Journal of Mathematical Inequalities
Volume 8, Number 4 (2014), 915–923 doi:10.7153/jmi-08-69

A NOTE ON THE HERMITE–HADAMARD INEQUALITY FOR CONVEX FUNCTIONS ON THE CO–ORDINATES

FEIXIANG CHEN

(*Communicated by S. Abramovich*)

Abstract. In this paper, we obtain some new Hermite-Hadamard-type inequalities for convex functions on the co-ordinates. We conclude that the results obtained in this work are the refinements of the earlier results.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a,b \in I$ with $a < b$, we have the following double inequality

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

This remarkable result is well known in the literature as the Hermite-Hadamard inequality for convex mapping.

Since then, some refinements of the Hermite-Hadamard inequality for convex functions have been extensively investigated by a number of authors (e.g., [3], [5], [6], [7] and [8]).

In ([7], 2010), A. E. Farissi established a simple proof and a new generalization of the Hermite-Hadamard inequality as follows:

THEOREM 1.1. ([7]) *Assume that $f : I \to \mathbb{R}$ is a convex function on I. Then for all $\lambda \in [0,1]$, we have*

$$f\Big(\frac{a+b}{2}\Big) \leqslant l(\lambda) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant L(\lambda) \leqslant \frac{f(a)+f(b)}{2},$$

where

$$l(\lambda) = \lambda f\Big(\frac{\lambda b+(2-\lambda)a}{2}\Big) + (1-\lambda) f\Big(\frac{(1+\lambda)b+(1-\lambda)a}{2}\Big),$$

and

$$L(\lambda) = \frac{1}{2}\Big(f(\lambda b+(1-\lambda)a) + \lambda f(a) + (1-\lambda) f(b)\Big).$$

Mathematics subject classification (2010): 26D15, 52A40.
Keywords and phrases: Hermite-Hadamard's inequality, convex function, convex functions on the co-ordinates.
This work is supported by Youth Project of Chongqing Three Gorges University of China.

Let us consider the bidimensional interval $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$, a mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leqslant \lambda f(x,y) + (1-\lambda)f(z,w)$$

holds for all (x,y), $(z,w) \in \Delta$ and $\lambda \in [0,1]$.

A function $f : \Delta \to \mathbb{R}$ is said to be co-ordinated convex on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex for all $y \in [c,d]$ and $x \in [a,b]$.

A formal definition for co-ordinated convex functions may be stated as follows:

DEFINITION 1.2. A function $f : \Delta \to \mathbb{R}$ is said to be convex on co-ordinates on Δ if the inequality

$$\begin{aligned} f(\lambda x + (1-\lambda)z, ty + (1-t)w) \leqslant\ & \lambda t f(x,y) + \lambda(1-t)f(x,w) \\ & + (1-\lambda)t f(z,y) + (1-t)(1-\lambda)f(z,w) \end{aligned}$$

holds for all (x,y), (x,w), (z,y), $(z,w) \in \Delta$ and $t \in [0,1]$, $\lambda \in [0,1]$.

S. S. Dragomir in [4] established the following Hadamard-type inequalities for co-ordinated convex functions in a rectangle from the plane $\mathbb{R}^2$.

THEOREM 1.3. *Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities*

$$\begin{aligned} & f\Big(\frac{a+b}{2}, \frac{c+d}{2}\Big) \\ \leqslant\ & \frac{1}{2}\Big[\frac{1}{b-a}\int_a^b f\Big(x, \frac{c+d}{2}\Big)dx + \frac{1}{d-c}\int_c^d f\Big(\frac{a+b}{2}, y\Big)dy\Big] \\ \leqslant\ & \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y)dydx \\ \leqslant\ & \frac{1}{4}\Big[\frac{1}{b-a}\int_a^b [f(x,c)+f(x,d)]dx + \frac{1}{d-c}\int_c^d [f(a,y)+f(b,y)]dy\Big] \\ \leqslant\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

Some new integral inequalities that are related to the Hermite-Hadamard type for co-ordinated convex functions are also established by many authors.

In ([12], 2012), M. E. Özdemir defined a new mapping associated with co-ordinated convexity and proved the following inequalities based on the properties of this mapping.

THEOREM 1.4. ([12]) *Suppose that $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is convex on the co-ordinates* *on* $\Delta = [a,b] \times [c,d]$. *Then:*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx \\
&\leqslant \frac{1}{4}\Big[\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
&+\frac{f\Big(\frac{a+b}{2},c\Big)+f\Big(\frac{a+b}{2},d\Big)+f\Big(a,\frac{c+d}{2}\Big)+f\Big(b,\frac{c+d}{2}\Big)}{2}+f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)\Big].
\end{aligned}$$

In ([1], 2008), M. Alomari and Darus defined co-ordinated s-convex functions and proved some inequalities based on this definition. In ([9], 2009), analogous results for h-convex functions on the co-ordinates were proved by M. A. Latif and M. Alomari.

In ([2], 2009), Alomari et al. established some Hadamard type inequalities for co-ordinated log-convex functions. In ([10], 2012), M. A. Latif and S. S. Dragomir obtained some new Hadamard type inequalities for differentiable co-ordinated convex and concave functions.

For recent results and generalizations concerning Hermite-Hadamard type inequality for co-ordinated convex functions see ([11], 2012) and the references given therein.

In this paper, we obtain some new Hermite-Hadamard-type inequalities for functions on the co-ordinates. These results refine the Hermite-Hadamard-type inequalities given in Theorem 1.3.

2. Main results

THEOREM 2.1. *Suppose that* $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ *is co-ordinated convex on* Δ. *Then for all* $\lambda \in [0,1]$, $t \in [0,1]$, *one has the inequalities*

$$\begin{aligned}
f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big) &\leqslant l(\lambda,t) \\
&\leqslant \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx \\
&\leqslant L(\lambda,t) \\
&\leqslant \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4},
\end{aligned}$$

where

$$\begin{aligned}
l(\lambda,t) = t\lambda f\Big(&\frac{(2-\lambda)a+\lambda b}{2},\frac{(2-t)c+td}{2}\Big) \\
&+\lambda(1-t)f\Big(\frac{(1-\lambda)a+(1+\lambda)b}{2},\frac{(2-t)c+td}{2}\Big) \\
&+t(1-\lambda)f\Big(\frac{(2-\lambda)a+\lambda b}{2},\frac{(1-t)c+(1+t)d}{2}\Big) \\
&+(1-\lambda)(1-t)f\Big(\frac{(1-\lambda)a+(1+\lambda)b}{2},\frac{(1-t)c+(1+t)d}{2}\Big)
\end{aligned}$$

and

$$L(\lambda,t) = \frac{t\lambda}{4}f(a,c)+\frac{\lambda(1-t)}{4}f(a,d)+\frac{t(1-\lambda)}{4}f(b,c)+\frac{(1-\lambda)(1-t)}{4}f(b,d)$$
$$+\frac{f((1-\lambda)a+\lambda b,(1-t)c+td)}{4}+\frac{\lambda}{4}f(a,(1-t)c+td)$$
$$+\frac{1-\lambda}{4}f(b,(1-t)c+td)+\frac{t}{4}f((1-\lambda)a+\lambda b,c)$$
$$+\frac{1-t}{4}f((1-\lambda)a+\lambda b,d).$$

Proof. For $\lambda \in [0,1]$, $t \in [0,1]$, we let

$$\Delta = [a,b]\times[c,d] = \Delta_1\cup\Delta_2\cup\Delta_3\cup\Delta_4,$$

where

$$\Delta_1 = [a,(1-\lambda)a+\lambda b]\times[c,(1-t)c+td],$$

$$\Delta_2 = [a,(1-\lambda)a+\lambda b]\times[(1-t)c+td,d],$$

$$\Delta_3 = [(1-\lambda)a+\lambda b,b]\times[c,(1-t)c+td],$$

and

$$\Delta_4 = [(1-\lambda)a+\lambda b,b]\times[(1-t)c+td,d].$$

Now by applying Theorem 1.3 to Δ_1, Δ_2, Δ_3, Δ_4, with $\lambda \neq 0,1$ and $t \neq 0,1$, respectively, we obtain

$$\begin{aligned} f\Big(\frac{(2-\lambda)a+\lambda b}{2},\frac{(2-t)c+td}{2}\Big) &\leqslant \frac{1}{t\lambda(b-a)(d-c)}\int_a^{(1-\lambda)a+\lambda b}\int_c^{(1-t)c+td} f(x,y)dydx \\ &\leqslant \frac{1}{4}\Big[f(a,c)+f(a,(1-t)c+td) \\ &\quad +f((1-\lambda)a+\lambda b,c)+f((1-\lambda)a+\lambda b,(1-t)c+td)\Big], \end{aligned} \tag{2}$$

$$\begin{aligned} f\Big(\frac{(2-\lambda)a+\lambda b}{2},\frac{(1-t)c+(1+t)d}{2}\Big) &\leqslant \frac{1}{\lambda(1-t)(b-a)(d-c)}\int_a^{(1-\lambda)a+\lambda b}\int_{(1-t)c+td}^{d} f(x,y)dydx \\ &\leqslant \frac{1}{4}\Big[f(a,(1-t)c+td)+f(a,d) \\ &\quad +f((1-\lambda)a+\lambda b,(1-t)c+td)+f((1-\lambda)a+\lambda b,d)\Big], \end{aligned} \tag{3}$$

$$
\begin{aligned}
& f\Big(\frac{(1-\lambda)a+(1+\lambda)b}{2},\frac{(2-t)c+td}{2}\Big)\\
&\quad\leqslant \frac{1}{(1-\lambda)t(b-a)(d-c)}\int_{(1-\lambda)a+\lambda b}^{b}\int_{c}^{(1-t)c+td} f(x,y)dydx\\
&\quad\leqslant \frac{1}{4}\Big[f((1-\lambda)a+\lambda b,c)+f((1-\lambda)a+\lambda b,(1-t)c+td)\\
&\qquad +f(b,c)+f(b,(1-t)c+td)\Big],
\end{aligned}
\tag{4}
$$

and

$$
\begin{aligned}
& f\Big(\frac{(1-\lambda)a+(1+\lambda)b}{2},\frac{(1-t)c+(1+t)d}{2}\Big)\\
&\quad\leqslant \frac{1}{(1-\lambda)(1-t)(b-a)(d-c)}\int_{(1-\lambda)a+\lambda b}^{b}\int_{(1-t)c+td}^{d} f(x,y)dydx\\
&\quad\leqslant \frac{1}{4}\Big[f((1-\lambda)a+\lambda b,d)+f((1-\lambda)a+\lambda b,(1-t)c+td)\\
&\qquad +f(b,(1-t)c+td)+f(b,d)\Big].
\end{aligned}
\tag{5}
$$

Multiplying (2) by $t\lambda$, (3) by $\lambda(1-t)$, (4) by $t(1-\lambda)$, (5) by $(1-\lambda)(1-t)$, respectively, and adding the resulting inequalities, we get

$$
\begin{aligned}
& t\lambda f\Big(\frac{(2-\lambda)a+\lambda b}{2},\frac{(2-t)c+td}{2}\Big)+\lambda(1-t)f\Big(\frac{(2-\lambda)a+\lambda b}{2},\frac{(1-t)c+(1+t)d}{2}\Big)\\
&\quad +t(1-\lambda)f\Big(\frac{(1-\lambda)a+(1+\lambda)b}{2},\frac{(2-t)c+td}{2}\Big)\\
&\quad +(1-\lambda)(1-t)f\Big(\frac{(1-\lambda)a+(1+\lambda)b}{2},\frac{(1-t)c+(1+t)d}{2}\Big)\\
&= l(\lambda,t)\\
&\leqslant \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\\
&\leqslant \frac{t\lambda}{4}\Big[f(a,c)+f(a,(1-t)c+td)\\
&\quad +f((1-\lambda)a+\lambda b,c)+f((1-\lambda)a+\lambda b,(1-t)c+td)\Big]\\
&\quad +\frac{\lambda(1-t)}{4}\Big[f(a,(1-t)c+td)+f(a,d)\\
&\quad +f((1-\lambda)a+\lambda b,(1-t)c+td)+f((1-\lambda)a+\lambda b,d)\Big]\\
&\quad +\frac{t(1-\lambda)}{4}\Big[f((1-\lambda)a+\lambda b,c)+f((1-\lambda)a+\lambda b,(1-t)c+td)\\
&\quad +f(b,c)+f(b,(1-t)c+td)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(1-\lambda)(1-t)}{4}\Big[f((1-\lambda)a+\lambda b,d)+f((1-\lambda)a+\lambda b,(1-t)c+td)\\
&+f(b,(1-t)c+td)+f(b,d)\Big]\\
=&\ \frac{t\lambda}{4}f(a,c)+\frac{\lambda(1-t)}{4}f(a,d)+\frac{t(1-\lambda)}{4}f(b,c)+\frac{(1-\lambda)(1-t)}{4}f(b,d)\\
&+\frac{f((1-\lambda)a+\lambda b,(1-t)c+td)}{4}+\frac{\lambda}{4}f(a,(1-t)c+td)+\frac{1-\lambda}{4}f(b,(1-t)c+td)\\
&+\frac{t}{4}f((1-\lambda)a+\lambda b,c)+\frac{1-t}{4}f((1-\lambda)a+\lambda b,d)\\
=&\ L(\lambda,t).
\end{aligned}
$$

Using the fact that $f(x,y)$ is a co-ordinated convex function, we obtain

$$
\begin{aligned}
&f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)\\
=&\ f\Big(\lambda\frac{(2-\lambda)a+\lambda b}{2}+(1-\lambda)\frac{(1-\lambda)a+(1+\lambda)b}{2},t\frac{(2-t)c+td}{2}+(1-t)\frac{(1-t)c+(1+t)d}{2}\Big)\\
\leqslant&\ t\lambda f\Big(\frac{(2-\lambda)a+\lambda b}{2},\frac{(2-t)c+td}{2}\Big)+t(1-\lambda)f\Big(\frac{(1-\lambda)a+(1+\lambda)b}{2},\frac{(2-t)c+td}{2}\Big)\\
&+\lambda(1-t)f\Big(\frac{(2-\lambda)a+\lambda b}{2},\frac{(1-t)c+(1+t)d}{2}\Big)\\
&+(1-\lambda)(1-t)f\Big(\frac{(1-\lambda)a+(1+\lambda)b}{2},\frac{(1-t)c+(1+t)d}{2}\Big)\\
=&\ l(\lambda,t).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
L(\lambda,t)=&\ \frac{t\lambda}{4}f(a,c)+\frac{\lambda(1-t)}{4}f(a,d)+\frac{t(1-\lambda)}{4}f(b,c)+\frac{(1-\lambda)(1-t)}{4}f(b,d)\\
&+\frac{f((1-\lambda)a+\lambda b,(1-t)c+td)}{4}+\frac{\lambda}{4}f(a,(1-t)c+td)\\
&+\frac{1-\lambda}{4}f(b,(1-t)c+td)+\frac{t}{4}f((1-\lambda)a+\lambda b,c)\\
&+\frac{1-t}{4}f((1-\lambda)a+\lambda b,d)\\
\leqslant&\ \frac{t\lambda}{4}f(a,c)+\frac{\lambda(1-t)}{4}f(a,d)+\frac{t(1-\lambda)}{4}f(b,c)+\frac{(1-\lambda)(1-t)}{4}f(b,d)\\
&+\frac{(1-\lambda)(1-t)}{4}f(a,c)+\frac{t(1-\lambda)}{4}f(a,d)+\frac{\lambda(1-t)}{4}f(b,c)++\frac{t\lambda}{4}f(b,d)\\
&+\frac{\lambda(1-t)}{4}f(a,c)+\frac{t\lambda}{4}f(a,d)+\frac{(1-\lambda)(1-t)}{4}f(b,c)++\frac{t(1-\lambda)}{4}f(b,d)\\
&+\frac{t(1-\lambda)}{4}f(a,c)+\frac{\lambda^2}{4}f(b,c)+\frac{(1-\lambda)(1-t)}{4}f(a,d)++\frac{\lambda(1-t)}{4}f(b,d)\\
=&\ \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}
$$

We have completed the proof. □

REMARK. Applying Theorem 2.1 for $\lambda = t = \frac{1}{2}$,

$$
\begin{aligned}
&L\Big(\frac{1}{2},\frac{1}{2}\Big)\\
&\leqslant \frac{1}{4}\Bigg[\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\\
&\quad+\frac{f\Big(\frac{a+b}{2},c\Big)+f\Big(\frac{a+b}{2},d\Big)+f\Big(a,\frac{c+d}{2}\Big)+f\Big(b,\frac{c+d}{2}\Big)}{2}+f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)\Bigg],
\end{aligned}
$$

we get the result of Theorem 1.4.

COROLLARY 2.2. *With notations above, we have the following inequality*

$$
\begin{aligned}
f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big) &\leqslant \sup_{0\leqslant\lambda\leqslant 1, 0\leqslant t\leqslant 1} l(\lambda,t)\\
&\leqslant \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\\
&\leqslant \inf_{0\leqslant\lambda\leqslant 1, 0\leqslant t\leqslant 1} L(\lambda,t)\\
&\leqslant \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4},
\end{aligned}
$$

where $l(\lambda,t)$, $L(\lambda,t)$ *are defined in Theorem 2.1.*

COROLLARY 2.3. *With notations above, we have the following inequality*

$$
\begin{aligned}
f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big) &\leqslant \max\Big\{\sup_{0\leqslant\lambda\leqslant 1, 0\leqslant t\leqslant 1} l(\lambda,t), A\Big\}\\
&\leqslant \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\\
&\leqslant \min\Big\{\inf_{0\leqslant\lambda\leqslant 1, 0\leqslant t\leqslant 1} L(\lambda,t), B\Big\}\\
&\leqslant \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4},
\end{aligned}
$$

where $l(\lambda,t)$, $L(\lambda,t)$ *are defined in Theorem 2.1 and*

$$A=\frac{1}{2}\Big[\frac{1}{b-a}\int_a^b f\Big(x,\frac{c+d}{2}\Big)dx+\frac{1}{d-c}\int_c^d f\Big(\frac{a+b}{2},y\Big)dy\Big],$$

$$B=\frac{1}{4}\Big[\frac{1}{b-a}\int_a^b [f(x,c)+f(x,d)]dx+\frac{1}{d-c}\int_c^d [f(a,y)+f(b,y)]dy\Big].$$

EXAMPLE. Let

$$\Delta = [0,1] \times [0,1]$$

and

$$f(x,y) = x^3y^3.$$

We get that $f(x,y) = x^3y^3$ is convex on the co-ordinates on Δ.

Now

$$\int_0^1 \int_0^1 f(x,y)dydx = \frac{1}{16},$$

$$A = \frac{1}{2}\Big[\int_0^1 f\Big(x,\frac{1}{2}\Big)dx + \int_0^1 f\Big(\frac{1}{2},y\Big)dy\Big] = \frac{1}{32},$$

$$l\Big(\frac{1}{2},\frac{1}{2}\Big) = \frac{1}{4}\Big[f\Big(\frac{1}{4},\frac{1}{4}\Big) + f\Big(\frac{3}{4},\frac{1}{4}\Big) + f\Big(\frac{1}{4},\frac{3}{4}\Big) + f\Big(\frac{3}{4},\frac{3}{4}\Big)\Big] = \frac{49}{1024},$$

$$L\Big(\frac{1}{2},\frac{1}{2}\Big) = \frac{1}{4}\Big[\frac{f(1,1)}{4} + \frac{f\big(\frac{1}{2},1\big) + f\big(1,\frac{1}{2}\big)}{2} + f\Big(\frac{1}{2},\frac{1}{2}\Big)\Big] = \frac{25}{256},$$

$$B = \frac{1}{4}\Big[\int_0^1 [f(x,0) + f(x,1)]dx + \int_0^1 [f(0,y) + f(1,y)]dy\Big] = \frac{1}{8}.$$

Therefore,

$$\begin{aligned} f\Big(\frac{1}{2},\frac{1}{2}\Big) &= \frac{1}{64} \\ &\leqslant A = \frac{1}{32} \\ &\leqslant l\Big(\frac{1}{2},\frac{1}{2}\Big) = \frac{49}{1024} \\ &\leqslant \sup_{0\leqslant\lambda\leqslant1,0\leqslant t\leqslant1} l(\lambda,t) \\ &\leqslant \int_0^1 \int_0^1 f(x,y)dydx = \frac{1}{16} \\ &\leqslant \inf_{0\leqslant\lambda\leqslant1,0\leqslant t\leqslant1} L(\lambda,t) \\ &\leqslant L\Big(\frac{1}{2},\frac{1}{2}\Big) = \frac{25}{256} \\ &\leqslant B = \frac{1}{8} \\ &\leqslant \frac{f(0,0) + f(0,1) + f(1,0) + f(1,1)}{4} = \frac{1}{4}, \end{aligned}$$

we get an estimation better than Theorem 1.3.

Acknowledgements. This work is supported by Youth Project of Chongqing Three Gorges University of China (No. 13QN11).

REFERENCES

[1] M. ALOMARI AND M. DARUS, *The Hadamard's inequality for s-convex functions of 2-variables on the co-ordinates*, Int. J. Math. Anal., **2**, 13 (2008), 629–638.
[2] M. ALOMARI AND M. DARUS, *On The Hadamard's inequality for log-convex functions on the coordinates*, J. Inequal. Appl., 2009, 2009: 283147.
[3] M. BESSENYEI AND ZSOLT PÁLES, *Hadamard-type inequalities for generalized convex functions*, Math. Inequal. Appl., **6**, 3 (2003), 379–392.
[4] S. S. DRAGOMIR, *On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane*, Taiwanese J. Math., **5**, 4 (2001), 775–788.
[5] S. S. DRAGOMIR, *Hermite-Hadamard's type inequalities for operator convex functions*, Appl. Math. Comput., **218**, 3 (2011), 766–772.
[6] S. S. DRAGOMIR, *Hermite-Hadamard's type inequalities for convex functions of selfadjoint operators in Hilbert spaces*, Linear Algebra Appl., **436**, 5 (2012), 1503–1515.
[7] ABDALLAH EL FARISSI, *Simple proof and refinement of Hermite-Hadamard inequality*, J. Math. Inequal., **4**, 3 (2010), 365–369.
[8] XIANG GAO, *A note on the Hermite-Hadamard inequality*, J. Math. Inequal., **4**, 4 (2010), 587–591.
[9] M. A. LATIF AND M. ALOMARI, *On Hadamard-type inequalities for h-convex functions on the co-ordinates*, Int. J. Math. Anal. **3**, 33 (2009), 1645–1656.
[10] M. A. LATIF AND S. S. DRAGOMIR, *On some new inequalities for differentiable co-ordinated convex functions*, J. Inequal. Appl., 2012, 2012:28.
[11] M. E. ÖZDEMIR, H. KAVURMACI, A. O. AKDEMIR AND M. AVCI, *Inequalities for convex and s-convex functions on $\Delta = [a,b] \times [c,d]$*, J. Inequal. Appl., 2012, 2012:20.
[12] M. E. ÖZDEMIR, ÇETIN YILDIZ AND A. O. AKDEMIR, *On some new Hadamard-Type inequalities for co-ordinated quasi-convex functions*, Hacet. J. Math. Stat. **41**, 5 (2012), 697–707.

(Received August 24, 2013)

Feixiang Chen
School of Mathematics and Statistics
Chongqing Three Gorges University
Wanzhou, Chongqing, 404000, P. R. China
e-mail: cfx2002@126.com