

HÖLDER AND MINKOWSKI TYPE INEQUALITIES WITH ALTERNATING SIGNS

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Abstract. We obtain new inequalities with alternating signs of Hölder and Minkowski type.

1. Introduction

Most classical inequalities are essentially concerned with positive terms. On the other hand, in different branches of analysis there is necessity to deal with sums and series with alternating signs. The main goal of this work is to obtain inequalities of Hölder and Minkowski type for such sums and series.

Let us start with formulating several known results. The first inequality with alternating signs was due to Szegő.

THEOREM 1. (Szegő [13]) *Let $0 \leq b_{2n+1} \leq b_{2n} \leq \dots \leq b_2 \leq b_1$ and the function $f = f(x)$ be convex on $[0; b_1]$, then*

$$f\left(\sum_{k=1}^{2n+1} (-1)^{k+1} b_k\right) \leq \sum_{k=1}^{2n+1} (-1)^{k+1} f(b_k). \quad (1)$$

Later on, Bellman using a simple geometrical method proved (1), where $2n + 1$ was replaced by $2n$ and convex f was such that $f(0) \leq 0$ (see [2] and [19]). However, this result has been already contained in Szegő's theorem. Indeed, it is sufficient to put $b_{2n+1} = 0$ in (1) and take into account that $f(0)$ in the right hand side is non-positive. In [18], Weinberger independently obtained a particular case of Theorem 1, which can be derived from (1) by putting $f(x) = x^p$, $p \geq 1$. Finally, Brunk and Olkin, using different technics, proved a weighted version of (1).

THEOREM 2. (Brunk [8], Olkin [14]) *Let*

$$0 \leq w_n \leq w_{n-1} \leq \dots \leq w_1 \leq 1, \quad 0 \leq b_n \leq b_{n-1} \leq \dots \leq b_2 \leq b_1,$$

and the function $f = f(x)$ be convex on $[0; b_1]$, then

$$f\left(\sum_{k=1}^n (-1)^{k+1} w_k b_k\right) \leq \left(1 - \sum_{k=1}^n (-1)^{k+1} w_k\right) f(0) + \sum_{k=1}^n (-1)^{k+1} w_k f(b_k).$$

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Note that Theorems 1 and 2 are of Jensen type. It was shown in [3] that these results were particular cases of more general statements. Namely, Theorem 1 follows from the majorization theorem [10, Theorem 108], and Theorem 2 is a corollary of Steffensen's inequality (see [16]). Analogues of other classical inequalities for sums with alternating signs were considered by Biernacki in [5]. He showed in particular that Chebyshev's sum inequality remains valid for such sums. Later on, several refinements of Theorems 1 and 2 were obtained in [6], [12], and [15].

Note that inequalities with alternating signs have numerous applications. For instance, Szegő proved Theorem 1 for purposes in generalization of Dirichlet integrals, the result of Weinberger was motivated by certain problems in symmetrization theory, etc. Close connection between sums with alternating signs and estimates of trigonometrical integrals was observed by Steffensen [16].

Now we give some notations and auxiliaries. In what follows, we denote non-negative sequences of real numbers in bold print, e.g. $\mathbf{a} = \{a_k\}_{k=1}^n$ or $\mathbf{b} = \{b_k\}_{k=1}^n$, where n is a positive integer or infinity (usually we omit number of elements). Sometimes properties of the sequences can be specified. Expressions like $\mathbf{a} \equiv 1$ mean that all elements of \mathbf{a} are equal to 1. From now on, we exclude cases of sequences such that denominators of fractions in inequalities vanish for them.

Further, let us recall some well-known inequalities for $\alpha, \beta \geq 0$, which we use:

$$(\alpha + \beta)^p \leq 2^{p-1}(\alpha^p + \beta^p), \quad p \geq 1 \quad (\text{Jensen's inequality}); \quad (2)$$

$$(\alpha + \beta)^p \geq 2^{p-1}(\alpha^p + \beta^p), \quad 0 \leq p \leq 1 \quad (\text{reverse Jensen's inequality}); \quad (3)$$

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p \geq 1 \quad (\text{Young's inequality}); \quad (4)$$

$$p\beta^{p-1} \leq \frac{\alpha^p - \beta^p}{\alpha - \beta} \leq p\alpha^{p-1}, \quad p \geq 1, \quad \alpha > \beta; \quad (5)$$

$$\alpha^p + \beta^p \leq (\alpha + \beta)^p, \quad p \geq 1. \quad (6)$$

For the inequality (5) we refer the reader to [10, Theorem 41]. The following result will be needed in Section 2.

LEMMA 1. *Let \mathbf{a} be non-increasing, \mathbf{b} be non-decreasing and such that $b_k \leq B$ for $k = 1, \dots, n$. Then*

$$\sum_{k=1}^n (-1)^{k+1} a_k b_k \leq B \sum_{k=1}^n (-1)^{k+1} a_k.$$

Proof. Since the sequences \mathbf{a} and $\{B - b_k\}$ are non-increasing, we have

$$B \sum_{k=1}^n (-1)^{k+1} a_k - \sum_{k=1}^n (-1)^{k+1} a_k b_k = \sum_{k=1}^n (-1)^{k+1} a_k (B - b_k) \geq 0.$$

It is easily seen that equality holds, for example, if $\mathbf{b} \equiv B$. \square

2. Hölder type inequalities

In this section, we show that there is no a direct analog of Hölder's inequality in case with alternating signs, but it is possible to obtain one of reverse Hölder's inequality. Note that reverse Hölder's inequalities for non-negative terms are well studied (see e.g. [20] and references there in).

THEOREM 1. *Let \mathbf{a} and \mathbf{b} be positive non-increasing and such that*

$$0 < a \leq a_k \leq A < \infty, \quad 0 < b \leq b_k \leq B < \infty, \quad k = 1, \dots, n,$$

then for $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$0 \leq \frac{\left(\sum_{k=1}^n (-1)^{k+1} a_k^q\right)^{1/q} \left(\sum_{k=1}^n (-1)^{k+1} b_k^p\right)^{1/p}}{\sum_{k=1}^n (-1)^{k+1} a_k b_k} \leq C_{\mathbf{a}, \mathbf{b}}, \quad (7)$$

where $C_{\mathbf{a}, \mathbf{b}} = A^{q-1}/b + B^{p-1}/a$ and $C_{\mathbf{a}, \mathbf{b}} \in (1; \infty)$. The left hand side of (7) should be read as there exists no positive constant, depending on a, A, b, B, p or q , which bounds the fraction in (7) from below.

Proof. From now on, F_H stands for the fraction in (7). The fact that there exist no positive constants bounding F_H from below, can be shown by the following example. Let n be even and $\mathbf{a} = \{a_1, a_1, a_3, a_3, \dots, a_n, a_n, \dots\}$ be positive and non-decreasing. The sequence \mathbf{b} is arbitrary except such that $b_{2k-1} - b_{2k} = 0$ for all $k = 1, \dots, n/2$. It follows that

$$F_H = \frac{0 \cdot \left(\sum_{k=1}^n (-1)^{k+1} b_k^p\right)^{1/p}}{\sum_{k=1}^{n/2} a_{2k-1} (b_{2k-1} - b_{2k})} = 0.$$

Thus F_H cannot be bounded from below by a positive absolute constant or a constant depending on p, q , maximum or minimum elements of \mathbf{a} and \mathbf{b} .

Now we prove the right hand side of (7). Here N_H denotes the numerator of F_H . From (4) we have

$$N_H \leq \frac{1}{q} \sum_{k=1}^n (-1)^{k+1} a_k^q + \frac{1}{p} \sum_{k=1}^n (-1)^{k+1} b_k^p.$$

Note that any sum with alternating signs can be written in the form

$$\sum_{k=1}^n (-1)^{k+1} \alpha_k = \sum_{k=1}^N (\alpha_{2k-1} - \alpha_{2k}), \quad (8)$$

where $N = n/2$ if n is even and $N = (n+1)/2$ if n is odd (we also assume that $\alpha_{n+1} = 0$).

Therefore from the right hand side of (5) we obtain

$$\begin{aligned}
 N_H &\leq \frac{1}{q} \sum_{k=1}^N (a_{2k-1}^q - a_{2k}^q) + \frac{1}{p} \sum_{k=1}^N (b_{2k-1}^p - b_{2k}^p) \\
 &\leq \frac{1}{q} \sum_{k=1}^N q a_{2k-1}^{q-1} (a_{2k-1} - a_{2k}) + \frac{1}{p} \sum_{k=1}^N p b_{2k-1}^{p-1} (b_{2k-1} - b_{2k}) \\
 &\leq A^{q-1} \sum_{k=1}^n (-1)^{k+1} a_k + B^{p-1} \sum_{k=1}^n (-1)^{k+1} b_k \\
 &= \sum_{k=1}^n (-1)^{k+1} \left(\frac{A^{q-1}}{b_k} + \frac{B^{p-1}}{a_k} \right) a_k b_k.
 \end{aligned}$$

In the last expression, sequences $\{A^{q-1}/b_k + B^{p-1}/a_k\}$ and $\{a_k b_k\}$ are non-decreasing and non-increasing, correspondingly, since \mathbf{a} and \mathbf{b} are non-increasing. Hence by Lemma

$$N_H \leq \max_k \left\{ \frac{A^{q-1}}{b_k} + \frac{B^{p-1}}{a_k} \right\} \sum_{k=1}^n (-1)^{k+1} a_k b_k \leq C_{\mathbf{a}, \mathbf{b}} \sum_{k=1}^n (-1)^{k+1} a_k b_k.$$

We conclude the proof by observing that $C_{\mathbf{a}, \mathbf{b}} \in (1, \infty)$. Indeed, it is easily seen that $C_{\mathbf{a}, \mathbf{b}} \rightarrow \infty$ as $a \rightarrow 0$ or $b \rightarrow 0$. We have $C_{\mathbf{a}, \mathbf{b}} > 1$, because, on the one hand, from (4) taking into account that $a \leq A$, $b \leq B$ we have

$$C_{\mathbf{a}, \mathbf{b}} \geq \frac{a^{q-1}}{b} + \frac{b^{p-1}}{a} > \frac{a^q/q + b^p/p}{ab} \geq 1,$$

and, on the other hand, $C_{\mathbf{a}, \mathbf{b}} \rightarrow 1$ from above as $b \rightarrow \infty$ if $\mathbf{a} \equiv 1$ and p is sufficiently close to 1. \square

REMARK 1. From Theorem 1, it is seen that $C_{\mathbf{a}, \mathbf{b}}$ tends to infinity as $a \rightarrow 0$ or $b \rightarrow 0$. Now we give an example of sequences that confirms it. Following the notation of Theorem 1, we suppose that the number of terms in the sums is odd, $\mathbf{a} \equiv 1$ and $b = b_{2n+1} = 0$ in \mathbf{b} . It gives

$$\begin{aligned}
 F_H &= \frac{(\sum_{k=1}^{2n+1} (-1)^{k+1} a_k^q)^{1/q} (\sum_{k=1}^{2n+1} (-1)^{k+1} b_k^p)^{1/p}}{\sum_{k=1}^{2n+1} (-1)^{k+1} a_k b_k} \\
 &= \frac{(\sum_{k=1}^{2n} (-1)^{k+1} b_k^p)^{1/p}}{\sum_{k=1}^{2n} (-1)^{k+1} b_k}.
 \end{aligned}$$

From (8) and the left hand side of (5) we deduce that

$$F_H = \frac{(\sum_{k=1}^n (b_{2k-1}^p - b_{2k}^p))^{1/p}}{\sum_{k=1}^n (b_{2k-1} - b_{2k})} \geq p^{1/p} \left(\frac{b_{2n}}{\sum_{k=1}^n (b_{2k-1} - b_{2k})} \right)^{1-1/p},$$

where the power is positive since $p > 1$ and, therefore, for a fixed positive b_{2n} the sum in the denominator can be made sufficiently small due to an appropriate choice of the sequence \mathbf{b} . Consequently, $F_{\mathbf{H}}$ can be arbitrarily large.

On the other hand, in some particular cases $F_{\mathbf{H}}$ can be bounded from above by an absolute constant even if \mathbf{a} and \mathbf{b} are decreasing to zero as $n \rightarrow \infty$. For instance, for harmonic series we have:

$$\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{q\alpha}} \right)^{1/q} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{p\beta}} \right)^{1/p} \leq \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{\alpha+\beta}}, \quad \alpha > 0, \quad \beta > 0. \quad (9)$$

Indeed, since $\sum_{k=1}^{\infty} (-1)^{k+1} k^{-s} = (1 - 2^{1-s})\zeta(s)$ for $s > 0$, in order to prove (9) it is sufficient to note that the function

$$F(\alpha, \beta) = \frac{((1 - 2^{1-q\alpha})\zeta(q\alpha))^{1/q} ((1 - 2^{1-p\beta})\zeta(p\beta))^{1/p}}{(1 - 2^{1-(\alpha+\beta)})\zeta(\alpha + \beta)}$$

has a maximum at $q\alpha = p\beta$ and moreover $\max F(\alpha, \beta) = 1$. The following inequality for geometric series holds:

$$\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{a^{qk}} \right)^{1/q} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{b^{pk}} \right)^{1/p} \leq \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(ab)^k}, \quad a > 1, \quad b > 1. \quad (10)$$

Certainly, the left and right hand sides of the inequality equal $(1 + a^q)^{-\frac{1}{q}}(1 + b^p)^{-\frac{1}{p}}$ and $(1 + ab)^{-1}$ respectively. Consequently, we have $1 + ab \leq (1 + a^q)^{\frac{1}{q}}(1 + b^p)^{\frac{1}{p}}$, which is true by Hölder's inequality.

It is clear that if $p = q = 2$ then the constant $C_{\mathbf{a}, \mathbf{b}}$ from Theorem 1 is equal to $A/b + B/a \geq a/b + b/a \geq 2$. Now we obtain a more precise constant for the case when the sequences \mathbf{a} and \mathbf{b} satisfy some additional conditions.

PROPOSITION 1. *Under the assumptions on the sequences \mathbf{a} and \mathbf{b} of Theorem 1, if moreover the sequence $\{a_k/b_k\}$ is monotone,*

$$0 \leq \frac{\sum_{k=1}^n (-1)^{k+1} a_k^2 \sum_{k=1}^n (-1)^{k+1} b_k^2}{(\sum_{k=1}^n (-1)^{k+1} a_k b_k)^2} \leq c_{\mathbf{a}, \mathbf{b}}^2, \quad (11)$$

where $c_{\mathbf{a}, \mathbf{b}} = \frac{1}{2} \max\{A/a + a/A; B/b + b/B\}$ and $c_{\mathbf{a}, \mathbf{b}} \in [1; \infty)$. The left hand side of (11) should be read as there exists no positive constant, depending on a, A, b or B , which bounds the fraction in (11) from below.

Proof. The left hand side inequality follows by the same method as in the proof of Theorem 1. Note that one can also find related examples on non-existence of a direct analogue of Cauchy's inequality for sums with alternating signs in [5].

Now we prove the right hand side of (11). Since (11) is linear homogeneous, we may consider the sequences $\mathbf{a}' = \mathbf{a}/A$ and $\mathbf{b}' = \mathbf{b}/B$ such that

$$0 < a/A \leq a'_k \leq 1, \quad 0 < b/B \leq b'_k \leq 1, \quad k = 1, \dots, n,$$

instead of \mathbf{a} and \mathbf{b} . Let N_C denote the numerator of the fraction in (11) for \mathbf{a}' and \mathbf{b}' . Applying (4) with $p = q = 2$ yields

$$\begin{aligned} N_C &\leq \frac{1}{4} \left(\sum_{k=1}^n (-1)^{k+1} (a_k'^2 + b_k'^2) \right)^2 \\ &= \frac{1}{4} \left(\sum_{k=1}^n (-1)^{k+1} \left(\frac{a_k'}{b_k'} + \frac{b_k'}{a_k'} \right) a_k' b_k' \right)^2. \end{aligned}$$

In the last expression, the sequence $\{c_k + 1/c_k\}$, where $c_k = a_k'/b_k'$, is non-decreasing. Indeed, $\{a_k/b_k\}$ is monotone (non-increasing or non-decreasing), consequently, $\{c_k\}$ is also monotone and moreover $c_1 = 1$. Since $f(x) = x + 1/x$ is convex for $x \in (0; \infty)$ and has a minimum at $x = 1$, the sequence $\{f(c_k)\} = \{c_k + 1/c_k\}$ is non-decreasing. From this by Lemma, we thus obtain

$$N_C \leq \frac{1}{4} (\max \{f(c_k)\})^2 \left(\sum_{k=1}^n (-1)^{k+1} a_k' b_k' \right)^2.$$

By convexity, $\max_{x \in [x_1, x_2]} f(x) = \max \{f(x_1), f(x_2)\}$ for each segment $x \in [x_1, x_2]$, $0 < x_1 \leq x_2 < \infty$. Hence, on account of the properties of \mathbf{a}' and \mathbf{b}' , we have $a/A \leq c_k \leq B/b$ and consequently

$$\max \{f(c_k)\} = \max \{a/A + A/a; B/b + b/B\}.$$

This implies the right hand side of (11). It is easily seen that equality holds if both sequences are constant. The fact that $c_{\mathbf{a}, \mathbf{b}} \geq 1$ is obvious. \square

The inequality (11) is an analogue of the following inequality for non-negative terms, where the left hand side is just Cauchy's inequality and the right hand side is a particular case of a general result due to Zhuang.

THEOREM 3. (Cauchy (1821), Zhuang [20]) *Let \mathbf{a} and \mathbf{b} be positive and such that $0 < a \leq a_k \leq A < \infty$ and $0 < b \leq b_k \leq B < \infty$ for $k = 1, \dots, n$, then*

$$1 \leq \frac{\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2}{(\sum_{k=1}^n a_k b_k)^2} \leq \zeta_{\mathbf{a}, \mathbf{b}},$$

where $\zeta_{\mathbf{a}, \mathbf{b}} = \frac{1}{2} \max (A/b + b/A; a/B + B/a)$ and $\zeta_{\mathbf{a}, \mathbf{b}} \in [1; \infty)$.

REMARK 2. Assume that $\mathbf{a} \equiv 1$, \mathbf{b} is such that $0 < b \leq b_k \leq B < \infty$ for all k , and n is odd in Theorem 1. Then it is clear that under these assumptions Theorems 1 and 1 can be summarized as follows

$$1 \leq \frac{(\sum_{k=1}^{2n+1} (-1)^{k+1} b_k^p)^{1/p}}{\sum_{k=1}^{2n+1} (-1)^{k+1} b_k} \leq (b^{-1} + B^{p-1}), \quad p \geq 1.$$

Let us write this inequality in another form. Suppose $b_k = x_k^r$, $p = R/r$, where R and r are positive integer numbers and $R \geq r$. Raising all expressions to the power $1/r$ and taking into account that $x^{-r} + X^{R-r} \leq x^{-r}(1 + X^R)$ implies

$$1 \leq \frac{(\sum_{k=1}^{2n+1} (-1)^{k+1} x_k^R)^{1/R}}{(\sum_{k=1}^{2n+1} (-1)^{k+1} x_k^r)^{1/r}} \leq \frac{1}{x} (1 + X^R)^{1/r}, \quad r, R \in \mathbb{N}, \quad r \leq R, \quad (12)$$

where $0 < x \leq x_k \leq X < \infty$ for each $k = 1, \dots, 2n+1$. Note that the right hand side of (12) cannot be appreciably improved from the point of view of Remark 1.

Following [18], we explain a geometrical meaning of (12), which has some applications in theory of symmetrization. Let x_k be the radii of concentric spheres in a space of dimension R . Then the value in the numerator is the radius of a single sphere having the total volume contained between the spheres of radius x_1 and x_2 , x_3 and x_4 , etc., and the value in the denominator is the equivalent radius in the same sense in a space of dimension r . Hence (12) states that the fraction of radii of these spheres cannot be small since it is bounded by a constant and can be as large as it is allowed by boundaries for the radii and dimensions r and R .

3. Minkowski type inequalities

In this section, we obtain sharp Minkowski type inequalities with alternating signs and sharp reverse Minkowski's inequality for non-negative terms.

THEOREM 2. Let \mathbf{a} and \mathbf{b} be non-negative non-increasing sequences. Then

$$0 \leq \frac{(\sum_{k=1}^n (-1)^{k+1} a_k^p)^{1/p} + (\sum_{k=1}^n (-1)^{k+1} b_k^p)^{1/p}}{(\sum_{k=1}^n (-1)^{k+1} (a_k + b_k)^p)^{1/p}} \leq 2^{1-1/p}, \quad p \geq 1. \quad (13)$$

The constant $2^{1-1/p}$ is best possible. The left hand side of (13) should be read as there exists no positive constant, depending only on p , which bounds the fraction in (13) from below. The fraction becomes reciprocal if $0 < p < 1$.

Proof. Throughout the proof, F_M denotes the fraction in (13). We begin by proving the left hand side of (13), i.e. by proving of non-existence of positive constant depending on p only, which bounds F_M from below. To prove this, it is sufficient to make the following observation. For each $p > 1$ there exists a sequence such that F_M tends

to zero. Indeed, suppose that $n \geq 2$, $\mathbf{a} = \{1, 1, 0, \dots, 0, \dots\}$ and $\mathbf{b} = \{b, 0, \dots, 0, \dots\}$ with some $b > 0$. From the left hand side of (5), we deduce that

$$F_M = \frac{b}{((1+b)^p - 1)^{1/p}} \leq \frac{b}{(pb)^{1/p}} < b^{1-\frac{1}{p}}.$$

In this way $F_M \rightarrow 0$ as $b \rightarrow 0$ since $1 - 1/p > 0$ for all $p > 1$.

Let us prove the right hand side of (13). From (2) we have

$$\left(\left(\sum_{k=1}^n (-1)^{k+1} a_k^p \right)^{1/p} + \left(\sum_{k=1}^n (-1)^{k+1} b_k^p \right)^{1/p} \right)^p \leq 2^{p-1} \left(\sum_{k=1}^n (-1)^{k+1} (a_k^p + b_k^p) \right).$$

Now it is enough to show that

$$\sum_{k=1}^n (-1)^{k+1} (a_k^p + b_k^p) \leq \sum_{k=1}^n (-1)^{k+1} (a_k + b_k)^p, \quad p \geq 1, \quad (14)$$

and extract the p th root. The inequality (14) is equivalent to

$$\sum_{k=1}^n (-1)^{k+1} ((a_k + b_k)^p - (a_k^p + b_k^p)) \geq 0,$$

which holds since $(a_k + b_k)^p - (a_k^p + b_k^p) \geq (a_{k+1} + b_{k+1})^p - (a_{k+1}^p + b_{k+1}^p)$ for each $k = 1, \dots, n-1$. The latter follows from the implication

$$f(a_k, y) \geq f(a_{k+1}, y), f(x, b_k) \geq f(x, b_{k+1}) \Rightarrow f(a_k, b_k) \geq f(a_{k+1}, b_{k+1}), \quad (15)$$

which is true since the function $f(x, y) = (x + y)^p - (x^p + y^p)$ with $p \geq 1$ is non-decreasing for $x \geq 0$ and $y \geq 0$ as a function of x or of y with a fixed y or x respectively, since $f'_x \geq 0$ and $f'_y \geq 0$. This completes the proof of (14).

The precision of the constant $2^{1-1/p}$ is come out from the following example. Let $\mathbf{a} = \{1, 1, 1, 0, \dots, 0, \dots\}$, $\mathbf{b} = \{b, \sqrt[p]{b^p - 1}, 0, \dots, 0, \dots\}$, $b > 1$, and $n \geq 3$. Then

$$F_M = \frac{2}{((1+b)^p - (1 + \sqrt[p]{b^p - 1})^p + 1)^{1/p}} = 2^{1-1/p} - \varepsilon_b,$$

where ε_b is positive and $\lim_{b \rightarrow \infty} \varepsilon_b = 0$ since $(1+b)^p - (1 + \sqrt[p]{b^p - 1})^p = 1 + o(1)$, $o(1) > 0$, as $b \rightarrow \infty$.

For $0 < p < 1$ the fraction F_M should be replaced by $1/F_M$. Indeed, we then use (3) instead of (2) and reversed version of (14) since $f(x, y) = (x^p + y^p) - (x + y)^p$, $0 < p < 1$, is non-decreasing for $x \geq 0$ and $y \geq 0$ as a function of x or of y , hence the implication (15) is still valid for this function. \square

REMARK 3. We leave it to the reader to verify that (13) has a weighed version if put $a_k = w_k^{1/p} \alpha_k$ and $b_k = w_k^{1/p} \beta_k$, where sequences $\{\alpha_k\}$, $\{\beta_k\}$ and weights $\{w_k\}$ are non-increasing and moreover $w_1 \leq 1$.

REMARK 4. By Theorem 2, for $0 < p < 1$ we have

$$\left(\sum_{k=1}^n (-1)^{k+1} (a_k + b_k)^p \right)^{\frac{1}{p}} \leq 2^{\frac{1}{p}-1} \left(\left(\sum_{k=1}^n (-1)^{k+1} a_k^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n (-1)^{k+1} b_k^p \right)^{\frac{1}{p}} \right),$$

where the constant is sharp. Because of this inequality, it is reasonable to suppose that a real non-negative functional

$$\|\mathbf{x}\| = \left(\sum_{k=1}^n (-1)^{k+1} |x_k|^p \right)^{1/p}, \quad 0 < p < 1,$$

is a quasi-norm on an appropriate (vector) space E of decreasing non-negative sequences $\mathbf{x} = \{x_k\}_{k=1}^n$ (including $\mathbf{x} = 0$) since it satisfies the axioms:

1. $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = 0$;
2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in E$;
3. there exist $K > 0$ such that $\|\mathbf{x} + \mathbf{y}\| \leq K(\|\mathbf{x}\| + \|\mathbf{y}\|)$ for all $\mathbf{x}, \mathbf{y} \in E$ (in our case $K = 2^{1/p-1} \geq 1$).

However, the space E equipped with regular operations of vector addition and scalar multiplication is not a vector space (the axiom on existence of an additive inverse element is not satisfied) but just a cone for which quasi-norms are usually defined in a different way and with $K = 1$ (see, e.g., [17]).

Using the same technics as in the proof of Theorem 2, we can easily obtain reverse Minkowski’s inequality with a sharp constant being independent of the sequences.

PROPOSITION 2. Let \mathbf{a} and \mathbf{b} be non-negative and belong to l_p , $p \geq 1$, then

$$1 \leq \frac{\left(\sum_{k=1}^{\infty} a_k^p \right)^{1/p} + \left(\sum_{k=1}^{\infty} b_k^p \right)^{1/p}}{\left(\sum_{k=1}^{\infty} (a_k + b_k)^p \right)^{1/p}} \leq 2^{1-1/p}, \quad p \geq 1. \tag{16}$$

Both constants are best possible.

Proof. The left hand side of (16) is just Minkowski’s inequality. Now we prove the right hand side. From (2) it follows that

$$\left(\left(\sum_{k=1}^{\infty} a_k^p \right)^{1/p} + \left(\sum_{k=1}^{\infty} b_k^p \right)^{1/p} \right)^p \leq 2^{p-1} \left(\sum_{k=1}^{\infty} (a_k^p + b_k^p) \right).$$

Applying the inequality (6) and extracting the p th root of both sides give (16).

The constant $2^{1-1/p}$ cannot be replaced by smaller one. Indeed, if n is positive integer, then for $\mathbf{a} = \{1, \dots, 1, 0, 0, \dots\}$ (first n elements are units) and $\mathbf{b} = \{n^{1/p}, 0, 0, \dots\}$ we obtain after some simplifications

$$\frac{(\sum_{k=1}^n a_k^p)^{1/p} + (\sum_{k=1}^n b_k^p)^{1/p}}{(\sum_{k=1}^n (a_k + b_k)^p)^{1/p}} = 2 \left(1 - \frac{1}{n} + \left(1 + \frac{1}{n^{1/p}} \right)^p \right)^{-1/p} = 2^{1-1/p} - \varepsilon_n,$$

where positive $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. \square

REMARK 5. Analogously we can obtain integral version of (16):

$$1 \leq \frac{\|f\|_p + \|g\|_p}{\|f + g\|_p} \leq 2^{1-1/p}, \quad p \geq 1, \quad (17)$$

where $f, g > 0$ belong to usual L_p -space with the norm $\|f\|_p = (f^p)^{1/p}$, $p \geq 1$.

To the best of our knowledge, the right hand side of (16) (and (17)) is not contained in common books on inequalities. However it can be also obtained from the following result of theory of quasi-normed spaces:

$$1 \leq \frac{\|g + f\|_p}{\|g\|_p + \|f\|_p} \leq 2^{1/p-1}, \quad g, f \in L_p, \quad 0 < p < 1, \quad (18)$$

where $f, g > 0$ belong to quasi-Banach space L_p with the quasi-norm $\|f\|_p = (f^p)^{1/p}$, $0 < p < 1$ (see, for example, [4, Appendix H], cf. Remark 4). Indeed, it is sufficient to apply to (18) quasi-linearization technics as in [1, §22].

Let us now compare the constant $2^{1-1/p}$ from (16) with ones in [7] and [9], where reverse integral Minkowski's inequalities for positive functions were obtained in terms of boundaries of their quotient. At first we formulate a result from [7].

THEOREM 4. (Bougoffa [7] (a discrete version)) *Let \mathbf{a} and \mathbf{b} be positive and such that $0 < m \leq a_k/b_k \leq M < \infty$, $k = 1, \dots, n$, then*

$$\left(\sum_{k=1}^n a_k^p \right)^{1/p} + \left(\sum_{k=1}^n b_k^p \right)^{1/p} \leq C_{m,M} \left(\sum_{k=1}^n (a_k + b_k)^p \right)^{1/p}, \quad p \geq 1,$$

where $C_{m,M} = 1 + \frac{1}{m+1} - \frac{1}{M+1}$ and $C_{m,M} \in [1; 2)$.

Obviously the right hand side of (16) is more precise for all \mathbf{a} and \mathbf{b} as above if

$$p < \ln 2 / (\ln 2 - \ln C_{m,M}).$$

For example, let m be close to zero and M be large enough, then $C_{m,M} = 2(1 - \varepsilon_{m,M})$, where positive $\varepsilon_{m,M}$ is small. Hence, the constant in the right hand side of (16) is stronger for all $p \in [1; p^*)$, where $p^* := -\ln 2 / \ln(1 - \varepsilon_{m,M})$ can be arbitrarily large for sufficiently small $\varepsilon_{m,M}$.

REMARK 6. Finally, we mention that it would be interesting to obtain inequalities with alternating signs for another monotone type sequences such as convex and general monotone (see, for example, [11]).

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