounds on unknown functions and extend some results of Pachpatte's inequalities. These inequalities can be used as basic tools to investigate the qualitative properties of certain impulsive differential equations and impulsive integral equations.

1. Introduction

Differential and integral inequalities play a fundamental role in the study of qualitative properties of solutions of differential equations, integral equations and integro-differential equations, such as existence, uniqueness, boundedness, stability, asymptotic behavior, oscillation etc. (see [1]–[13]). Pachpatte's book [14] gave many results on linear and nonlinear integral inequalities and their applications. For example of linear case, if u, f and g are nonnegative continuous functions on $\mathbb{R}_+$ satisfying

$$u(t) \leqslant u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g(\sigma)u(\sigma)d\sigma\right)ds, \qquad t \in \mathbb{R}_+, \tag{1}$$

where u_0 is a nonnegative constant, then

$$u(t) \leqslant u_0\left[1 + \int_0^t f(s)\exp\left(\int_0^s [f(\sigma)+g(\sigma)]d\sigma\right)ds\right]. \tag{2}$$

Impulsive equations and inequalities appear as a description of many real world phenomena which have a short-term rapid change of their states at certain moments, see [15]–[18]. For some details of this subject, we need to introduce the following notations. Assume that $0 \leqslant t_0 < t_1 < t_2 < \cdots$, $\lim_{k\to\infty} t_k = \infty$, $\mathbb{R}_+ = [0,+\infty)$ and $I \subset \mathbb{R}$. Define $PC(\mathbb{R}_+, I) = \{u : \mathbb{R}_+ \to I; u(t)$ is continuous for $t \neq t_k$, and $u(0^+)$, $u(t_k^-)$ and $u(t_k^+)$ exist, and $u(t_k^-) = u(t_k)$, $k = 1,2,\ldots\}$ and $PC^1(\mathbb{R}_+, I) = \{u \in PC(\mathbb{R}_+, I) : u'(t)$ is continuous everywhere for $t \neq t_k$, and $u'(0^+)$, $u'(t_k^+)$ and $u'(t_k^-)$ exist, and $u'(t_k^-) =$

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$u'(t_k)$, $k=1,2,\ldots\}$. For some recent works on impulsive inequalities, we refer the reader to the papers [19]–[25].

The aim of this article is to extend some results of Pachpatte's integral inequalities in [14] by including integral impulses effects in that one. So, for $t\in\mathbb{R}_+$, inequality (1) with integral impulses can be written as

$$u(t)\leqslant u_0+\int_0^t f(s)u(s)ds+\int_0^t f(s)\left(\int_0^s g(\sigma)u(\sigma)d\sigma\right)ds$$
$$+\sum_{0<t_k<t}\beta_k\int_{t_k-\tau_k}^{t_k-\theta_k}u(s)ds, \tag{3}$$

where $\beta_k\geqslant 0$, $0\leqslant\theta_k\leqslant\tau_k\leqslant t_k-t_{k-1}$, $k=1,2,3,\ldots$. Theorems 1.7.1–1.7.5 in [14] are modified by our results. To prove our main results, we need the following theorem [24].

THEOREM 1. *Assume that:*

(H_0) *the sequence* $\{t_k\}$ *satisfies* $0\leqslant t_0<t_1<t_2<\cdots$, $\lim_{k\to\infty}t_k=\infty$*;*

(H_1) $u\in PC^1[\mathbb{R}_+,\mathbb{R}_+]$ *and* $u(t)$ *is left-continuous at* t_k*,* $k=1,2,\ldots$*;*

(H_2) *for* $k=1,2,\ldots$*,* $t\geqslant t_0$*,*

$$\begin{aligned}&u'(t)\leqslant p(t)u(t)+q(t),\quad t\neq t_k,\\&u(t_k^+)\leqslant d_ku(t_k)+c_k\textstyle\int_{t_k-\tau_k}^{t_k-\theta_k}u(s)ds+b_k,\end{aligned}\tag{4}$$

where $q,p\in C[\mathbb{R}_+,\mathbb{R}]$*,* $c_k,d_k\geqslant 0$*,* $0\leqslant\theta_k\leqslant\tau_k\leqslant t_k-t_{k-1}$ *and* b_k *are constants.*

Then,

$$u(t)\leqslant\left\{u(t_0)\prod_{t_0<t_k<t}A_k+\sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t}A_jB_k\right)\right\}e^{\int_{t_\alpha}^t p(\xi)d\xi}$$
$$+\int_{t_\alpha}^t q(s)e^{\int_s^t p(\xi)d\xi}ds,\quad t\geqslant t_0,$$

where

$$A_k=d_ke^{\int_{t_{k-1}}^{t_k}p(\xi)d\xi}+c_k\int_{t_k-\tau_k}^{t_k-\theta_k}e^{\int_{t_{k-1}}^{s}p(\xi)d\xi}ds,$$
$$B_k=d_k\int_{t_{k-1}}^{t_k}q(s)e^{\int_s^{t_k}p(\xi)d\xi}ds+c_k\int_{t_k-\tau_k}^{t_k-\theta_k}\int_{t_{k-1}}^{s}q(r)e^{\int_r^s p(\xi)d\xi}drds+b_k,$$

and $\alpha=\max\{k:\ t\geqslant t_k,k=1,2,3,\ldots\}$.

In the last section, we present an example to illustrate the advantage of our results.

2. Main results

THEOREM 2. *Let conditions* (H_0)–(H_1) *hold and* f, g *be nonnegative continuous functions defined on* $\mathbb{R}_+$. *If*

$$u(t) \leqslant u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g(\sigma)u(\sigma)d\sigma\right)ds + \sum_{0<t_k<t} \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s)ds, \quad t \geqslant 0, \tag{5}$$

holds, where $\beta_k \geqslant 0$ *and* $0 \leqslant \theta_k \leqslant \tau_k \leqslant t_k - t_{k-1}$ *for* $k = 1,2,\ldots$ *and* u_0 *is a nonnegative constant. Then*

$$u(t) \leqslant u_0 \prod_{0<t_k<t} C_k \exp\left(\int_{t_\alpha}^t f(s)\left[1+\int_0^s g(\sigma)d\sigma\right]ds\right), \quad t \geqslant 0, \tag{6}$$

where

$$C_k = \exp\left(\int_{t_{k-1}}^{t_k} f(s)\left[1+\int_0^s g(\sigma)d\sigma\right]ds\right) + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} \exp\left(\int_{t_{k-1}}^{s} f(\sigma)\left[1+\int_0^\sigma g(\xi)d\xi\right]d\sigma\right)ds. \tag{7}$$

Proof. Firstly, we define a function $v(t)$ by the right-hand side of (5). Note that the function $v(t)$ is nondecreasing, $u(t) \leqslant v(t)$ and $v(0) = u_0$. Then, for $t \neq t_k$, we have

$$\begin{aligned} v'(t) &= f(t)u(t) + f(t)\int_0^t g(\sigma)u(\sigma)d\sigma \\ &\leqslant f(t)\left(v(t) + \int_0^t g(\sigma)v(\sigma)d\sigma\right) \\ &\leqslant f(t)\left(1+\int_0^t g(\sigma)d\sigma\right)v(t). \end{aligned} \tag{8}$$

For $t = t_k^+$, we get

$$\begin{aligned} v(t_k^+) &= u_0 + \int_0^{t_k^+} f(s)u(s)ds + \int_0^{t_k^+} f(s)\left(\int_0^s g(\sigma)u(\sigma)d\sigma\right)ds \\ &\quad + \sum_{0<t_n<t_k^+} \beta_n \int_{t_n-\tau_n}^{t_n-\theta_n} u(s)ds \\ &= u_0 + \int_0^{t_k} f(s)u(s)ds + \int_0^{t_k} f(s)\left(\int_0^s g(\sigma)u(\sigma)d\sigma\right)ds \\ &\quad + \sum_{0<t_n<t_k} \beta_n \int_{t_n-\tau_n}^{t_n-\theta_n} u(s)ds + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s)ds \end{aligned}$$

$$
\begin{aligned}
&= v(t_k) + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s)ds \\
&\leqslant v(t_k) + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} v(s)ds. \qquad (9)
\end{aligned}
$$

Applying Theorem 1 for (8) and (9), we obtain

$$
v(t) \leqslant v(0) \prod_{0<t_k<t} C_k \exp\left(\int_{t_\alpha}^{t} f(s) \left[1 + \int_0^s g(\sigma)d\sigma \right] ds \right), \quad t \geqslant 0,
$$

which results in (6). □

THEOREM 3. *Let conditions* (H_0)–(H_1) *hold and* f, g, h, p *be nonnegative continuous functions defined on* $\mathbb{R}_+$, *constants* $\beta_k \geqslant 0$ *and* $0 \leqslant \theta_k \leqslant \tau_k \leqslant t_k - t_{k-1}$ *for* $k = 1, 2, \ldots$ *and* u_0 *is a nonnegative constant.*

(i) *If*

$$
\begin{aligned}
u(t) \leqslant u_0 &+ \int_0^t (f(s)u(s) + p(s))\, ds + \int_0^t f(s) \left(\int_0^s g(\sigma)u(\sigma)d\sigma \right) ds \\
&+ \sum_{0<t_k<t} \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s)ds, \quad t \geqslant 0, \qquad (10)
\end{aligned}
$$

then

$$
\begin{aligned}
u(t) \leqslant & \left\{ u_0 \prod_{0<t_k<t} C_k + \sum_{0<t_k<t} \left(\prod_{t_k<t_j<t} C_j D_k \right) \right\} \\
& \times \exp\left(\int_{t_\alpha}^{t} f(s) \left[1 + \int_0^s g(\sigma)d\sigma \right] ds \right) \\
& + \int_{t_\alpha}^{t} p(s) \exp\left(\int_s^t f(\sigma) \left[1 + \int_0^\sigma g(\xi)d\xi \right] d\sigma \right) ds, \quad t \geqslant 0, \qquad (11)
\end{aligned}
$$

where C_k *is defined by* (7) *and*

$$
\begin{aligned}
D_k = & \int_{t_{k-1}}^{t} p(s) \exp\left(\int_s^{t_k} f(\sigma) \left[1 + \int_0^\sigma g(\xi)d\xi \right] d\sigma \right) ds \\
& + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} \int_{t_{k-1}}^{s} p(r) \exp\left(\int_r^s f(\sigma) \left[1 + \int_0^\sigma g(\xi)d\xi \right] d\sigma \right) drds. \qquad (12)
\end{aligned}
$$

(ii) *If*

$$
\begin{aligned}
u(t) \leqslant u_0 &+ \int_0^t f(s)u(s)ds + \int_0^t f(s) \left(\int_0^s [g(\sigma)u(\sigma) + p(\sigma)]d\sigma \right) ds \\
&+ \sum_{0<t_k<t} \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s)ds, \quad t \geqslant 0, \qquad (13)
\end{aligned}
$$

then

$$u(t) \leqslant \left\{ u_0 \prod_{0<t_k<t} C_k + \sum_{0<t_k<t} \left(\prod_{t_k<t_j<t} C_j E_k \right) \right\} \times \exp\left(\int_{t_\alpha}^{t} f(s) \left[1 + \int_0^s g(\sigma) d\sigma \right] ds \right) + \int_{t_\alpha}^{t} f(s) \left(\int_0^s p(\sigma) d\sigma \right) \times \exp\left(\int_s^t f(\sigma) \left[1 + \int_0^\sigma g(\xi) d\xi \right] d\sigma \right) ds, \quad t \geqslant 0, \tag{14}$$

where C_k *is defined by* (7) *and*

$$E_k = \int_{t_{k-1}}^{t} f(s) \left(\int_0^s p(\sigma) d\sigma \right) \times \exp\left(\int_s^{t_k} f(\sigma) \left[1 + \int_0^\sigma g(\xi) d\xi \right] d\sigma \right) ds + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} \int_{t_{k-1}}^{s} f(r) \left(\int_0^r p(\sigma) d\sigma \right) \times \exp\left(\int_r^s f(\sigma) \left[1 + \int_0^\sigma g(\xi) d\xi \right] d\sigma \right) dr ds. \tag{15}$$

(*iii*) *If*

$$u(t) \leqslant u_0 + \int_0^t f(s) u(s) ds + \int_0^t g(s) \left(u(s) + \int_0^s h(\sigma) u(\sigma) d\sigma \right) ds + \sum_{0<t_k<t} \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s) ds, \quad t \geqslant 0, \tag{16}$$

then for $t \geqslant 0$,

$$u(t) \leqslant u_0 \prod_{0<t_k<t} F_k \exp\left\{ \int_{t_\alpha}^{t} \left(f(s) + g(s) \left[1 + \int_0^s h(\sigma) d\sigma \right] \right) ds \right\}, \tag{17}$$

where

$$F_k = \exp\left\{ \int_{t_{k-1}}^{t_k} \left(f(s) + g(s) \left[1 + \int_0^s h(\sigma) d\sigma \right] \right) ds \right\} + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} \exp\left\{ \int_{t_{k-1}}^{s} \left(f(r) + g(r) \left[1 + \int_0^r h(\sigma) d\sigma \right] \right) dr \right\} ds. \tag{18}$$

(*iv*) *If*

$$u(t) \leqslant h(t) + p(t) \left\{ \int_0^t f(s) u(s) ds + \int_0^t f(s) p(s) \left(\int_0^s g(\sigma) u(\sigma) d\sigma \right) ds \right\} + \sum_{0<t_k<t} \beta_k p(t) \int_{t_k-\tau_k}^{t_k-\theta_k} u(s) ds, \quad t \geqslant 0, \tag{19}$$

then

$$\begin{aligned} u(t) \leqslant h(t) + p(t) \Bigg\{ & \Bigg(\sum_{0<t_k<t} \Bigg[\prod_{t_k<t_j<t} G_j H_k \Bigg] \Bigg) \\ & \times \exp \left(\int_{t_\alpha}^{t} f(s)p(s) \left[1 + \int_0^s g(\sigma)p(\sigma)d\sigma \right] ds \right) \\ & + \int_{t_\alpha}^{t} f(s) \left(h(s) + p(s) \int_0^s g(\sigma)h(\sigma)d\sigma \right) \\ & \times \exp \left(\int_s^t f(\sigma)p(\sigma) \left[1 + \int_0^\sigma g(\xi)p(\xi)d\xi \right] d\sigma \right) ds \Bigg\}, \; t \geqslant 0, \end{aligned} \tag{20}$$

where

$$G_k = \left(1 + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} p(s)ds \right) \exp \left(\int_{t_{k-1}}^{t_k} f(s)p(s) \left[1 + \int_0^s g(\sigma)p(\sigma)d\sigma \right] ds \right), \tag{21}$$

$$\begin{aligned} H_k = & \left(1 + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} p(s)ds \right) \int_{t_{k-1}}^{t_k} f(s) \left(h(s) + p(s) \int_0^s g(\sigma)h(\sigma)d\sigma \right) \\ & \times \exp \left(\int_s^{t_k} f(\sigma)p(\sigma) \left[1 + \int_0^\sigma g(\xi)p(\xi)d\xi \right] d\sigma \right) ds + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} h(s)ds. \end{aligned} \tag{22}$$

Proof. (i) We define a function $v(t)$ as the right-hand side of (10). Hence, the function $v(t)$ is nondecreasing, $u(t) \leqslant v(t)$ on $\mathbb{R}_+$ and $v(0) = u_0$. Then we get that

$$\begin{aligned} v'(t) &= f(t)u(t) + p(t) + f(t) \int_0^t g(\sigma)u(\sigma)d\sigma \\ &\leqslant f(t)v(t) + p(t) + f(t) \int_0^t g(\sigma)v(\sigma)d\sigma \\ &\leqslant p(t) + f(t) \left(1 + \int_0^t g(\sigma)d\sigma \right) v(t), \quad t \neq t_k. \end{aligned} \tag{23}$$

For $t = t_k^+$, we obtain

$$\begin{aligned} v(t_k^+) = u_0 &+ \int_0^{t_k^+} (f(s)u(s) + p(s))\, ds + \int_0^{t_k^+} f(s) \left(\int_0^s g(\sigma)u(\sigma)d\sigma \right) ds \\ &+ \sum_{0<t_n<t_k^+} \beta_n \int_{t_n-\tau_n}^{t_n-\theta_n} u(s)ds \\ = u_0 &+ \int_0^{t_k} (f(s)u(s) + p(s))\, ds + \int_0^{t_k} f(s) \left(\int_0^s g(\sigma)u(\sigma)d\sigma \right) ds \\ &+ \sum_{0<t_n<t_k} \beta_n \int_{t_n-\tau_n}^{t_n-\theta_n} u(s)ds + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s)ds \end{aligned}$$

$$
\begin{aligned}
&= v(t_k) + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s)ds \\
&\leqslant v(t_k) + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} v(s)ds. \qquad (24)
\end{aligned}
$$

Using Theorem 1 for (23) and (24), we get

$$
\begin{aligned}
v(t) \leqslant &\left\{ v(0) \prod_{0<t_k<t} C_k + \sum_{0<t_k<t} \left(\prod_{t_k<t_j<t} C_j D_k \right) \right\} \exp\left(\int_{t_\alpha}^{t} f(s) \left[1 + \int_0^s g(\sigma)d\sigma \right] ds \right) \\
&+ \int_{t_\alpha}^{t} p(s) \exp\left(\int_s^t f(\sigma) \left[1 + \int_0^\sigma g(\xi)d\xi \right] d\sigma \right) ds, \quad t \geqslant 0.
\end{aligned}
$$

Now by using $u(t) \leqslant v(t)$ and $v(0) = u_0$, we get the required inequality in (11).

The proofs of (ii)–(iii) are similar to that of (i) and therefore omitted.

(iv) We define a function $v(t)$ by

$$
v(t) = \int_0^t f(s)u(s)ds + \int_0^t f(s)p(s) \left(\int_0^s g(\sigma)u(\sigma)d\sigma \right) ds + \sum_{0<t_k<t} \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s)ds,
$$

then the function $v(t)$ is nondecreasing, $v(0) = 0$ and $u(t) \leqslant h(t) + p(t)v(t)$. Thus, for $t \neq t_k$, we get

$$
\begin{aligned}
v'(t) &= f(t)u(t) + f(t)p(t) \int_0^t g(\sigma)u(\sigma)d\sigma \\
&\leqslant f(t)\left[h(t) + p(t)v(t)\right] + f(t)p(t) \int_0^t g(\sigma)\left[h(\sigma) + p(\sigma)v(\sigma)\right] d\sigma \\
&= f(t) \left[h(t) + p(t) \int_0^t g(\sigma)h(\sigma)d\sigma \right] \\
&\quad + f(t)p(t) \left[v(t) + \int_0^t g(\sigma)p(\sigma)v(\sigma)d\sigma \right] \\
&\leqslant f(t) \left[h(t) + p(t) \int_0^t g(\sigma)h(\sigma)d\sigma \right] \\
&\quad + f(t)p(t) \left[1 + \int_0^t g(\sigma)p(\sigma)d\sigma \right] v(t). \qquad (25)
\end{aligned}
$$

For $t = t_k^+$, we obtain

$$
\begin{aligned}
v(t_k^+) = &\int_0^{t_k^+} f(s)u(s)ds + \int_0^{t_k^+} f(s)p(s) \left(\int_0^s g(\sigma)u(\sigma)d\sigma \right) ds \\
&+ \sum_{0<t_n<t_k^+} \beta_n \int_{t_n-\tau_n}^{t_n-\theta_n} u(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{t_k} f(s)u(s)ds + \int_0^{t_k} f(s)p(s)\left(\int_0^s g(\sigma)u(\sigma)d\sigma\right)ds \\
&\quad + \sum_{0<t_n<t_k} \beta_n \int_{t_n-\tau_n}^{t_n-\theta_n} u(s)ds + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s)ds \\
&= v(t_k) + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s)ds \\
&\leqslant v(t_k) + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} [h(s)+p(s)v(s)]\,ds \\
&\leqslant v(t_k) + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} h(s)ds + \beta_k v(t_k-\theta_k) \int_{t_k-\tau_k}^{t_k-\theta_k} p(s)ds \\
&\leqslant \left(1+\beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} p(s)ds\right) v(t_k) + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} h(s)ds. \qquad (26)
\end{aligned}
$$

From (25)–(26) and Theorem 1, we have

$$
\begin{aligned}
v(t) &\leqslant \left(\sum_{0<t_k<t}\left[\prod_{t_k<t_j<t} G_j H_k\right]\right) \exp\left(\int_{t_\alpha}^t f(s)p(s)\left[1+\int_0^s g(\sigma)p(\sigma)d\sigma\right]ds\right) \\
&\quad + \int_{t_\alpha}^t f(s)\left(h(s)+p(s)\int_0^s g(\sigma)h(\sigma)d\sigma\right) \\
&\quad \times \exp\left(\int_s^t f(\sigma)p(\sigma)\left[1+\int_0^\sigma g(\xi)p(\xi)d\xi\right]d\sigma\right)ds, \quad t \geqslant 0, \qquad (27)
\end{aligned}
$$

which results in (20). This completes the proof. □

THEOREM 4. *Let conditions* (H_0)–(H_1) *hold. Assume that* q, f, g, h, p *are nonnegative continuous functions defined on* $\mathbb{R}_+$, *and constants* $\beta_k \geqslant 0$, $0 \leqslant \theta_k \leqslant \tau_k \leqslant t_k - t_{k-1}$, *for* $k = 1,2,\ldots$, *and* u_0 *is a nonnegative constant.*

(*i*) *If*

$$
\begin{aligned}
u(t) &\leqslant u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g(\sigma)u(\sigma)d\sigma\right)ds \\
&\quad + \int_0^t f(s)\left[\int_0^s g(\sigma)\left(\int_0^\sigma h(\xi)u(\xi)d\xi\right)d\sigma\right]ds \\
&\quad + \sum_{0<t_k<t} \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s)ds, \quad t \geqslant 0, \qquad (28)
\end{aligned}
$$

then

$$
\begin{aligned}
u(t) &\leqslant u_0 \prod_{0<t_k<t} I_k \exp\left(\int_{t_\alpha}^t f(s)\left\{1+\int_0^s g(\sigma)\right.\right. \\
&\qquad \left.\left. \times\left[1+\int_0^\sigma h(\xi)d\xi\right]d\sigma\right\}ds\right), \quad t \geqslant 0, \qquad (29)
\end{aligned}
$$

where

$$I_k = \exp\left(\int_{t_{k-1}}^{t_k} f(s)\left\{1+\int_0^s g(\sigma)\left[1+\int_0^\sigma h(\xi)d\xi\right]d\sigma\right\}ds\right)$$
$$+\beta_k\int_{t_k-\tau_k}^{t_k-\theta_k}\exp\left(\int_{t_{k-1}}^s f(r)\left\{1+\int_0^r g(\sigma)\left[1+\int_0^\sigma h(\xi)d\xi\right]d\sigma\right\}dr\right)ds. \quad (30)$$

(*ii*) *If*

$$u(t)\leqslant q(t)+p(t)\left[\int_0^t f(s)u(s)ds+\int_0^t f(s)p(s)\left(\int_0^s g(\sigma)u(\sigma)d\sigma\right)ds\right.$$
$$\left.+\int_0^t f(s)p(s)\left[\int_0^s g(\sigma)p(\sigma)\left(\int_0^\sigma h(\xi)u(\xi)d\xi\right)d\sigma\right]ds\right]$$
$$+\sum_{0<t_k<t}\beta_k p(t)\int_{t_k-\tau_k}^{t_k-\theta_k}u(s)ds, \quad t\geqslant 0, \quad (31)$$

then

$$u(t)\leqslant q(t)+p(t)\left\{\sum_{0<t_k<t}\left(\prod_{t_k<t_j<t}L_jM_k\right)\right.$$
$$\times\exp\left(\int_{t_\alpha}^t f(s)p(s)\left[1+\int_0^s g(\sigma)p(\sigma)\left\{1+\int_0^\sigma h(\xi)p(\xi)d\xi\right\}d\sigma\right]ds\right)$$
$$+\int_{t_\alpha}^t f(s)\left[q(s)+p(s)\int_0^s g(\sigma)\left\{q(\sigma)+p(\sigma)\int_0^\sigma h(\xi)q(\xi)d\xi\right\}d\sigma\right]$$
$$\times\exp\left(\int_s^t f(r)p(r)\left[1+\int_0^r g(\sigma)p(\sigma)\right.\right.$$
$$\left.\left.\times\left\{1+\int_0^\sigma h(\xi)p(\xi)d\xi\right\}d\sigma\right]dr\right)ds, \quad t\geqslant 0, \quad (32)$$

where

$$L_k=\left(1+\beta_k\int_{t_k-\tau_k}^{t_k-\theta_k}p(s)ds\right)$$
$$\times\exp\left(\int_{t_{k-1}}^{t_k} f(s)p(s)\left[1+\int_0^s g(\sigma)p(\sigma)\left\{1+\int_0^\sigma h(\xi)p(\xi)d\xi\right\}d\sigma\right]ds\right), \quad (33)$$

$$M_k=\left(1+\beta_k\int_{t_k-\tau_k}^{t_k-\theta_k}p(s)ds\right)$$
$$\times\int_{t_{k-1}}^{t_k} f(s)\left[q(s)+p(s)\int_0^s g(\sigma)\left\{q(\sigma)+p(\sigma)\int_0^\sigma h(\xi)q(\xi)d\xi\right\}d\sigma\right]$$
$$\times\exp\left(\int_s^{t_k} f(r)p(r)\left[1+\int_0^r g(\sigma)p(\sigma)\left\{1+\int_0^\sigma h(\xi)p(\xi)d\xi\right\}d\sigma\right]dr\right)ds$$
$$+\beta_k\int_{t_k-\tau_k}^{t_k-\theta_k}q(s)ds. \quad (34)$$

Proof. The proof of this theorem follows the same arguments as that of inequalities in Theorems 2 and 3. Therefore, we omitted it. □

THEOREM 5. *Let conditions* (H_0)–(H_1) *hold. Assume that* f, g *are nonnegative continuous functions defined on* $\mathbb{R}_+$ *and* $n(t)$ *is a positive and nondecreasing continuous function defined on* $\mathbb{R}_+$. *In addition, assume that constants* $\beta_k \geqslant 0$, $0 \leqslant \theta_k \leqslant \tau_k \leqslant t_k - t_{k-1}$ *for* $k = 1, 2, \ldots$.

If

$$u(t) \leqslant n(t) + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g(\sigma)u(\sigma)d\sigma\right)ds + \sum_{0<t_k<t} \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s)ds, \quad t \geqslant 0, \tag{35}$$

then

$$u(t) \leqslant n(t) \prod_{0<t_k<t} C_k \exp\left(\int_{t_\alpha}^t f(s)\left[1 + \int_0^s g(\sigma)d\sigma\right]ds\right), \quad t \geqslant 0, \tag{36}$$

where C_k *is defined by* (7).

Proof. Since $n(t)$ is positive and nondecreasing, we get from (35) that

$$\frac{u(t)}{n(t)} \leqslant 1 + \int_0^t f(s)\frac{u(s)}{n(s)}ds + \int_0^t f(s)\left(\int_0^s g(\sigma)\frac{u(\sigma)}{n(\sigma)}d\sigma\right)ds + \sum_{0<t_k<t} \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} \frac{u(s)}{n(s)}ds, \quad t \geqslant 0.$$

Applying Theorem 2, we get the desired inequality in (36). □

THEOREM 6. *Let conditions* (H_0)–(H_1) *hold. Assume that* f, g, h, p *are nonnegative continuous functions defined on* $\mathbb{R}_+$ *and constants* $\beta_k \geqslant 0$, $0 \leqslant \theta_k \leqslant \tau_k \leqslant t_k - t_{k-1}$, *for* $k = 1, 2, \ldots$, *and* u_0 *is a nonnegative constant.*

(*i*) *If*

$$u(t) \leqslant u_0 + \int_0^t f(s)\left(h(s) + \int_0^s p(\sigma)u(\sigma)d\sigma\right)ds + \sum_{0<t_k<t} \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s)ds, \quad t \geqslant 0, \tag{37}$$

then

$$u(t) \leqslant \left\{u_0 \prod_{0<t_k<t} N_k + \sum_{0<t_k<t}\left(\prod_{t_k<t_j<t} N_j O_k\right)\right\} \exp\left(\int_{t_\alpha}^t f(s)\left(\int_0^s p(\sigma)d\sigma\right)ds\right) + \int_{t_\alpha}^t f(s)h(s)\exp\left(\int_s^t f(r)\left(\int_0^r p(\sigma)d\sigma\right)dr\right)ds, \quad t \geqslant 0, \tag{38}$$

where

$$N_k = \exp\left(\int_{t_{k-1}}^{t_k} f(s)\left(\int_0^s p(\sigma)d\sigma\right)ds\right) + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} \exp\left(\int_{t_{k-1}}^{s} f(r)\left(\int_0^r p(\sigma)d\sigma\right)dr\right)ds, \quad (39)$$

$$O_k = \int_{t_{k-1}}^{t} f(s)h(s)\exp\left(\int_s^{t_k} f(r)\left(\int_0^r p(\sigma)d\sigma\right)dr\right)ds + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k}\int_{t_{k-1}}^{s} f(r)h(r)\exp\left(\int_r^s f(\sigma)\left(\int_0^\sigma p(\xi)d\xi\right)d\sigma\right)drds. \quad (40)$$

(ii) *If*

$$u(t) \leqslant u_0 + \int_0^t f(s)\left[h(s) + \int_0^s g(r)\left(\int_0^r p(\sigma)u(\sigma)d\sigma\right)dr\right]ds + \sum_{0<t_k<t} \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s)ds, \quad t \geqslant 0, \quad (41)$$

then, for $t \geqslant 0$,

$$u(t) \leqslant \left\{u_0 \prod_{0<t_k<t} P_k + \sum_{0<t_k<t}\left(\prod_{t_k<t_j<t} P_j Q_k\right)\right\} \times \exp\left(\int_{t_\alpha}^t f(s)\left[\int_0^s g(r)\left(\int_0^r p(\sigma)d\sigma\right)dr\right]ds\right) + \int_{t_\alpha}^t f(s)h(s)\exp\left(\int_s^t f(r)\left[\int_0^r g(\sigma)\left(\int_0^\sigma p(\xi)d\xi\right)d\sigma\right]dr\right)ds, \quad (42)$$

where

$$P_k = \exp\left(\int_{t_{k-1}}^{t_k} f(s)\left[\int_0^s g(r)\left(\int_0^r p(\sigma)d\sigma\right)dr\right]ds\right) + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} \exp\left(\int_{t_{k-1}}^{s} f(r)\left[\int_0^r g(\sigma)\left(\int_0^\sigma p(\xi)d\xi\right)d\sigma\right]dr\right)ds, \quad (43)$$

$$Q_k = \int_{t_{k-1}}^{t} f(s)h(s)\exp\left(\int_s^{t_k} f(r)\left[\int_0^r g(\sigma)\left(\int_0^\sigma p(\xi)d\xi\right)d\sigma\right]dr\right)ds + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k}\int_{t_{k-1}}^{s} f(r)h(r)\exp\left(\int_r^s f(\eta)\left[\int_0^\eta g(\sigma)\left(\int_0^\sigma p(\xi)d\xi\right)d\sigma\right]d\eta\right)drds. \quad (44)$$

Proof. We shall give the details of the proof of (ii) only, the proof of (i) can be completed similarly.

(ii) Define a function $v(t)$ by the right-hand side of (41). Therefore, the function $v(t)$ is nondecreasing, $u(t) \leqslant v(t)$ and $v(0) = u_0$. Then, for $t \neq t_k$, we have

$$\begin{aligned} v'(t) &= f(t)h(t) + f(t)\left[\int_0^t g(s)\left(\int_0^s p(\sigma)u(\sigma)d\sigma\right)ds\right] \\ &\leqslant f(t)h(t) + f(t)\left[\int_0^t g(s)\left(\int_0^s p(\sigma)v(\sigma)d\sigma\right)ds\right] \\ &\leqslant f(t)h(t) + f(t)\left[\int_0^t g(s)\left(\int_0^s p(\sigma)d\sigma\right)ds\right]v(t). \end{aligned} \tag{45}$$

For $t = t_k^+$, we get

$$\begin{aligned} v(t_k^+) &= u_0 + \int_0^{t_k^+} f(s)h(s)ds + \int_0^{t_k^+} f(s)\left[\int_0^s g(r)\left(\int_0^r p(\sigma)u(\sigma)d\sigma\right)dr\right]ds \\ &\quad + \sum_{0<t_n<t_k^+} \beta_n \int_{t_n-\tau_n}^{t_n-\theta_n} u(s)ds \\ &= u_0 + \int_0^{t_k} f(s)u(s)ds + \int_0^{t_k} f(s)\left[\int_0^s g(r)\left(\int_0^r p(\sigma)u(\sigma)d\sigma\right)dr\right]ds \\ &\quad + \sum_{0<t_n<t_k} \beta_n \int_{t_n-\tau_n}^{t_n-\theta_n} u(s)ds + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s)ds \\ &= v(t_k) + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} u(s)ds \\ &\leqslant v(t_k) + \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} v(s)ds. \end{aligned} \tag{46}$$

Applying Theorem 1 for (45) and (46), we have for $t \geqslant 0$ that

$$\begin{aligned} v(t) \leqslant &\left\{ v(0) \prod_{0<t_k<t} P_k + \sum_{0<t_k<t}\left(\prod_{t_k<t_j<t} P_j Q_k\right)\right\} \\ &\times \exp\left(\int_{t_\alpha}^t f(s)\left[\int_0^s g(r)\left(\int_0^r p(\sigma)d\sigma\right)dr\right]ds\right) \\ &+ \int_{t_\alpha}^t f(s)h(s)\exp\left(\int_s^t f(r)\left[\int_0^r g(\sigma)\left(\int_0^\sigma p(\xi)d\xi\right)d\sigma\right]dr\right)ds, \end{aligned}$$

which results in (42). $\square$

3. An example

In this section, in order to illustrate our results, we consider an example.

EXAMPLE 1. Consider the following impulsive integro-differential equation with integral impulses

$$\begin{aligned} x'(t) &= V\left(t, x(t), \int_0^t U(t,s,x(s))\,ds\right), \quad t \neq t_k, \quad t \in [0,\infty), \\ \Delta x(t_k) &= \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} x(s)ds, \qquad k = 1,2,\ldots, \\ x(0) &= x_0, \end{aligned} \tag{47}$$

where $0 < t_1 < t_2 < \cdots$ with $\lim_{k\to\infty} t_k = \infty$, $U : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}$ is continuous, $V : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is continuous at $t \neq t_k$, $\lim_{t\to t_k^+} V(t,\cdot,\cdot)$ and $\lim_{t\to t_k^-} V(t,\cdot,\cdot)$ exist and $\lim_{t\to t_k^-} V(t,\cdot,\cdot) = V(t,\cdot,\cdot)$, $\beta_k \geqslant 0$, $0 \leqslant \theta_k \leqslant \tau_k \leqslant t_k - t_{k-1}$ $(k = 1,2,\ldots)$. Here, we assume that the solution $x(t)$ of (47) exists on $\mathbb{R}_+$, and we obtain

$$x(t) = x(0) + \int_0^t V\left(s, x(s), \int_0^s U(s,\sigma,x(\sigma))\,d\sigma\right)ds + \sum_{0<t_k<t} \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} x(s)ds. \tag{48}$$

We assume that

$$|U(t,s,x(s))| \leqslant f(t)g(s)|x(s)|, \quad |V(t,x(t),\gamma)| \leqslant f(t)|x(t)| + |\gamma|, \tag{49}$$

where $f, g \in C(\mathbb{R}_+,\mathbb{R}_+)$. From (48) and (49), we obtain

$$\begin{aligned} |x(t)| &\leqslant |x_0| + \int_0^t \left|V\left(s, x(s), \int_0^s U(s,\sigma,x(\sigma))\,d\sigma\right)\right| ds + \sum_{0<t_k<t} \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} |x(s)|ds \\ &\leqslant |x_0| + \int_0^t f(s)|x(s)|ds + \int_0^t f(s)\left(\int_0^s g(\sigma)|x(\sigma)|d\sigma\right)ds \\ &\quad + \sum_{0<t_k<t} \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} |x(s)|ds. \end{aligned} \tag{50}$$

Hence Theorem 2 yields the estimate

$$|x(t)| \leqslant |x_0| \prod_{0<t_k<t} C_k \exp\left(\int_{t_\alpha}^t f(s)\left[1 + \int_0^s g(\sigma)d\sigma\right]ds\right), \quad t \geqslant 0, \tag{51}$$

where C_k is defined by (7). The inequality (51) gives the bound on the solution $x(t)$ of (47).

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