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SOME GENERALIZATIONS OF OPERATOR INEQUALITIES

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Abstract. In this paper, we generalize some operator inequalities as follows: Let A, A_i $(i=1,\dots,n)$ be positive operators on a Hilbert space with $0<m\leqslant A,\ A_i\leqslant M$ $(i=1,\dots,n)$. Then for $1\leqslant p<\infty$ and every positive unital linear map Φ,

$$\Phi^p(A^{-1})\Phi^p(A)+\Phi^p(A)\Phi^p(A^{-1})\leqslant\frac{(M+m)^{2p}}{2M^pm^p},$$

and

$$\left(\frac{A_1+\cdots+A_n}{n}\right)^{2p}\leqslant\left(\frac{(M+m)^{2p}}{4M^pm^p}\right)^2G^{2p}(A_1,\dots,A_n),$$

where $G(A_1,\dots,A_n)$ is Ando-Li-Mathias geometric mean [1].

1. Introduction

Let M,m be scalars and I be the identity operator. Other capital letters are used to denote the general elements of the C^* -algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators acting on a Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle)$. The operator norm is defined by $\|\cdot\|$. We write $A\geqslant 0$ to mean that the operator A is positive. If $A-B\geqslant 0$ $(A-B\leqslant 0)$, then we say that $A\geqslant B$ $(A\leqslant B)$. A^* stands for the adjoint of A. We denote the absolute value operator of A by $|A|$, that is, $|A|=(A^*A)^{\frac{1}{2}}$.

A linear map Φ is positive if $\Phi(A)\geqslant 0$ whenever $A\geqslant 0$. It is said to be unital if $\Phi(I)=I$.

For $A,B>0$, the geometric mean $A\sharp B$ is defined by

$$A\sharp B=A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

We denote the Ando-Li-Mathias geometric mean [1] for $A_1,\dots,A_n>0$ by $G(A_1,\dots,A_n)$. There is no explicit formula for $G(A_1,\dots,A_n)$ in terms of $A_1,\dots,A_n$ when $n\geqslant 3$. However, the only basic property that we need is

$$G(A_1,\dots,A_n)\leqslant\frac{A_1+\cdots+A_n}{n}.$$

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It is well known that for two positive operator A, B,

$$A \geqslant B \nRightarrow A^p \geqslant B^p \qquad \text{for} \quad p > 1.$$

Let $0 < m \leqslant A \leqslant M$ and Φ be positive unital linear map. Lin [5, Theorem 2.10.] proved the following operator inequalities:

$$|\Phi(A^{-1})\Phi(A) + \Phi(A)\Phi(A^{-1})| \leqslant \frac{(M+m)^2}{2Mm} \tag{1.1}$$

and

$$\Phi(A^{-1})\Phi(A) + \Phi(A)\Phi(A^{-1}) \leqslant \frac{(M+m)^2}{2Mm}. \tag{1.2}$$

Let $A_1, A_2, \ldots, A_n$ be positive operators on a Hilbert space with $0 < m \; A_i \leqslant M \;\; (i = 1, \ldots, n)$. Fujii et al. [4, (12)] showed a reverse arithmetic-geometric mean inequality of operators

$$\frac{A_1 + \cdots + A_n}{n} \leqslant \frac{(M+m)^2}{4Mm} G(A_1, \cdots, A_n). \tag{1.3}$$

To our surprise, Lin [6, Theorem 3.2] showed that the reverse AM-GM inequality (1.3) can be squared:

$$\left(\frac{A_1 + \cdots + A_n}{n}\right)^2 \leqslant \left(\frac{(M+m)^2}{4Mm}\right)^2 G^2(A_1, \ldots, A_n). \tag{1.4}$$

In this paper, we will present some operator inequalities which are generalizations of (1.1), (1.2) and (1.4).

2. Main results

We give some Lemmas before we give the main theorems of this paper:

LEMMA 1. ([3]) *Let $A, B > 0$. Then the following norm inequality holds:*

$$\|AB\| \leqslant \frac{1}{4}\|A+B\|^2. \tag{2.1}$$

LEMMA 2. ([2]) *Let A and B be positive operators. Then for $1 \leqslant r < \infty$*

$$\|A^r + B^r\| \leqslant \|(A+B)^r\|. \tag{2.2}$$

LEMMA 3. ([5]) *For any bounded operator X,*

$$|X| \leqslant tI \Leftrightarrow \|X\| \leqslant t \Leftrightarrow \begin{bmatrix} tI & X \\ X^* & tI \end{bmatrix} \geqslant 0. \tag{2.3}$$

Now we present the first main result in the following theorem.

THEOREM 4. *Let $A \geqslant 0$ and $p \geqslant 1$. Then for every positive unital linear map Φ,*

$$|\Phi^p(A^{-1})\Phi^p(A) + \Phi^p(A)\Phi^p(A^{-1})| \leqslant \frac{(M+m)^{2p}}{2M^p m^p} \tag{2.4}$$

and

$$\Phi^p(A^{-1})\Phi^p(A) + \Phi^p(A)\Phi^p(A^{-1}) \leqslant \frac{(M+m)^{2p}}{2M^p m^p}. \tag{2.5}$$

Proof. Compute

$$\begin{aligned}
\|\Phi^p(A)M^p m^p \Phi^p(A^{-1})\| &\leqslant \frac{1}{4}\|\Phi^p(A) + M^p m^p \Phi^p(A^{-1})\|^2 && \text{(by (2.1))} \\
&\leqslant \frac{1}{4}\|\Phi(A) + Mm\Phi(A^{-1})\|^{2p} && \text{(by (2.2))} \\
&\leqslant \frac{1}{4}(M+m)^{2p}. && \text{(by [6, (2.3)])}
\end{aligned}$$

So

$$\|\Phi^p(A)\Phi^p(A^{-1})\| \leqslant \frac{(M+m)^{2p}}{4M^p m^p}. \tag{2.6}$$

By (2.6) and Lemma 3, we obtain

$$\begin{bmatrix} \frac{(M+m)^{2p}}{4M^p m^p} I & \Phi^p(A)\Phi^p(A^{-1}) \\ \Phi^p(A^{-1})\Phi^p(A) & \frac{(M+m)^{2p}}{4M^p m^p} I \end{bmatrix} \geqslant 0,$$

and

$$\begin{bmatrix} \frac{(M+m)^{2p}}{4M^p m^p} I & \Phi^p(A^{-1})\Phi^p(A) \\ \Phi^p(A)\Phi^p(A^{-1}) & \frac{(M+m)^{2p}}{4M^p m^p} I \end{bmatrix} \geqslant 0.$$

Summing up these two operator matrices, we have

$$\begin{bmatrix} \frac{(M+m)^{2p}}{2M^p m^p} I & \Phi^p(A^{-1})\Phi^p(A) + \Phi^p(A)\Phi^p(A^{-1}) \\ \Phi^p(A)\Phi^p(A^{-1}) + \Phi^p(A^{-1})\Phi^p(A) & \frac{(M+m)^{2p}}{2M^p m^p} I \end{bmatrix} \geqslant 0.$$

By Lemma 3, we achieve the below operator inequality

$$|\Phi^p(A^{-1})\Phi^p(A) + \Phi^p(A)\Phi^p(A^{-1})| \leqslant \frac{(M+m)^{2p}}{2M^p m^p}. \tag{2.7}$$

□

REMARK 1. As $|X| \geqslant X$ for any sel-adjoint X, we find that (2.4) is stronger than (2.5). Since (2.4) is different from (2.5), thus we present them, separately.

REMARK 2. When $p=1$, by (2.4) and (2.5) we obtain (1.1) and (1.2), respectively. Thus (2.4) and (2.5) are the generalizations of (1.1) and (1.2), respectively.

In the next theorem, we show a generalization of the reverse operator AM-GM inequality (1.4):

THEOREM 5. *Let* $0 < m \leqslant A_i \leqslant M \quad (i=1,\dots,n)$ *and* $p \geqslant 1$. *Then for every positive unital linear map* Φ,

$$\left(\frac{A_1+\cdots+A_n}{n}\right)^{2p} \leqslant \left(\frac{(M+m)^{2p}}{4M^p m^p}\right)^2 G^{2p}(A_1,\dots,A_n). \tag{2.8}$$

Proof. By [7, p. 40], we know that (2.8) is equivalent to

$$\left\|\left(\frac{A_1+\cdots+A_n}{n}\right)^{p} G^{-p}(A_1,\dots,A_n)\right\| \leqslant \frac{(M+m)^{2p}}{4M^p m^p}. \tag{2.9}$$

Compute

$$\begin{aligned}
&\left\|\left(\frac{A_1+\cdots+A_n}{n}\right)^{p} M^p m^p G^{-p}(A_1,\dots,A_n)\right\| \\
&\quad \leqslant \frac{1}{4}\left\|\left(\frac{A_1+\cdots+A_n}{n}\right)^{p} + M^p m^p G^{-p}(A_1,\dots,A_n)\right\|^2 \quad \text{(by (2.1))} \\
&\quad \leqslant \frac{1}{4}\left\|\left(\frac{A_1+\cdots+A_n}{n}\right) + MmG^{-1}(A_1,\dots,A_n)\right\|^{2p} \quad \text{(by (2.2))} \\
&\quad \leqslant \frac{1}{4}(M+m)^{2p}. \qquad \text{(by [6, Theorem 3.2])}
\end{aligned}$$

That is

$$\left\|\left(\frac{A_1+\cdots+A_n}{n}\right)^{p} G^{-p}(A_1,\dots,A_n)\right\| \leqslant \frac{(M+m)^{2p}}{4M^p m^p}.$$

Thus, (2.9) holds. $\square$

REMARK 3. When $p=1$, by (2.8) we obtain (1.4). Thus (2.8) is a generalization of (1.4).

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