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ULAM'S TYPE STABILITY OF A FUNCTIONAL EQUATION DERIVING FROM QUADRATIC AND ADDITIVE FUNCTIONS

ABASALT BODAGHI AND SANG OG KIM

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Abstract. In this paper, we continue the investigation of functional equation which is begun by the authors in the first part. We also prove the Hyers-Ulam stability for the following mixed quadratic-additive functional equation in quasi-Banach spaces.

$$f(x+my)+f(x-my) = \begin{cases} 2f(x)-2m^2f(y)+m^2f(2y) & m \text{ is even} \\ f(x+y)+f(x-y)-2(m^2-1)f(y)+(m^2-1)f(2y), & m \text{ is odd.} \end{cases}$$

1. Introduction

We first introduce some basic facts concerning quasi-Banach spaces which are taken from [2] and [19]. Let X be a real linear space. A quasi-norm is a real-valued function on X satisfying the following:

(i) $\|x\| \geqslant 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$;

(iii) There is a constant $M \geqslant 1$ such that $\|x+y\| \leqslant M(\|x\|+\|y\|)$ for all $x,y \in X$.

It is easily verified that the condition (iii) implies that

$$\left\|\sum_{j=1}^{2n} x_j\right\| \leqslant M^n \sum_{j=1}^{2n} \|x_j\| \quad \text{and} \quad \left\|\sum_{j=1}^{2n+1} x_j\right\| \leqslant M^{n+1} \sum_{j=1}^{2n+1} \|x_j\|,$$

for all $n \geqslant 1$ and $x_1, x_2, \ldots, x_{2n+1} \in X$. The pair $(X, \|\cdot\|)$ is called a quasi-normed space if $\|\cdot\|$ is a quasi-norm on X. The smallest possible M is called the *modulus of concavity* of $\|\cdot\|$. A quasi-Banach space is a complete quasi-normed space. A quasi-norm $\|\cdot\|$ is called a p-norm $(0 < p \leqslant 1)$ if $\|x+y\|^p \leqslant \|x\|^p + \|y\|^p$, for all $x,y \in X$. In this case, a quasi-Banach space is called a p-Banach space.

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A functional equation is called *stable* if any approximate solution to the functional equation is near a true solution of that functional equation. In [22], Ulam proposed the stability problem for functional equations concerning the stability of group homomorphisms. In [15], Hyers considered the case of approximate additive mappings in Banach spaces and satisfying the well-known weak Hyers inequality controlled by a positive constant. Bourgin [7] was the next author who treated this problem for additive mappings (see also [1]). In [18], Th. M. Rassias provided a generalization of Hyers' theorem which allows the Cauchy difference to be unbounded. Găvruta then generalized the Rassias' result in [14] for the unbounded Cauchy difference. Subsequently, various approaches to this problem have been studied by a number of authors (see for instance, [3], [4], [5], [8], [9], [10], [11], [12] and [20]).

In [13], Eskandani et al. determined the general solution of the following mixed type additive and quadratic functional equation

$$f(x+2y)+f(x-2y)+8f(y)=2f(x)+4f(2y). \tag{1.1}$$

They investigated the Hyers-Ulam stability of the equation (1.1) in non-Archimedean Banach modules over a unital Banach algebra. In [17], Najati and Moghimi established the general solution of the mixed type additive and quadratic functional equation

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+2f(2x)-2f(x) \tag{1.2}$$

The stability of equation (1.2) in quasi-Banach spaces and in random normed spaces is proved in [17] and [16], respectively.

In this paper we consider the following functional equation which is introduced in [6].

$$f(x+my)+f(x-my)$$
$$=\begin{cases} 2f(x)-2m^2f(y)+m^2f(2y) & m \text{ is even} \\ f(x+y)+f(x-y)-2(m^2-1)f(y)+(m^2-1)f(2y), & m \text{ is odd} \end{cases} \tag{1.3}$$

where m is an integer with $m \neq 0, \pm 1$. It is easy to check that the function $f(x) = ax^2+bx$ is a solution of the functional equation (1.3).

In the current work, we establish the Hyers-Ulam stability problem for the functional equation (1.3) in quasi-Banach spaces.

2. Hyers-Ulam stability of (1.3) in quasi-Banach spaces

In this section, we investigate the generalized Hyers-Ulam stability problem for the functional equation (1.3). Let X be a quasi-Banach space. Given a p-norm, the formula $d(x,y) := \|x-y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz Theorem [19] (see also [2]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms, here and subsequently, we restrict our attention mainly to p-norms. Moreover, Tabor [21] has investigated a version of Hyers-Rassias-Gajda Theorem in quasi-Banach spaces.

Let m be an integer such that with $m \neq 0, \pm 1$. We use the abbreviation for the given mapping $f : X \longrightarrow Y$ as follows:

$$\mathscr{D}_m f(x,y) = \begin{cases} f(x+my) + f(x-my) - 2f(x) + 2m^2 f(y) - m^2 f(2y) & m \text{ is even} \\ f(x+my) + f(x-my) - f(x+y) & \\ \qquad -f(x-y) + 2(m^2-1)f(y) - (m^2-1)f(2y), & m \text{ is odd} \end{cases}$$

for all $x, y \in X$.

We have the following result which is analogous to [6, Theorem 2] for the functional equation (1.3). Since the proof is similar, it is omitted.

THEOREM 2.1. *Let X and Y be real vector spaces. Then a mapping $f : X \longrightarrow Y$ satisfies the functional equation* (1.1) *if and only if it satisfies the functional equation $\mathscr{D}_m f(x,y) = 0$ where m is an integer with $m \neq 0, \pm 1$.*

LEMMA 2.2. *Let X and Y be real vector spaces.*

(i) If an odd function $f : X \longrightarrow Y$ satisfies the functional equation (1.3)*, then f is additive;*

(ii) If an even function $f : X \longrightarrow Y$ satisfies the functional equation (1.3)*, then f is quadratic.*

Proof. The result follows from Theorem 2.1 and [13, Lemma 2.1 and Lemma 2.2]. □

LEMMA 2.3. *Let $0 \leqslant p \leqslant 1$ and let $x_1, x_2, \cdots, x_n$ be non-negative real numbers. Then*

$$\left(\sum_{j=1}^{n} x_j\right)^p \leqslant \sum_{j=1}^{n} x_j^p.$$

From now on, let X be a normed real linear space with norm $\|\cdot\|_X$ and Y be a real p-Banach space with norm $\|\cdot\|_Y$. In this section, by using an idea of Găvruta [14] we prove the stability of (1.3) in the spirit of Hyers, Ulam and Rassias.

THEOREM 2.4. *Let $l \in \{1, -1\}$ be fixed and let $\phi : X \times X \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{1}{2^{klp}} \phi^p(0, 2^{kl}x) < \infty, \lim_{k\to\infty} \frac{1}{2^{kl}} \phi(2^{kl}x, 2^{kl}y) = 0 \tag{2.1}$$

for all $x, y \in X$. Suppose that $f : X \longrightarrow Y$ is an odd mapping satisfying the inequality

$$\|\mathscr{D}_m f(x,y)\|_Y \leqslant \phi(x,y) \tag{2.2}$$

for all $x,y \in X$, where m is an integer with $m \neq 0,\pm 1$. Then there exists a unique additive mapping $A: X \longrightarrow Y$ such that

$$\|f(x)-A(x)\|_Y \leqslant \begin{cases} \frac{1}{2m^2}\left(\widetilde{\phi}_o(x)\right)^{\frac{1}{p}}, & m \text{ is even} \\ \frac{1}{2(m^2-1)}\left(\widetilde{\phi}_o(x)\right)^{\frac{1}{p}}, & m \text{ is odd} \end{cases} \tag{2.3}$$

where

$$\widetilde{\phi}_o(x) := \sum_{k=\frac{|l-1|}{2}}^{\infty} \frac{1}{2^{klp}} \phi^p(0, 2^{kl}x) \tag{2.4}$$

for all $x \in X$.

Proof. Let $l=1$. We prove the result only in the case that m is a non-zero even integer. Replacing (x,y) by $(0,x)$ in (2.2), we have

$$\|2f(x)-f(2x)\|_Y \leqslant \frac{1}{m^2}\phi(0,x) \tag{2.5}$$

for all $x \in X$. Replacing x by $2^n x$ in (2.5) and then dividing both sides by 2^{n+1}, we get

$$\left\|\frac{1}{2^n}f(2^n x) - \frac{1}{2^{n+1}}f(2^{n+1}x)\right\|_Y \leqslant \frac{\phi(0,2^n x)}{m^2 2^{n+1}} \tag{2.6}$$

for all $x \in X$ and all non-negative integers n. Since Y is a p-Banach space,

$$\begin{aligned} \left\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^k x)}{2^k}\right\|_Y^p &= \left\|\sum_{j=k}^{n} \frac{f(2^{j+1}x)}{2^{j+1}} - \frac{f(2^j x)}{2^j}\right\|_Y^p \\ &\leqslant \sum_{j=k}^{n} \left\|\frac{f(2^{j+1}x)}{2^{j+1}} - \frac{f(2^j x)}{2^j}\right\|_Y^p \\ &\leqslant \frac{1}{m^{2p}} \sum_{j=k}^{n} \frac{\phi^p(0,2^j x)}{2^{(j+1)p}}. \end{aligned} \tag{2.7}$$

for all $x \in X$ and all integers $n \geqslant k \geqslant 0$. Thus the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}$ is Cauchy by (2.1) and (2.7). Since Y is complete, this sequence converges for all $x \in X$. So one can define the mapping $A: X \longrightarrow Y$ so that

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = A(x) \qquad (x \in X). \tag{2.8}$$

It follows from (2.1) and (2.8) that

$$\|\mathscr{D}_m A(x,y)\|_Y \leqslant \lim_{n\to\infty} \frac{1}{2^n}\|\mathscr{D}_m f(2^n x, 2^n y)\|_Y \leqslant \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{2^n} = 0.$$

for all $x,y\in X$. Hence, the mapping A satisfies in (1.3). Thus, by the part (i) of Lemma 2.2, this mapping is additive. Putting $k=0$ and letting n go to infinity in (2.7), we see that (2.3) holds when m is non-zero even. For the uniqueness of A, assume that $\mathscr{A}:X\longrightarrow Y$ is another additive mapping that satisfies (2.3). Then

$$\begin{aligned}\|A(x)-\mathscr{A}(x)\|_Y^p &= \lim_{n\to\infty}\frac{1}{2^{np}}\|f(2^nx)-\mathscr{A}(2^nx)\|_Y^p\\ &\leqslant \lim_{n\to\infty}\frac{1}{2^{(n+1)p}m^{2p}}\sum_{k=0}^{\infty}\frac{1}{2^{kp}}\phi^p(0,2^{k+n}x)\\ &=\frac{1}{m^{2p}2^p}\lim_{n\to\infty}\sum_{k=n}^{\infty}\frac{1}{2^{kp}}\phi^p(0,2^kx),\end{aligned}$$

for all $x\in X$. The above relations and (2.1) imply that $A=\mathscr{A}$. Similar to the above considerations, we can obtain the result for the odd case. For $l=-1$, we obtain

$$\left\|f(x)-2^nf\left(\frac{x}{2^n}\right)\right\|_Y^p\leqslant\frac{2}{2^pm^2}\sum_{j=1}^{n}2^{jp}\phi^p\left(0,\frac{x}{2^j}\right)\tag{2.9}$$

from which one can prove the result by a similar technique. □

COROLLARY 2.5. *Let α,r,s,p and q be non-negative real numbers such that $p+q\neq 1\neq r,s$. Suppose that $f:X\longrightarrow Y$ is an odd mapping fulfilling*

$$\|\mathscr{D}_mf(x,y)\|_Y\leqslant\alpha(\|x\|_X^p\|y\|_X^q+\|x\|_X^r+\|y\|_X^s)\tag{2.10}$$

for all $x,y\in X$, where m is an integer with $m\neq 0,\pm1$. Then there exists a unique additive mapping $A:X\longrightarrow Y$ such that

$$\|f(x)-A(x)\|\leqslant\begin{cases}\dfrac{\alpha\|x\|_X^s}{m^2\sqrt[p]{|2^p-2^{sp}|}}, & m \text{ is even}\\[2ex] \dfrac{\alpha\|x\|_X^s}{(m^2-1)\sqrt[p]{|2^p-2^{sp}|}}, & m \text{ is odd}\end{cases}\tag{2.11}$$

for all $x\in X$.

Proof. The result follows from Theorem 2.4 by letting $\phi(x,y)=\alpha(\|x\|_X^p\|y\|_X^q+\|x\|_X^r+\|y\|_X^s)$. □

In analogy with Theorem 2.4 we have the following result for the stability of functional equation (1.3) when f is an even mapping.

THEOREM 2.6. *Let $l\in\{1,-1\}$ be fixed and let $\phi:X\times X\longrightarrow[0,\infty)$ be a function such that*

$$\sum_{k=0}^{\infty}\frac{1}{4^{klp}}\phi^p(2^{kl}x,2^{kl}y)<\infty,\ \lim_{k\to\infty}\frac{1}{4^{kl}}\phi(2^{kl}x,2^{kl}y)=0\tag{2.12}$$

for all $x,y \in X$. Suppose that $f : X \longrightarrow Y$ is an even mapping with $f(0) = 0$ satisfying the inequality

$$\|\mathscr{D}_m f(x,y)\|_Y \leqslant \phi(x,y) \tag{2.13}$$

for all $x,y \in X$, where m is an even integer with $m \neq 0$. Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ such that

$$\|f(x) - Q(x)\|_Y \leqslant M\left[\widetilde{\phi}_{\mathrm{e}}(x)\right]^{\frac{1}{p}} \tag{2.14}$$

where

$$\widetilde{\phi}_{\mathrm{e}}(x) := \sum_{k=\frac{|l-1|}{2}}^{\infty} \frac{1}{4^{klp}} \Phi^p(2^{kl}x) \tag{2.15}$$

in which $\Phi(x) = \frac{1}{4}\left[\phi(0,\frac{x}{m}) + \phi(x,\frac{x}{m})\right]$ for all $x \in X$.

Proof. Let $l = 1$. Putting $x = 0$ in (2.13) and interchanging y into x, we have

$$\|2f(mx) + 2m^2 f(x) - m^2 f(2x)\|_Y \leqslant \phi(0,x) \tag{2.16}$$

for all $x \in X$. Substituting x,y by mx,x in (2.13), respectively, we get

$$\|f(2mx) - 2f(mx) + 2m^2 f(x) - m^2 f(2x)\|_Y \leqslant \phi(mx,x) \tag{2.17}$$

for all $x \in X$. It follows from (2.16) and (2.17) that

$$\|f(2mx) - 4f(mx)\|_Y \leqslant M\left[\phi(0,x) + \phi(mx,x)\right]$$

for all $x \in X$. Thus we have

$$\left\|\frac{f(2x)}{4} - f(x)\right\|_Y \leqslant M\Phi(x) \tag{2.18}$$

for all $x \in X$ and all non-negative integers n for which

$$\Phi(x) = \frac{1}{4}\left[\phi\left(0,\frac{x}{m}\right) + \phi\left(x,\frac{x}{m}\right)\right] \quad (x \in X). \tag{2.19}$$

Replacing x by $2^n x$ in (2.18) and then dividing both sides by 4^n, we get

$$\left\|\frac{1}{4^{n+1}} f(2^{n+1}x) - \frac{1}{4^n} f(2^n x)\right\|_Y \leqslant \frac{M}{4^n}\Phi(2^n x) \tag{2.20}$$

for all $x \in X$ and all non-negative integers n. Since Y is a p-Banach space,

$$\begin{aligned}
\left\|\frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^k x)}{4^k}\right\|_Y^p &= \left\|\sum_{j=k}^{n} \frac{f(2^{j+1}x)}{4^{j+1}} - \frac{f(2^j x)}{4^j}\right\|_Y^p \\
&\leqslant \sum_{j=k}^{n} \left\|\frac{f(2^{j+1}x)}{4^{j+1}} - \frac{f(2^j x)}{4^j}\right\|_Y^p \\
&\leqslant M^p \sum_{j=k}^{n} \frac{\Phi^p(2^j x)}{4^{jp}}.
\end{aligned} \tag{2.21}$$

for all $x \in X$ and all integers $n \geqslant k \geqslant 0$. Now since $0 < p \leqslant 1$, by Lemma 2.3 we deduce from (2.19) that

$$\Phi^p(x) \leqslant \frac{1}{4^p}\left[\phi^p\left(0, \frac{x}{m}\right) + \phi^p\left(x, \frac{x}{m}\right)\right] \tag{2.22}$$

for all $x \in X$. Therefore it follows from (2.12) and (2.22) that $\sum_{j=1}^{\infty} \frac{\Phi^p(2^j x)}{4^{jp}} < \infty$ for all $x \in X$. The last inequality and (2.21) imply that $\left\{\frac{f(2^n x)}{4^n}\right\}$ is a Cauchy sequence. Due to the completeness of Y, the sequence $\left\{\frac{f(2^n x)}{4^n}\right\}$ is convergent to the mapping $Q : X \longrightarrow Y$, that is

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \qquad (x \in X). \tag{2.23}$$

Letting $k = 0$ and passing to the limit $n \longrightarrow \infty$ in (2.21), we get

$$\|Q(x) - f(x)\|_Y^p \leqslant M^p \sum_{j=0}^{\infty} \frac{\Phi^p(2^j x)}{4^{jp}}. \tag{2.24}$$

for all $x \in X$. Therefore (2.14) follows from (2.22) and (2.24) when m is even. Now, we show that Q is quadratic. Employing (2.12), (2.13) and (2.23), we obtain

$$\|\mathscr{D}_m Q(x,y)\|_Y = \lim_{n\to\infty} \frac{1}{4^n} \|\mathscr{D}_m f(2^n x, 2^n y)\|_Y \leqslant \lim_{n\to\infty} \frac{1}{4^n} \phi(2^n x, 2^n y) = 0.$$

Hence, the mapping Q satisfies (1.3). It follows from the part (ii) of Lemma 2.2 that the mapping Q is quadratic. Since $\lim_{n\to\infty} \sum_{k=n}^{\infty} \frac{1}{4^{kp}} \Phi^p(2^k x) = 0$, the proof of the the uniqueness of Q is similar to the proof of Theorem 2.4. For $l = -1$, one can deduce that

$$\left\|f(x) - 4^n f\left(\frac{x}{2^n}\right)\right\|_Y^p \leqslant M^p \sum_{j=1}^{n} 4^{jp} \Phi^p\left(\frac{x}{2^j}\right) \tag{2.25}$$

for all $x \in X$. The above process can be repeated to get the result. □

COROLLARY 2.7. *Let α and s be non-negative real numbers such that $s \neq 2$. Suppose that $f : X \longrightarrow Y$ is an even mapping fulfilling*

$$\|\mathscr{D}_m f(x,y)\|_Y \leqslant \alpha(\|x\|_X^s + \|y\|_X^s)$$

for all $x, y \in X$, where m is an even integer with $m \neq 0$. Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ such that

$$\|f(x) - Q(x)\|_Y \leqslant M\left(1 + \frac{2}{|m|^s}\right) \frac{\alpha \|x\|_X^s}{\sqrt[p]{|4^p - 2^{sp}|}}$$

for all $x \in X$.

Proof. Defining $\phi(x,y) = \alpha(\|x\|_X^s + \|y\|_X^s)$ and applying Theorem 2.6, one can obtain the desired result. $\square$

THEOREM 2.8. *Let $l \in \{1,-1\}$ be fixed and let $\phi : X \times X \longrightarrow [0,\infty)$ be a function satisfying*

$$\sum_{k=0}^{\infty} \frac{1}{m^{2klp}} \phi^p(m^{kl}x, m^{kl}y) < \infty, \lim_{k\to\infty} \frac{1}{m^{2kl}} \phi(m^{kl}x, m^{kl}y) = 0 \tag{2.26}$$

for all $x,y \in X$. Suppose that $f : X \longrightarrow Y$ is an even mapping with $f(0) = 0$ satisfying the inequality (2.13) for all $x,y \in X$, where m is an odd integer with $m \neq \pm 1$. Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ such that

$$\|f(2x) - 2f(x) - Q(x)\|_Y \leqslant \frac{M^2}{m^2} \left[\sum_{j=\frac{|l-1|}{2}}^{\infty} \frac{\Psi^p(m^j x)}{m^{2jp}} \right]^{\frac{1}{p}} \tag{2.27}$$

where the mappings $\mathcal{Q}(x)$ and $\Psi(x)$ are defined by

$$\mathcal{Q}(x) = \lim_{n\to\infty} \frac{1}{m^{2n}} \{f(2m^n x) - 2f(m^n x)\}$$

and

$$\Psi(x) = [\phi(0,x) + \phi(x,x) + \phi(mx,x)] \tag{2.28}$$

for all $x \in X$.

Proof. We bring the details only for the case $l = 1$. Other case is similar. Replacing (x,y) by $(0,x)$ in (2.13), we have

$$\|2f(mx) + 2(m^2-2)f(x) - (m^2-1)f(2x)\|_Y \leqslant \phi(0,x) \tag{2.29}$$

for all $x \in X$. Putting $x = y$ by in (2.13), we get

$$\|f((m+1)x) + f((m-1)x) + 2(m^2-1)f(x) - m^2 f(2x)\|_Y \leqslant \phi(x,x) \tag{2.30}$$

for all $x \in X$. Once more, interchanging (x,y) into (mx,x) in (2.13), we have

$$\|f(2mx) - f((m+1)x) - f((m-1)x) + 2(m^2-1)f(x) - (m^2-1)f(2x)\|_Y \leqslant \phi(mx,x) \tag{2.31}$$

for all $x \in X$. It follows from (2.29), (2.30) and (2.31) that

$$\|f(2mx) - 2f(mx) + 2m^2 f(x) - m^2 f(2x)\|_Y \leqslant M^2 [\phi(0,x) + \phi(x,x) + \phi(mx,x)]$$

for all $x \in X$. The above relation implies that

$$\left\| \frac{g(mx)}{m^2} - g(x) \right\|_Y \leqslant \frac{M^2}{m^2} \Psi(x) \qquad (x \in X) \tag{2.32}$$

where $g(x) = f(2x) - 2f(x)$ and $\Psi(x) = [\phi(0,x) + \phi(x,x) + \phi(mx,x)]$ for all $x \in X$. In general for any positive integer n, we get

$$\left\| \frac{1}{m^{2n}} g(m^n x) - g(x) \right\|_Y^p \leqslant \frac{M^{2p}}{m^{2p}} \sum_{j=0}^{n-1} \frac{\Psi^p(m^j x)}{m^{2jp}} \tag{2.33}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{ \frac{g(m^n x)}{m^{2n}} \right\}$, replace x by $m^k x$ and divide by m^{2k} in (2.33). For any $k, n > 0$, we have

$$\left\| \frac{g(m^{n+k} x)}{m^{2(n+k)}} - \frac{g(m^k x)}{m^{2k}} \right\|_Y^p \leqslant \frac{M^{2p}}{m^{2p}} \sum_{j=0}^{n-1} \frac{\Psi^p(m^{j+k} x)}{m^{2(j+k)p}} \tag{2.34}$$

for all $x \in X$. Similar to the first part the right hand side of the inequality (2.34) tends to 0 as k tends to infinity. Thus the sequence $\left\{ \frac{g(m^n x)}{m^{2n}} \right\}$ is Cauchy. The completeness of Y allows us to assume that there exists a map $\mathscr{Q} : X \longrightarrow Y$ such that

$$\mathscr{Q}(x) = \lim_{n\to\infty} \frac{g(m^n x)}{m^{2n}} \quad (x \in X). \tag{2.35}$$

Letting $n \to \infty$ in (2.33) and using (2.35), we see that (2.27) holds when m is an odd integer with $m \neq \pm 1$. On the other hand it follows from (2.13), (2.26) and (2.35) that

$$\begin{aligned}
\frac{1}{m^{2n}} \|\mathscr{D}_m g(m^n x, m^n y)\|_Y &= \frac{1}{m^{2n}} \|\mathscr{D}_m f(2m^n x, 2m^n y) - 2\mathscr{D}_m f(m^n x, m^n y)\|_Y \\
&\leqslant \frac{M}{m^{2n}} \|\mathscr{D}_m f(2m^n x, 2m^n y)\|_Y + \frac{2M}{m^{2n}} \|\mathscr{D}_m f(m^n x, m^n y)\|_Y \\
&\leqslant M \frac{\phi(2m^n x, 2m^n y)}{m^{2n}} + 2M \frac{\phi(m^n x, m^n y)}{m^{2n}}
\end{aligned}$$

for all $x, y \in X$. Taking $n \to \infty$ in the above inequality and using (2.26), we observe that $\mathscr{D}_m \mathscr{Q}(x,y) = 0$ for all $x, y \in X$. Therefore, by the part (ii) of Lemma 2.2, $\mathscr{Q}$ is a quadratic mapping. The rest of the proof is similar to the proof of Theorem 2.6. □

COROLLARY 2.9. *Let α and s be non-negative real numbers such that $s \neq 2$. Suppose that $f : X \longrightarrow Y$ is an even mapping fulfilling*

$$\|\mathscr{D}_m f(x,y)\|_Y \leqslant \alpha(\|x\|_X^s + \|y\|_X^s)$$

for all $x, y \in X$, where m is an odd integer with $m \neq \pm 1$. Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ such that

$$\|f(2x) - 2f(x) - Q(x)\|_Y \leqslant M^2 (4 + |m|^s) \frac{\alpha \|x\|_X^s}{\sqrt[p]{|m^{2p} - m^{sp}|}}$$

for all $x \in X$.

THEOREM 2.10. *Let $l \in \{1,-1\}$ be fixed and let $\phi : X \times X \longrightarrow [0,\infty)$ be a function satisfying (2.1) and (2.12) for all $x,y \in X$. Suppose that $f : X \longrightarrow Y$ is a mapping with $f(0) = 0$ satisfying the inequality*

$$\|\mathscr{D}_m f(x,y)\|_Y \leqslant \phi(x,y) \tag{2.36}$$

for all $x,y \in X$, where m is a non-zero even integer. Then there exist a unique additive mapping $A : X \longrightarrow Y$ and a unique quadratic mapping $Q : X \longrightarrow Y$ such that

$$\|f(x) - A(x) - Q(x)\| \leqslant \frac{M}{4m^2}\Big(\widetilde{\phi}_{\mathrm{o}}(x) + \widetilde{\phi}_{\mathrm{o}}(-x)\Big)^{\frac{1}{p}} + \frac{M^2}{2}\Big(\widetilde{\phi}_{\mathrm{e}}(x) + \widetilde{\phi}_{\mathrm{e}}(-x)\Big)^{\frac{1}{p}} \tag{2.37}$$

for all $x \in X$, where $\widetilde{\phi}_{\mathrm{o}}(x)$ and $\widetilde{\phi}_{\mathrm{e}}(x)$ are defined in (2.4) and (2.15), respectively.

Proof. We decompose f into the even part and odd part by setting

$$f_{\mathrm{e}}(x) = \frac{f(x) + f(-x)}{2}, \qquad f_{\mathrm{o}}(x) = \frac{f(x) - f(-x)}{2}.$$

Obviously, $f(x) = f_{\mathrm{e}}(x) + f_{\mathrm{o}}(x)$ and $f_{\mathrm{e}}(0) = 0$ for all $x \in X$. Then

$$\|\mathscr{D}_m f_{\mathrm{e}}(x,y)\|_Y, \|\mathscr{D}_m f_{\mathrm{o}}(x,y)\|_Y \leqslant \Psi(x,y)$$

where $\Psi(x,y) = \frac{M}{2}\Big(\phi(x,y) + \phi(-x,-y)\Big)$ for all $x \in X$. So

$$\lim_{k\to\infty} \frac{1}{2^{kl}} \Psi(2^{kl}x, 2^{kl}y) = 0 \text{ and } \lim_{k\to\infty} \frac{1}{4^{kl}} \Psi(2^{kl}x, 2^{kl}y) = 0 \tag{2.38}$$

for all $x,y \in X$. Since

$$\Psi^p(x,y) \leqslant \frac{M^p}{2^p}\Big(\phi^p(x,y) + \phi^p(-x,-y)\Big) \qquad (x,y \in X) \tag{2.39}$$

we have

$$\sum_{k=0}^{\infty} \frac{1}{2^{klp}} \Psi^p(0, 2^{kl}x) < \infty \text{ and } \sum_{k=0}^{\infty} \frac{1}{4^{klp}} \Psi^p(2^{kl}x, 2^{kl}y) < \infty \tag{2.40}$$

for all $x,y \in X$. Hence, in view of Theorems 2.4 and 2.6, there exists a unique additive mapping $A : X \longrightarrow Y$ and a unique quadratic mapping $Q : X \longrightarrow Y$ such that

$$\|f_{\mathrm{o}}(x) - A(x)\|_Y \leqslant \frac{1}{2m^2}\Big[\widetilde{\Phi}_{\mathrm{o}}(x)\Big]^{\frac{1}{p}} \text{ and } \|f_{\mathrm{e}}(x) - Q(x)\|_Y \leqslant M\Big[\widetilde{\Phi}_{\mathrm{e}}(x)\Big]^{\frac{1}{p}} \tag{2.41}$$

where $\widetilde{\Phi}_{\mathrm{o}}(x) := \sum_{k=\frac{|l-1|}{2}}^{\infty} \frac{1}{2^{klp}} \Psi^p(0, 2^{kl}x)$ and $\widetilde{\Phi}_{\mathrm{e}}(x) := \sum_{k=\frac{|l-1|}{2}}^{\infty} \frac{1}{4^{klp}} \Psi_{\mathrm{e}}^p(2^{kl}x)$ for all $x \in X$. We also have

$$\widetilde{\Phi}_{\mathrm{o}}(x) \leqslant \frac{M^p}{2^p}\Big(\widetilde{\phi}_{\mathrm{o}}(x) + \widetilde{\phi}_{\mathrm{o}}(-x)\Big) \text{ and } \widetilde{\Phi}_{\mathrm{e}}(x) \leqslant \frac{M^p}{2^p}\Big(\widetilde{\phi}_{\mathrm{e}}(x) + \widetilde{\phi}_{\mathrm{e}}(-x)\Big)$$

for all $x \in X$. Hence, the relations in (2.41) imply that

$$\|f_{\mathrm{o}}(x) - A(x)\|_Y \leqslant \frac{M}{4m^2}\left(\widetilde{\phi}_{\mathrm{o}}(x) + \widetilde{\phi}_{\mathrm{o}}(-x)\right)^{\frac{1}{p}} \tag{2.42}$$

and

$$\|f_{\mathrm{e}}(x) - Q(x)\|_Y \leqslant \frac{M^2}{2}\left(\widetilde{\phi}_{\mathrm{e}}(x) + \widetilde{\phi}_{\mathrm{e}}(-x)\right)^{\frac{1}{p}} \tag{2.43}$$

for all $x \in X$. Therefore (2.37) follows from (2.42) and (2.43). □

COROLLARY 2.11. *Let α and s be nonnegative real numbers such that $s \neq 1,2$. Suppose that $f : X \longrightarrow Y$ is a mapping fulfilling*

$$\|\mathscr{D}_m f(x,y)\|_Y \leqslant \alpha(\|x\|_X^s + \|y\|_X^s)$$

for all $x, y \in X$, where m is an even integer with $m \neq 0$. Then there exist a unique additive mapping $A : X \longrightarrow Y$ and a unique quadratic mapping $Q : X \longrightarrow Y$ such that

$$\|f(x) - A(x) - Q(x)\|_Y \leqslant \left[\frac{M}{2m^2}\sqrt[p]{\frac{2}{|2^p - 2^{sp}|}} + \frac{M^2}{2}\left(1 + \frac{2}{|m|^s}\right)\sqrt[p]{\frac{2}{|2^{2p} - 2^{sp}|}}\right]\alpha\|x\|_X^s$$

for all $x \in X$.

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Abasalt Bodaghi
Department of Mathematics, Garmsar Branch
Islamic Azad University
Garmsar, Iran
e-mail: abasalt.bodaghi@gmail.com

Sang Og Kim
Department of Mathematics, Hallym University
Chuncheon 200-702, Republic of Korea
e-mail: sokim@hallym.ac.kr