

PARTIAL SUMS OF GENERALIZED BESSEL FUNCTIONS

HALIT ORHAN AND NIHAT YAGMUR

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Abstract. Let $(g_{p,b,c})_n(z) = z + \sum_{m=1}^n b_m z^{m+1}$ be the sequence of partial sums of generalized and normalized Bessel functions $g_{p,b,c}(z) = z + \sum_{m=1}^{\infty} b_m z^{m+1}$ where $b_m = \frac{(-c/4)^m}{m!(\kappa)_m}$ and $\kappa := p + (b+1)/2 \neq 0, -1, -2, \dots$. The purpose of the present paper is to determine lower bounds for $\Re \left\{ \frac{g_{p,b,c}(z)}{(g_{p,b,c})_n(z)} \right\}$, $\Re \left\{ \frac{(g_{p,b,c})_n(z)}{g_{p,b,c}(z)} \right\}$, $\Re \left\{ \frac{g'_{p,b,c}(z)}{(g_{p,b,c})_n(z)} \right\}$ and $\Re \left\{ \frac{(g_{p,b,c})'_n(z)}{g'_{p,b,c}(z)} \right\}$. Further we give lower bounds for $\Re \left\{ \frac{\mathbb{A}[g_{p,b,c}](z)}{(\mathbb{A}[g_{p,b,c}])_n(z)} \right\}$ and $\Re \left\{ \frac{(\mathbb{A}[g_{p,b,c}])_n(z)}{\mathbb{A}[g_{p,b,c}](z)} \right\}$, where $\mathbb{A}[g_{p,b,c}]$ is the Alexander transform of $g_{p,b,c}$.

1. Introduction and preliminary results

Let \mathcal{A} denote the class of functions f normalized by

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \quad (1.1)$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and satisfy the usual normalization condition $f(0) = f'(0) - 1 = 0$. Let \mathcal{S} denote the subclass of \mathcal{A} contains all functions which are univalent in \mathcal{U} . Also let $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$ and $\mathcal{H}(\alpha)$ denote the subclasses of \mathcal{A} consisting of functions which are, respectively, starlike, convex and close-to-convex of order α in \mathcal{U} ($0 \leq \alpha < 1$).

The Alexander transform $\mathbb{A}[f] : \mathcal{U} \rightarrow \mathbb{C}$ of f is defined by

$$\mathbb{A}[f](z) = \int_0^z \frac{f(t)}{t} dt = z + \sum_{m=2}^{\infty} \frac{a_m}{m} z^m. \quad (1.2)$$

We consider the following second-order linear homogeneous differential equation (see, for details, [16] and [2]):

$$z^2 w''(z) + bz w'(z) + [cz^2 - p^2 + (1-b)p] w(z) = 0 \quad (1.3)$$

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where $b, c, p \in \mathbb{C}$.

A particular solution of the differential equation (1.3), which is denoted by $w_{p,b,c}(z)$, is called the generalized Bessel function of the first kind of order p . In fact we have the following series representation for the function $w_{p,b,c}(z)$:

$$w_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n! \Gamma(p+n+\frac{b+1}{2})} \left(\frac{z}{2}\right)^{2n+p} \quad (z \in \mathbb{C}), \tag{1.4}$$

where $\Gamma(z)$ stands for Euler gamma function. The series in (1.4) permits us to study the Bessel, the modified Bessel and the spherical Bessel functions in a unified manner. Each of these particular cases of the function $w_{p,b,c}(z)$ is worthy of mention here.

- For $b = c = 1$ in (1.4), we obtain the familiar Bessel function $J_p(z)$ defined by (see [16] and [2]):

$$J_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(p+n+1)} \left(\frac{z}{2}\right)^{2n+p} \quad (z \in \mathbb{C}). \tag{1.5}$$

- For $b = -c = 1$ in (1.4), we obtain the modified Bessel function $I_p(z)$ defined by (see [16] and [2]):

$$I_p(z) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(p+n+1)} \left(\frac{z}{2}\right)^{2n+p} \quad (z \in \mathbb{C}). \tag{1.6}$$

- For $b - 1 = c = 1$ in (1.4), we obtain the spherical Bessel function $S_p(z)$ defined by (see [16] and [2]):

$$S_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(p+n+\frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p} \quad (z \in \mathbb{C}). \tag{1.7}$$

We now consider the function $g_{p,b,c}(z)$ defined, in terms of the generalized Bessel function $w_{p,b,c}(z)$, by (see [8]):

$$g_{p,b,c}(z) = 2^p \Gamma\left(p + \frac{b+1}{2}\right) z^{1-\frac{p}{2}} w_{p,b,c}(\sqrt{z}). \tag{1.8}$$

By using the Pochhammer (or Appell) symbol, defined in terms of Euler’s gamma functions, by $(\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda) = \lambda(\lambda + 1)\dots(\lambda + n - 1)$, we obtain the following series representation for the function $g_{p,b,c}(z)$ given by (1.8):

$$g_{p,b,c}(z) = z + \sum_{m=1}^{\infty} b_m z^{m+1} \tag{1.9}$$

where $b_m = \frac{(-c/4)^m}{m!(\kappa)_m}$ and $\kappa := p + (b + 1)/2 \neq 0, -1, -2, \dots$

For further results on this relative $g_{p,b,c}(z)$ of the generalized Bessel function $w_{p,b,c}(z)$, we refer the reader to the recent papers (see, for example, [2, 3, 4, 5, 6, 8, 9]).

In this note, we will examine the ratio of a function of the form (1.9) to its sequence of partial sums $(g_{p,b,c})_n(z) = \sum_{m=0}^n b_m z^{m+1}$ when the coefficients of $g_{p,b,c}$ satisfy some

conditions. We will determine lower bounds for $\Re \left\{ \frac{g_{p,b,c}(z)}{(g_{p,b,c})_n(z)} \right\}$, $\Re \left\{ \frac{(g_{p,b,c})_n(z)}{g_{p,b,c}(z)} \right\}$, $\Re \left\{ \frac{g'_{p,b,c}(z)}{(g_{p,b,c})'_n(z)} \right\}$, $\Re \left\{ \frac{(g_{p,b,c})'_n(z)}{g'_{p,b,c}(z)} \right\}$, $\Re \left\{ \frac{\mathbb{A}[g_{p,b,c}](z)}{(\mathbb{A}[g_{p,b,c}])_n(z)} \right\}$ and $\Re \left\{ \frac{(\mathbb{A}[g_{p,b,c}])_n(z)}{\mathbb{A}[g_{p,b,c}](z)} \right\}$, where $\mathbb{A}[g_{p,b,c}]$ is the Alexander transform of $g_{p,b,c}$.

For various interesting developments concerning partial sums of analytic univalent functions, the reader may be (for examples) referred to the works of Brickman et al. [7], Lin and Owa [10], Orhan and Gunes [11], Owa et.al [12], Sheil-Small [13], Silverman [14], Silvia [15].

LEMMA 1.1. *If the parameters $b, p \in \mathbb{R}$ and $c \in \mathbb{C}$ are so constrained that $\kappa > \frac{|c|}{8}$, then the function*

$$g_{p,b,c} : \mathcal{U} \longrightarrow \mathbb{C}$$

given by (1.9) satisfies the following inequalities:

(i) *If $\kappa > \frac{|c|}{8}$ then*

$$|g_{p,b,c}(z)| \leq \frac{8\kappa + |c|}{8\kappa - |c|} \quad (z \in \mathcal{U}),$$

(ii) *If $\kappa > \frac{|c|-4}{4}$ then*

$$|g'_{p,b,c}(z)| \leq \frac{4\kappa(\kappa + 1) + (\kappa + 2)|c|}{\kappa[4(\kappa + 1) - |c|]} \quad (z \in \mathcal{U}),$$

(iii) *If $\kappa > \frac{|c|}{8}$ then*

$$|\mathbb{A}[g_{p,b,c}](z)| \leq \frac{8\kappa}{8\kappa - |c|} \quad (z \in \mathcal{U}).$$

Proof. (i) By using the well-known triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

and the inequalities

$$m! \geq 2^{m-1}, \quad (\kappa)_m \geq \kappa^m \quad (m \in \mathbb{N} = \{1, 2, \dots\}),$$

we have

$$\begin{aligned} |g_{p,b,c}(z)| &= \left| z + \sum_{m=1}^{\infty} \frac{(-c/4)^m}{m! (\kappa)_m} z^{m+1} \right| \leq 1 + \sum_{m=1}^{\infty} \frac{|c|^m}{m! 4^m (\kappa)_m} \\ &= 1 + \frac{|c|}{4\kappa} \sum_{m=1}^{\infty} \left(\frac{|c|}{8\kappa} \right)^{m-1} \\ &= \frac{8\kappa + |c|}{8\kappa - |c|}, \quad \left(\kappa > \frac{|c|}{8} \right). \end{aligned}$$

(ii) Suppose that $\kappa > \frac{|c|-2}{4}$, by using well-known triangle inequality and the following inequality:

$$2m!(\kappa + 1)_{m-1} \geq (m + 1)(\kappa + 1)^{m-1} \quad (m \in \mathbb{N}),$$

we get

$$\begin{aligned} |g'_{p,b,c}(z)| &= \left| 1 + \sum_{m=1}^{\infty} \frac{(m + 1)(-c/4)^m}{m!(\kappa)_m} z^m \right| \leq 1 + \sum_{m=1}^{\infty} \frac{(m + 1)|c|^m}{m!4^m(\kappa)_m} \\ &= 1 + \frac{|c|}{2\kappa} \sum_{m=1}^{\infty} \frac{(m + 1)|c|^{m-1}}{2m!4^{m-1}(\kappa + 1)_{m-1}} \leq 1 + \frac{|c|}{2\kappa} \sum_{m=1}^{\infty} \left(\frac{|c|}{4(\kappa + 1)} \right)^{m-1} \\ &= \frac{4\kappa(\kappa + 1) + (\kappa + 2)|c|}{\kappa[4(\kappa + 1) - |c|]} \quad \left(\kappa > \frac{|c| - 4}{4} \right). \end{aligned}$$

(iii) In order to prove the part (iii) of Lemma 1.1, we make use of the well-known triangle inequality and the inequalities

$$(m + 1)! \geq 2^m, (\kappa)_m \geq \kappa^m \quad (m \in \mathbb{N}).$$

We thus find

$$\begin{aligned} |\mathbb{A}[g_{p,b,c}](z)| &= \left| z + \sum_{m=1}^{\infty} \frac{(-c/4)^m}{(m + 1)!(\kappa)_m} z^{m+1} \right| \leq 1 + \sum_{m=1}^{\infty} \frac{|c|^m}{(m + 1)!4^m(\kappa)_m} \\ &= 1 + \frac{|c|}{8\kappa} \sum_{m=1}^{\infty} \left(\frac{|c|}{8\kappa} \right)^{m-1} \\ &= \frac{8\kappa}{8\kappa - |c|}, \quad \left(\kappa > \frac{|c|}{8} \right). \quad \square \end{aligned}$$

2. Main results

THEOREM 2.1. *If the parameters $b, p \in \mathbb{R}$, $c \in \mathbb{C}$ and, $\kappa = p + (b + 1)/2 \neq 0, -1, -2, \dots$ are so constrained that $\kappa > \frac{3|c|}{8}$, then*

$$\Re \left\{ \frac{g_{p,b,c}(z)}{(g_{p,b,c})_n(z)} \right\} \geq \frac{8\kappa - 3|c|}{8\kappa - |c|} \quad (z \in \mathcal{U}), \tag{2.1}$$

and

$$\Re \left\{ \frac{(g_{p,b,c})_n(z)}{g_{p,b,c}(z)} \right\} \geq \frac{8\kappa - |c|}{8\kappa + |c|} \quad (z \in \mathcal{U}). \tag{2.2}$$

Proof. We observe from part (i) of Lemma 1.1 that

$$1 + \sum_{m=1}^{\infty} |b_m| \leq \frac{8\kappa + |c|}{8\kappa - |c|},$$

which is equivalent to

$$\frac{8\kappa - |c|}{2|c|} \sum_{m=1}^{\infty} |b_m| \leq 1,$$

where $b_m = \frac{(-c/4)^m}{m!(\kappa)_m}$.

Now, we may write

$$\begin{aligned} & \frac{8\kappa - |c|}{2|c|} \left[\frac{g_{p,b,c}(z)}{(g_{p,b,c})_n(z)} - \frac{8\kappa - 3|c|}{8\kappa - |c|} \right] \\ &= \frac{1 + \sum_{m=1}^n b_m z^m + \frac{8\kappa - |c|}{2|c|} \sum_{m=n+1}^{\infty} b_m z^m}{1 + \sum_{m=1}^n b_m z^m} \\ &:= \frac{1 + A(z)}{1 + B(z)}. \end{aligned}$$

Set $(1 + A(z)) / (1 + B(z)) = (1 + w(z)) / (1 - w(z))$, so that $w(z) = (A(z) - B(z)) / (2 + A(z) + B(z))$. Then

$$w(z) = \frac{\frac{8\kappa - |c|}{2|c|} \sum_{m=n+1}^{\infty} b_m z^m}{2 + 2 \sum_{m=1}^n b_m z^m + \frac{8\kappa - |c|}{2|c|} \sum_{m=n+1}^{\infty} b_m z^m}$$

and

$$|w(z)| \leq \frac{\frac{8\kappa - |c|}{2|c|} \sum_{m=n+1}^{\infty} |b_m|}{2 - 2 \sum_{m=1}^n |b_m| - \frac{8\kappa - |c|}{2|c|} \sum_{m=n+1}^{\infty} |b_m|}.$$

Now $|w(z)| \leq 1$ if and only if

$$\frac{8\kappa - |c|}{|c|} \sum_{m=n+1}^{\infty} |b_m| \leq 2 - 2 \sum_{m=1}^n |b_m|,$$

which is equivalent to

$$\sum_{m=1}^n |b_m| + \frac{8\kappa - |c|}{2|c|} \sum_{m=n+1}^{\infty} |b_m| \leq 1. \tag{2.3}$$

It suffices to show that the left hand side of (2.3) is bounded above by $\frac{8\kappa - |c|}{2|c|} \sum_{m=1}^{\infty} |b_m|$, which is equivalent to

$$\frac{8\kappa - 3|c|}{2|c|} \sum_{m=1}^n |b_m| \geq 0.$$

To prove the result (2.2), we write

$$\begin{aligned} & \frac{8\kappa + |c|}{2|c|} \left[\frac{(g_{p,b,c})_n(z)}{g_{p,b,c}(z)} - \frac{8\kappa - |c|}{8\kappa + |c|} \right] \\ &= \frac{1 + \sum_{m=1}^n b_m z^m - \frac{8\kappa - |c|}{2|c|} \sum_{m=n+1}^\infty b_m z^m}{1 + \sum_{m=1}^\infty b_m z^m} \\ &:= \frac{1 + w(z)}{1 - w(z)} \end{aligned}$$

where

$$|w(z)| \leq \frac{\frac{8\kappa + |c|}{2|c|} \sum_{m=n+1}^\infty |b_m|}{2 - 2 \sum_{m=1}^n |b_m| - \frac{8\kappa - 3|c|}{2|c|} \sum_{m=n+1}^\infty |b_m|} \leq 1.$$

The last inequality is equivalent to

$$\sum_{m=1}^n |b_m| + \frac{8\kappa - |c|}{2|c|} \sum_{m=n+1}^\infty |b_m| \leq 1 \tag{2.4}$$

Since the left hand side of (2.4) is bounded above by $\frac{8\kappa - |c|}{2|c|} \sum_{m=1}^\infty |b_m|$, the proof is completed. \square

THEOREM 2.2. *If the parameters $b, p \in \mathbb{R}$, $c \in \mathbb{C}$ and, $\kappa = p + (b + 1)/2 \neq 0, -1, -2, \dots$ are so constrained that*

$$\kappa > \frac{3|c| - 4 + \sqrt{9|c|^2 + 8|c| + 16}}{8},$$

then

$$\Re \left\{ \frac{g'_{p,b,c}(z)}{(g_{p,b,c})'_n(z)} \right\} \geq \frac{4\kappa(\kappa + 1) - (3\kappa + 2)|c|}{\kappa[4(\kappa + 1) - |c|]} \quad (z \in \mathcal{U}), \tag{2.5}$$

and

$$\Re \left\{ \frac{(g_{p,b,c})'_n(z)}{g'_{p,b,c}(z)} \right\} \geq \frac{\kappa[4(\kappa + 1) - |c|]}{4\kappa(\kappa + 1) + (\kappa + 2)|c|} \quad (z \in \mathcal{U}). \tag{2.6}$$

Proof. From part (ii) of Lemma 1.1 we observe that

$$1 + \sum_{m=1}^\infty (m + 1) |b_m| \leq \frac{4\kappa(\kappa + 1) + (\kappa + 2)|c|}{\kappa[4(\kappa + 1) - |c|]},$$

which is equivalent to

$$\frac{\kappa[4(\kappa + 1) - |c|]}{2(\kappa + 1)|c|} \sum_{m=1}^\infty (m + 1) |b_m| \leq 1,$$

where $b_m = \frac{(-c/4)^m}{m!(\kappa)_m}$.
 Now, we write

$$\begin{aligned} & \frac{\kappa [4(\kappa + 1) - |c|]}{2(\kappa + 1)|c|} \left[\frac{g'_{p,b,c}(z)}{(g_{p,b,c})'_n(z)} - \frac{4\kappa(\kappa + 1) - (3\kappa + 2)|c|}{\kappa [4(\kappa + 1) - |c|]} \right] \\ &= \frac{1 + \sum_{m=1}^n (m + 1)b_m z^m + \frac{\kappa [4(\kappa + 1) - |c|]}{2(\kappa + 1)|c|} \sum_{m=n+1}^{\infty} (m + 1)b_m z^m}{1 + \sum_{m=1}^n (m + 1)b_m z^m} \\ &:= \frac{1 + w(z)}{1 - w(z)}, \end{aligned}$$

where

$$|w(z)| \leq \frac{\frac{\kappa [4(\kappa + 1) - |c|]}{2(\kappa + 1)|c|} \sum_{m=n+1}^{\infty} (m + 1)|b_m|}{2 - 2 \sum_{m=1}^n (m + 1)|b_m| - \frac{\kappa [4(\kappa + 1) - |c|]}{2(\kappa + 1)|c|} \sum_{m=n+1}^{\infty} (m + 1)|b_m|} \leq 1.$$

The last inequality is equivalent to

$$\sum_{m=1}^n (m + 1)|b_m| + \frac{\kappa [4(\kappa + 1) - |c|]}{2(\kappa + 1)|c|} \sum_{m=n+1}^{\infty} (m + 1)|b_m| \leq 1. \tag{2.7}$$

It suffices to show that the left hand side of (2.7) is bounded above by $\frac{\kappa [4(\kappa + 1) - |c|]}{2(\kappa + 1)|c|} \sum_{m=1}^{\infty} (m + 1)|b_m|$, which is equivalent to

$$\frac{4\kappa(\kappa + 1) - (3\kappa + 2)|c|}{2(\kappa + 1)|c|} \sum_{m=1}^n (m + 1)|b_m| \geq 0.$$

To prove the result (2.6), we write

$$\begin{aligned} & \frac{4\kappa(\kappa + 1) + (\kappa + 2)|c|}{2(\kappa + 1)|c|} \left[\frac{(g_{p,b,c})'_n(z)}{g'_{p,b,c}(z)} - \frac{\kappa [4(\kappa + 1) - |c|]}{4\kappa(\kappa + 1) + (\kappa + 2)|c|} \right] \\ &:= \frac{1 + w(z)}{1 - w(z)}, \end{aligned}$$

where

$$|w(z)| \leq \frac{\frac{4\kappa(\kappa + 1) + (\kappa + 2)|c|}{2(\kappa + 1)|c|} \sum_{m=n+1}^{\infty} (m + 1)|b_m|}{2 - 2 \sum_{m=1}^n (m + 1)|b_m| - \frac{4\kappa(\kappa + 1) - (3\kappa + 2)|c|}{2(\kappa + 1)|c|} \sum_{m=n+1}^{\infty} (m + 1)|b_m|} \leq 1.$$

The last inequality is equivalent to

$$\sum_{m=1}^n (m + 1)|b_m| + \frac{\kappa [4(\kappa + 1) - |c|]}{2(\kappa + 1)|c|} \sum_{m=n+1}^{\infty} (m + 1)|b_m| \leq 1. \tag{2.8}$$

Since the left hand side of (2.8) is bounded above by $\frac{\kappa [4(\kappa + 1) - |c|]}{2(\kappa + 1)|c|} \sum_{m=1}^{\infty} (m + 1)|b_m|$, the proof is completed. \square

THEOREM 2.3. *If the parameters $b, p \in \mathbb{R}$, $c \in \mathbb{C}$ and, $\kappa = p + (b + 1)/2 \neq$*

0, -1, -2, ... are so constrained that $\kappa > \frac{|c|}{4}$, then

$$\Re \left\{ \frac{\mathbb{A}[g_{p,b,c}](z)}{(\mathbb{A}[g_{p,b,c}])_n(z)} \right\} \geq \frac{8\kappa - 2|c|}{8\kappa - |c|} \quad (z \in \mathcal{U}), \tag{2.9}$$

and

$$\Re \left\{ \frac{(\mathbb{A}[g_{p,b,c}])_n(z)}{\mathbb{A}[g_{p,b,c}](z)} \right\} \geq \frac{8\kappa - |c|}{8\kappa} \quad (z \in \mathcal{U}), \tag{2.10}$$

where $\mathbb{A}[g_{p,b,c}]$ is the Alexander transform of $g_{p,b,c}$.

Proof. We prove only (2.9), which is similar in spirit to the proof of Theorem 2.1. The proof of (2.10) follows the pattern of that in (2.2).

We consider from part (iii) of Lemma 1.1 that

$$1 + \sum_{m=1}^{\infty} \frac{|b_m|}{m+1} \leq \frac{8\kappa}{8\kappa - |c|},$$

which is equivalent to

$$\frac{8\kappa - |c|}{|c|} \sum_{m=1}^{\infty} \frac{|b_m|}{m+1} \leq 1,$$

where $b_m = \frac{(-c/4)^m}{m!(\kappa)_m}$.

We may write

$$\begin{aligned} & \frac{8\kappa - |c|}{|c|} \left[\frac{\mathbb{A}[g_{p,b,c}](z)}{(\mathbb{A}[g_{p,b,c}])_n(z)} - \frac{8\kappa - 2|c|}{8\kappa - |c|} \right] \\ &= \frac{1 + \sum_{m=1}^n \frac{b_m}{m+1} z^m + \frac{8\kappa - |c|}{2|c|} \sum_{m=n+1}^{\infty} \frac{b_m}{m+1} z^m}{1 + \sum_{m=1}^n \frac{b_m}{m+1} z^m} \\ &:= \frac{1 + w(z)}{1 - w(z)}, \end{aligned}$$

where

$$|w(z)| \leq \frac{\frac{8\kappa - |c|}{2|c|} \sum_{m=n+1}^{\infty} \frac{|b_m|}{m+1} z^m}{2 - 2 \sum_{m=1}^n \frac{|b_m|}{m+1} - \frac{8\kappa - |c|}{|c|} \sum_{m=n+1}^{\infty} \frac{|b_m|}{m+1}} \leq 1.$$

The last inequality is equivalent to

$$\sum_{m=1}^n \frac{|b_m|}{m+1} + \frac{8\kappa - |c|}{|c|} \sum_{m=n+1}^{\infty} \frac{|b_m|}{m+1} \leq 1. \tag{2.11}$$

It suffices to show that the left hand side of (2.11) is bounded above by $\frac{8\kappa - |c|}{|c|} \sum_{m=1}^{\infty} \frac{|b_m|}{m+1}$, which is equivalent to

$$\frac{8\kappa - 2|c|}{|c|} \sum_{m=1}^n \frac{|b_m|}{m+1} \geq 0. \quad \square$$

2.1. Bessel functions

Choosing $b = c = 1$, in (1.3) or (1.4), we obtain the Bessel function $J_p(z)$ of the first kind of order p defined by (1.5). Let $\mathcal{J}_p : \mathcal{U} \rightarrow \mathbb{C}$ be defined by

$$\mathcal{J}_p(z) = 2^p \Gamma(p+1) z^{1-\frac{p}{2}} J_p(\sqrt{z}).$$

We observe that

$$\mathcal{J}_{-1/2}(z) = z \cos \sqrt{z}, \quad \mathcal{J}_{1/2}(z) = \sqrt{z} \sin \sqrt{z}, \quad \mathcal{J}_{3/2}(z) = \frac{3 \sin \sqrt{z}}{\sqrt{z}} - 3 \cos \sqrt{z}.$$

In particular, the results of Theorems 2.1-2.3 become:

COROLLARY 2.4. *The following assertions hold true:*

(i) *If $p > -\frac{5}{8}$, then*

$$\Re \left\{ \frac{\mathcal{J}_p(z)}{(\mathcal{J}_p)_n(z)} \right\} \geq \frac{8p+5}{8p+7} \quad (z \in \mathcal{U}), \tag{2.12}$$

and

$$\Re \left\{ \frac{(\mathcal{J}_p)_n(z)}{\mathcal{J}_p(z)} \right\} \geq \frac{8p+7}{8p+9} \quad (z \in \mathcal{U}). \tag{2.13}$$

(ii) *If $p > \frac{-9+\sqrt{33}}{8} \approx -0.40693$ then*

$$\Re \left\{ \frac{\mathcal{J}'_p(z)}{(\mathcal{J}'_p)_n(z)} \right\} \geq \frac{4p^2+9p+3}{4p^2+11p+7} \quad (z \in \mathcal{U}), \tag{2.14}$$

and

$$\Re \left\{ \frac{(\mathcal{J}'_p)_n(z)}{\mathcal{J}'_p(z)} \right\} \geq \frac{4p^2+11p+7}{4p^2+13p+11} \quad (z \in \mathcal{U}). \tag{2.15}$$

(iii) *If $p > -\frac{3}{4}$, then*

$$\Re \left\{ \frac{\mathbb{A}[\mathcal{J}_p](z)}{(\mathbb{A}[\mathcal{J}_p])_n(z)} \right\} \geq \frac{8p+6}{8p+7} \quad (z \in \mathcal{U}), \tag{2.16}$$

and

$$\Re \left\{ \frac{(\mathbb{A}[\mathcal{J}_p])_n(z)}{\mathbb{A}[\mathcal{J}_p](z)} \right\} \geq \frac{8p+7}{8p+8} \quad (z \in \mathcal{U}). \tag{2.17}$$

REMARK 2.5. For $p = -1/2$ we get $\mathcal{J}_{-1/2}(z) = z \cos \sqrt{z}$, and for $n = 0$, we have $(\mathcal{J}_{-1/2})_0(z) = z$, so,

$$\Re \{ \cos \sqrt{z} \} \geq \frac{1}{3} \quad (z \in \mathcal{U}), \tag{2.18}$$

and

$$\Re \{ 1/\cos \sqrt{z} \} \geq \frac{3}{5} \quad (z \in \mathcal{U}). \tag{2.19}$$

REMARK 2.6. If we take $p = 1/2$ we have $\mathcal{J}_{1/2}(z) = \sqrt{z} \sin \sqrt{z}$ and $\mathcal{J}'_{1/2}(z) = \frac{\sin \sqrt{z} + \sqrt{z} \cos \sqrt{z}}{2\sqrt{z}}$, and for $n = 0$, we get $(\mathcal{J}_{1/2})_0(z) = z$, so,

$$\Re \left\{ \frac{\sin \sqrt{z}}{\sqrt{z}} \right\} \geq \frac{9}{11} \approx 0,82 \quad (z \in \mathcal{U}), \tag{2.20}$$

$$\Re \left\{ \frac{\sqrt{z}}{\sin \sqrt{z}} \right\} \geq \frac{11}{13} \approx 0,85 \quad (z \in \mathcal{U}), \tag{2.21}$$

$$\Re \left\{ \frac{\sin \sqrt{z}}{\sqrt{z}} + \cos \sqrt{z} \right\} \geq \frac{34}{27} \approx 1,26 \quad (z \in \mathcal{U}), \tag{2.22}$$

and

$$\Re \left\{ \frac{\sqrt{z}}{\sin \sqrt{z} + \sqrt{z} \cos \sqrt{z}} \right\} \geq \frac{27}{74} \approx 0,36 \quad (z \in \mathcal{U}). \tag{2.23}$$

REMARK 2.7. For $p = 3/2$, we have $\mathcal{J}_{3/2}(z) = 3 \left(\frac{\sin \sqrt{z}}{\sqrt{z}} - \cos \sqrt{z} \right)$, $\mathcal{J}'_{3/2}(z) = \frac{3}{2} \left(\frac{\cos \sqrt{z}}{z} + \frac{(z-1) \sin \sqrt{z}}{z\sqrt{z}} \right)$ and $(\mathcal{J}_{3/2})_0(z) = z$. Hence,

$$\Re \left\{ \frac{\sin \sqrt{z}}{z\sqrt{z}} - \frac{\cos \sqrt{z}}{z} \right\} \geq \frac{17}{57} \approx 0,30 \quad (z \in \mathcal{U}), \tag{2.24}$$

$$\Re \left\{ \frac{z\sqrt{z}}{\sin \sqrt{z} - \sqrt{z} \cos \sqrt{z}} \right\} \geq \frac{19}{7} \approx 2,71 \quad (z \in \mathcal{U}), \tag{2.25}$$

$$\Re \left\{ \frac{\cos \sqrt{z}}{z} + \frac{(z-1) \sin \sqrt{z}}{z\sqrt{z}} \right\} \geq \frac{102}{195} \approx 0,52 \quad (z \in \mathcal{U}), \tag{2.26}$$

and

$$\Re \left\{ \frac{z\sqrt{z}}{\sqrt{z} \cos \sqrt{z} + (z-1) \sin \sqrt{z}} \right\} \geq \frac{195}{158} \approx 1,23 \quad (z \in \mathcal{U}). \tag{2.27}$$

2.2. Modified Bessel functions

Taking $b = 1$ and $c = -1$, in (1.3) or (1.4), we obtain the modified Bessel function $I_p(z)$ of the first kind of order p defined by (1.6). Let the function $\mathcal{I}_p : \mathcal{U} \rightarrow \mathbb{C}$ be defined by

$$\mathcal{I}_p(z) = 2^p \Gamma(p+1) z^{1-\frac{p}{2}} I_p(\sqrt{z}).$$

We observe that

$$\mathcal{I}_{-1/2}(z) = z \cosh \sqrt{z}, \quad \mathcal{I}_{1/2}(z) = \sqrt{z} \sinh \sqrt{z}, \quad \mathcal{I}_{3/2}(z) = 3 \cosh \sqrt{z} - \frac{3 \sinh \sqrt{z}}{\sqrt{z}}.$$

The properties of the function \mathcal{I}_p are the same like for the function \mathcal{J}_p , because in this case we have $|c| = 1$. More precisely, we have the following results.

COROLLARY 2.8. *The following assertions are true:*

(i) *If $p > -\frac{5}{8}$, then*

$$\Re \left\{ \frac{\mathcal{J}_p(z)}{(\mathcal{J}_p)_n(z)} \right\} \geq \frac{8p+5}{8p+7} \quad (z \in \mathcal{U}), \tag{2.28}$$

and

$$\Re \left\{ \frac{(\mathcal{J}_p)_n(z)}{\mathcal{J}_p(z)} \right\} \geq \frac{8p+7}{8p+9} \quad (z \in \mathcal{U}). \tag{2.29}$$

(ii) *If $p > \frac{-9+\sqrt{33}}{8} \approx -0.40693$ then*

$$\Re \left\{ \frac{\mathcal{J}'_p(z)}{(\mathcal{J}'_p)_n(z)} \right\} \geq \frac{4p^2+9p+3}{4p^2+11p+7} \quad (z \in \mathcal{U}), \tag{2.30}$$

and

$$\Re \left\{ \frac{(\mathcal{J}'_p)_n(z)}{\mathcal{J}'_p(z)} \right\} \geq \frac{4p^2+11p+7}{4p^2+13p+11} \quad (z \in \mathcal{U}). \tag{2.31}$$

(iii) *If $p > -\frac{3}{4}$, then*

$$\Re \left\{ \frac{\mathbb{A}[\mathcal{J}_p](z)}{(\mathbb{A}[\mathcal{J}_p])_n(z)} \right\} \geq \frac{8p+6}{8p+7} \quad (z \in \mathcal{U}), \tag{2.32}$$

and

$$\Re \left\{ \frac{(\mathbb{A}[\mathcal{J}_p])_n(z)}{\mathbb{A}[\mathcal{J}_p](z)} \right\} \geq \frac{8p+7}{8p+8} \quad (z \in \mathcal{U}). \tag{2.33}$$

REMARK 2.9. For $p = -1/2$ we get $\mathcal{J}_{-1/2}(z) = z \cosh \sqrt{z}$, and for $n = 0$, we have $(\mathcal{J}_{-1/2})_0(z) = z$, so,

$$\Re \{ \cosh \sqrt{z} \} \geq \frac{1}{3} \quad (z \in \mathcal{U}), \tag{2.34}$$

and

$$\Re \{ 1/\cosh \sqrt{z} \} \geq \frac{3}{5} \quad (z \in \mathcal{U}). \tag{2.35}$$

REMARK 2.10. If we take $p = 1/2$ we have $\mathcal{J}_{1/2}(z) = \sqrt{z} \sinh \sqrt{z}$ and $\mathcal{J}'_{1/2}(z) = \frac{\sinh \sqrt{z} + \sqrt{z} \cosh \sqrt{z}}{2\sqrt{z}}$, and for $n = 0$, we get $(\mathcal{J}_{1/2})_0(z) = z$, so,

$$\Re \left\{ \frac{\sinh \sqrt{z}}{\sqrt{z}} \right\} \geq \frac{9}{11} \approx 0,82 \quad (z \in \mathcal{U}), \tag{2.36}$$

$$\Re \left\{ \frac{\sqrt{z}}{\sinh \sqrt{z}} \right\} \geq \frac{11}{13} \approx 0,85 \quad (z \in \mathcal{U}), \tag{2.37}$$

$$\Re \left\{ \frac{\sinh \sqrt{z}}{\sqrt{z}} + \cosh \sqrt{z} \right\} \geq \frac{34}{27} \approx 1,26 \quad (z \in \mathcal{U}), \tag{2.38}$$

and

$$\Re \left\{ \frac{\sqrt{z}}{\sinh \sqrt{z} + \sqrt{z} \cosh \sqrt{z}} \right\} \geq \frac{27}{74} \approx 0,36 \quad (z \in \mathcal{U}). \tag{2.39}$$

REMARK 2.11. For $p = 3/2$, we have $\mathcal{S}_{3/2}(z) = 3 \left(\cosh \sqrt{z} - \frac{\sinh \sqrt{z}}{\sqrt{z}} \right)$, $\mathcal{S}'_{3/2}(z) = \frac{3}{2} \left(\frac{(1+z)\sinh \sqrt{z}}{z\sqrt{z}} - \frac{\cosh \sqrt{z}}{z} \right)$ and $(\mathcal{S}_{3/2})_0(z) = z$. Hence,

$$\Re \left\{ \frac{\cosh \sqrt{z}}{z} - \frac{\sinh \sqrt{z}}{z\sqrt{z}} \right\} \geq \frac{17}{57} \approx 0,30 \quad (z \in \mathcal{U}), \tag{2.40}$$

$$\Re \left\{ \frac{z\sqrt{z}}{\sqrt{z} \cosh \sqrt{z} - \sinh \sqrt{z}} \right\} \geq \frac{19}{7} \approx 2,71 \quad (z \in \mathcal{U}), \tag{2.41}$$

$$\Re \left\{ \frac{(1+z)\sinh \sqrt{z}}{z\sqrt{z}} - \frac{\cosh \sqrt{z}}{z} \right\} \geq \frac{102}{195} \approx 0,52 \quad (z \in \mathcal{U}), \tag{2.42}$$

and

$$\Re \left\{ \frac{z\sqrt{z}}{(1+z)\sinh \sqrt{z} - \sqrt{z} \cosh \sqrt{z}} \right\} \geq \frac{195}{158} \approx 1,23 \quad (z \in \mathcal{U}). \tag{2.43}$$

2.3. Spherical Bessel functions

If we take $b = 2$ and $c = 1$, in (1.3) or (1.4), we obtain the spherical Bessel function $S_p(z)$ of the first kind of order p defined by (1.7).

COROLLARY 2.12. Let $\mathbb{S}_p : \mathcal{U} \rightarrow \mathbb{C}$ be defined by

$$\mathbb{S}_p(z) = 2^p \Gamma(p+1) z^{1-\frac{p}{2}} S_p(\sqrt{z}).$$

Then the following assertions are true:

(i) If $p > -\frac{9}{8}$, then

$$\Re \left\{ \frac{\mathbb{S}_p(z)}{(\mathbb{S}_p)_n(z)} \right\} \geq \frac{8p+9}{8p+11} \quad (z \in \mathcal{U}), \tag{2.44}$$

and

$$\Re \left\{ \frac{(\mathbb{S}_p)_n(z)}{\mathbb{S}_p(z)} \right\} \geq \frac{8p+11}{8p+13} \quad (z \in \mathcal{U}). \tag{2.45}$$

(ii) If $p > \frac{-13+\sqrt{33}}{8} \approx -0.90693$ then

$$\Re \left\{ \frac{\mathbb{S}'_p(z)}{(\mathbb{S}'_p)_n(z)} \right\} \geq \frac{8p^2+26p+17}{8p^2+30p+27} \quad (z \in \mathcal{U}), \tag{2.46}$$

and

$$\Re \left\{ \frac{(\mathbb{S}'_p)_n(z)}{\mathbb{S}'_p(z)} \right\} \geq \frac{8p^2 + 30p + 27}{8p^2 + 34p + 37} \quad (z \in \mathcal{U}). \tag{2.47}$$

(iii) If $p > -\frac{5}{4}$, then

$$\Re \left\{ \frac{\mathbb{A}[\mathbb{S}_p](z)}{(\mathbb{A}[\mathbb{S}_p])_n(z)} \right\} \geq \frac{8p + 10}{8p + 11} \quad (z \in \mathcal{U}), \tag{2.48}$$

and

$$\Re \left\{ \frac{(\mathbb{A}[\mathbb{S}_p])_n(z)}{\mathbb{A}[\mathbb{S}_p](z)} \right\} \geq \frac{8p + 11}{8p + 12} \quad (z \in \mathcal{U}). \tag{2.49}$$

3. Illustrative examples and image domains

In this section, we present several illustrative examples along with the geometrical descriptions of the image domains of the unit disk by the ratio of Bessel (modified Bessel) function to its sequence of partial sums or the ratio of its sequence of partial sums to the function which we considered in our remarks in section 2.

EXAMPLE 3.1. The image domains of $f_1(z) = \cos \sqrt{z}$ and $f_2(z) = \frac{1}{\cos \sqrt{z}}$ are shown in Figure 1.

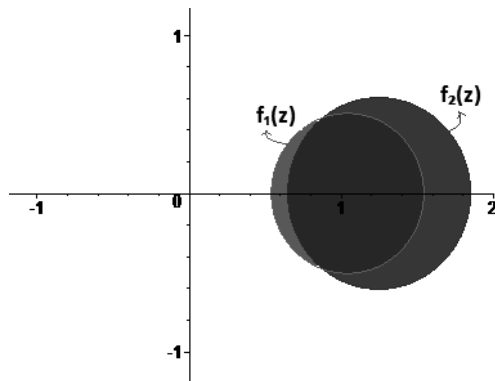


Figure 1.

EXAMPLE 3.2. We present the image domains of $f_3(z) = \frac{\sin\sqrt{z}}{\sqrt{z}}$, $f_4(z) = \frac{\sqrt{z}}{\sin\sqrt{z}}$ and $f_5(z) = \frac{\sin\sqrt{z}}{\sqrt{z}} + \cos\sqrt{z}$ in Figure 2.

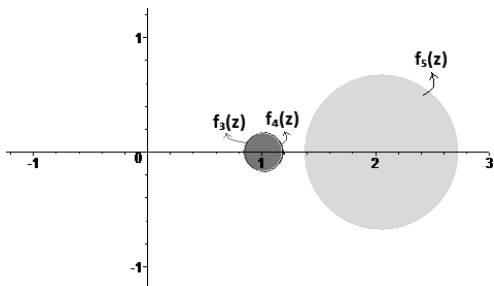


Figure 2.

EXAMPLE 3.3. We have the image domains of $f_6(z) = \frac{\sin\sqrt{z}}{z\sqrt{z}} - \frac{\cos\sqrt{z}}{z}$ and $f_7(z) = \frac{\cos\sqrt{z}}{z} + \frac{(z-1)\sin\sqrt{z}}{z\sqrt{z}}$ in Figure 3.

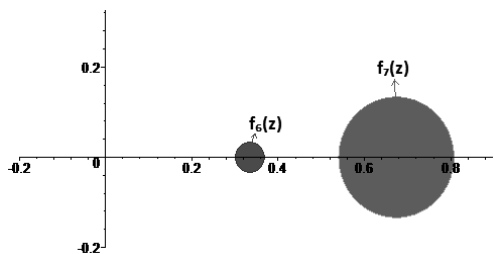


Figure 3.

EXAMPLE 3.4. The image domains of $f_8(z) = \frac{1}{\cosh\sqrt{z}}$, $f_9(z) = \frac{\sinh\sqrt{z}}{\sqrt{z}} + \cosh\sqrt{z}$ and $f_{10}(z) = \frac{z\sqrt{z}}{(1+z)\sinh\sqrt{z} - \sqrt{z}\cosh\sqrt{z}}$ are shown in Figure 4.

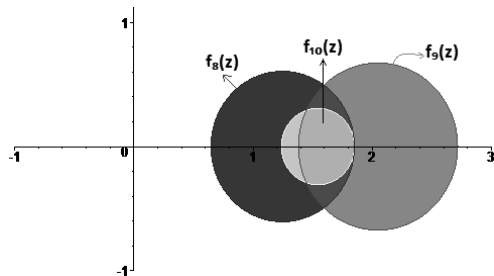


Figure 4.

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Halit Orhan
 Department of Mathematics, Faculty of Science
 Atatürk University
 Erzurum, 25240, Turkey
 e-mail: orhanhalit607@gmail.com

Nihat Yagmur
 Department of Mathematics, Faculty of Science and Art
 Erzincan University
 Erzincan, 24000, Turkey
 e-mail: nhtyagmur@gmail.com