NEW ERROR BOUNDS OF THE CHEBYSHEV FUNCTIONAL AND APPLICATION TO THE TWO–POINT INTEGRAL FORMULA

S. Kovač, J. Pečarić and S. Tipurić-Spužević

(*Communicated by S. Varošanec*)

Abstract. In this paper we develop some new inequalities regarding Chebyshev functional and obtaine some new application to the two-point integral formula.

1. Introduction

Let us begin by well-known spaces of functions. $L_p[a,b]$, $1 \leqslant p < \infty$ stands for the space of the functions $f : [a,b] \to \mathbb{R}$ which are $p-$integrable. Thus they are equipped with the $p-$norm

$$\|f\|_p = \left[\int_a^b |f(t)|^p \, dt \right]^{\frac{1}{p}}.$$

Further, $L_\infty[a,b]$ stands for space of the functions $f : [a,b] \to \mathbb{R}$ which are essentially bounded and equipped with the $\infty-$norm

$$\|f\|_\infty = \text{ess sup}_{t \in [a,b]} |f(t)|.$$

Let $f, g : [a,b] \to \mathbb{R}$ be functions such $f, g, f \cdot g \in L_1[a,b]$. The Chebyshev functional $T(f,g)$ is defined by

$$T(f,g) := \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx.$$

For $c_0 \in [a,b]$ set $D(c_0)$ which stands for the class of all continuous functions $f : [a,b] \to \mathbb{R}$ differentiable on $\langle a, c_0 \rangle \bigcup \langle c_0, b \rangle$ such that

$$M_l := \sup_{t \in \langle a,c_0 \rangle} |f'(t)| < \infty \text{ and } M_r := \sup_{t \in \langle c_0,b \rangle} |f'(t)| < \infty.$$

If $c_0 = a$ we set $M_l = 0$ and if $c_0 = b$ we set $M_r = 0$.

In [1] the generalization of the M. Niezgoda result ([2]) is obtained for the weighted integral formula. Given a subdivision $\sigma = \{a = x_0 < x_1 < \ldots < x_{m-1} < x_m = b\}$ of the

interval $[a,b]$, let us consider different $w-$ harmonic sequences of functions $\{w_{jk}\}_{k\in\mathbb{N}}$ on each interval $[x_{j-1},x_j]$, $j \in \{1,2,\ldots,m\}$. Define

$$W_{n,w}(t,\sigma) = \begin{cases} w_{1n}(t), \ t \in [a,x_1] \\ w_{2n}(t), \ t \in (x_1,x_2] \\ \vdots \\ w_{mn}(t), \ t \in (x_{m-1},b], \end{cases} \tag{1.1}$$

Let us denote

$$Tf(x_0,x_1,x_2,\ldots,x_m) := \int_a^b w(t)f(t)\mathrm{d}t - \sum_{k=1}^n (-1)^{k-1}\Big[w_{mk}(b)f^{(k-1)}(b)$$

$$+ \sum_{j=1}^{m-1}\big[w_{jk}(x_j) - w_{j+1,k}(x_j)\big]f^{(k-1)}(x_j) - w_{1k}(a)f^{(k-1)}(a)\Big].$$

The following has been established:

THEOREM 1. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)} \in D(c_0)$, for some $n \in \mathbb{N}$ and*

$$M_l = \sup_{t\in\langle a,c_0\rangle} \left|f^{(n)}(t)\right| < \infty \quad and \ M_r = \sup_{t\in\langle c_0,b\rangle} \left|f^{(n)}(t)\right| < \infty.$$

Then the following inequality holds:

$$|Tf(x_0,x_1,x_2,\ldots,x_m)| \leqslant \begin{cases} \big[M_l^p(c_0-a) + M_r^p(b-c_0)\big]^{1/p} \cdot \|W_{n,w}(\cdot,\sigma)\|_q, \ 1 \leqslant p < \infty \\ \max\{M_l,M_r\} \cdot \|W_{n,w}(\cdot,\sigma)\|_1, \qquad\qquad\quad p = \infty. \end{cases}$$

THEOREM 2. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n)} \in D(c_0)$, for some $n \in \mathbb{N}$ and*

$$M_l = \sup_{t\in\langle a,c_0\rangle} \left|f^{(n+1)}(t)\right| < \infty \quad and \ M_r = \sup_{t\in\langle c_0,b\rangle} \left|f^{(n+1)}(t)\right| < \infty.$$

Assume $1 \leqslant p,q \leqslant \infty$ are conjugate exponents ($\frac{1}{p} + \frac{1}{q} = 1$) and $W_{n,w}(\cdot,\sigma)$ defined by (1.1). Then the following inequality holds:

$$\left|Tf(x_0,x_1,x_2,\ldots,x_m) - (-1)^n\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\int_a^b W_{n,w}(t,\sigma)dt\right|$$

$$\leqslant \begin{cases} \frac{1}{(p+1)^{1/p}}\big[M_l^p(c_0-a)^{p+1} + M_r^p(b-c_0)^{p+1}\big]^{1/p} \cdot \|W_{n,w}(\cdot,\sigma) - \eta\|_{q,[a,b]}, \ 1 \leqslant p < \infty \\ \max\{M_l(c_0-a), M_r(b-c_0)\} \cdot \|W_{n,w}(\cdot,\sigma) - \eta\|_{1,[a,b]}, \qquad\quad p = \infty, \ q = 1, \end{cases}$$

where $\eta = \frac{1}{b-a}\int_a^b W_{n,w}(t,\sigma)dt$.

THEOREM 3. *Let* $f : [a,b] \to \mathbb{R}$ *be a function such that* $f^{(n)} \in D(c_0)$, *for some* $n \in \mathbb{N}$ *and*

$$M_l = \sup_{t \in \langle a,c_0 \rangle} \left| f^{(n+1)}(t) \right| < \infty \quad \text{and } M_r = \sup_{t \in \langle c_0,b \rangle} \left| f^{(n+1)}(t) \right| < \infty.$$

Assume $1 \leqslant p, q \leqslant \infty$ *are conjugate exponents* ($\frac{1}{p} + \frac{1}{q} = 1$) *and* $W_{n,w}(\cdot, \sigma)$ *defined by* (1.1). *Then the following inequality holds:*

$$\left| Tf(x_0, x_1, x_2, \ldots, x_m) - (-1)^n f^{(n)}(c_0) \cdot \int_a^b W_{n,w}(t, \sigma) dt \right|$$

$$\leqslant \begin{cases} \frac{1}{(p+1)^{1/p}} \left[M_l^p (c_0 - a)^{p+1} + M_r^p (b - c_0)^{p+1} \right]^{1/p} \cdot \| W_{n,w}(\cdot, \sigma) \|_{q,[a,b]}, \ 1 \leqslant p < \infty \\ \\ \max\{ M_l(c_0 - a), M_r(b - c_0) \} \cdot \| W_{n,w}(\cdot, \sigma) \|_{1,[a,b]}, \qquad\qquad p = \infty, \ q = 1. \end{cases}$$

Let us observe the general integral two-point formula obtained in [3]. Let $w : [a,b] \to \mathbb{R}$ be some integrable function and $x \in [a, \frac{a+b}{2}]$. Consider a subdivision

$$\sigma := \{ x_0 = a, x_1 = x, x_2 = a + b - x < x_3 = b \}$$

of $[a,b]$. Let $\{ Q_{k,x} \}_{k \in \mathbb{N}}$ be sequence of polynomials such that $\deg Q_{k,x} \leqslant k - 1$, $Q'_{k,x}(t) = Q_{k-1,x}(t)$, $k \in \mathbb{N}$ and $Q_{0,x} \equiv 0$. Define functions $w_{jk}(t)$ on $[x_{j-1}, x_j]$, for $j = 1, 2, 3$ and $k \in \mathbb{N}$:

$$w_{1k}(t) = \frac{1}{(k-1)!} \int_a^t (t-s)^{k-1} w(s) ds$$

$$w_{2k}(t) = \frac{1}{(k-1)!} \int_x^t (t-s)^{k-1} w(s) ds + Q_{k,x}(t) \qquad (1.2)$$

$$w_{3k}(t) = -\frac{1}{(k-1)!} \int_t^b (t-s)^{k-1} w(s) ds.$$

Obviously, $\{ w_{jk} \}_{k \in \mathbb{N}}$ are sequences of $w-$harmonic functions on $[x_{j-1}, x_j]$, for every $j = 1, 2, 3$. Let us define coefficients $A_k(x)$ and $B_k(x)$ by following:

$$A_k(x) = (-1)^{k-1} \left[\frac{1}{(k-1)!} \int_a^x (x-s)^{k-1} w(s) ds - Q_{k,x}(x) \right], \qquad (1.3)$$

and

$$B_k(x) = (-1)^{k-1} \left[\frac{1}{(k-1)!} \int_x^b (a+b-x-s)^{k-1} w(s) ds + Q_{k,x}(a+b-x) \right]. \quad (1.4)$$

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[a,b]$ for some $n \in \mathbb{N}$. We introduce the following notation:

$$T_{n,w}(x) = 0, \quad \text{for } n = 1$$

$$T_{n,w}(x) := \sum_{k=2}^n \left[A_k(x) f^{(k-1)}(x) + B_k(x) f^{(k-1)}(a+b-x) \right], \quad \text{for } n \geqslant 2.$$

In [3] the weighted version of two-point integral formula is established:

THEOREM 4. *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n)}$ *is piecewise continuous on* $[a,b]$, *for some* $n \in \mathbb{N}$. *Then*

$$\int_a^b w(t)f(t)dt = A_1(x)f(x) + B_1(x)f(a+b-x) + T_{n,w}(x) + (-1)^n \int_a^b W_{n,w}(t,x)f^{(n)}(t)dt, \tag{1.5}$$

where

$$W_{n,w}(t,x) = \begin{cases} w_{1n}(t) \ \text{for } t \in [a,x], \\ \\ w_{2n}(t) \ \text{for } t \in (x,a+b-x], \\ \\ w_{3n}(t) \ \text{for } t \in (a+b-x,b] \end{cases} \tag{1.6}$$

and $\{w_{kn}\}$ *are* $w-$ *harmonic sequences of functions.*

This theorem is the generalization of the Guessab and Schmeisser two-point formula (see [4] and [5]).

The aim of this paper is to derive the analogues error bound for the two-point version of integral formula obtained in [4] with some special cases for $x \in [a, \frac{a+b}{2}]$.

2. Main result

For $x \in [a, \frac{a+b}{2}]$ let us denote

$$T_w f(a,x,a+b-x,b) := \int_a^b w(t)f(t)dt - A_1(x)f(x) - B_1(x)f(a+b-x) - T_{n,w}(x).$$

We shall derive the upper bound for $T_w f(a,x,a+b-x,b)$ for functions f such that $f^{(n-1)} \in D(c_0)$.

THEOREM 5. *Let* $f : [a,b] \to \mathbb{R}$ *be a function such that* $f^{(n-1)} \in D(c_0)$, *for some* $n \in \mathbb{N}$ *and*

$$M_l = \sup_{t \in \langle a,c_0 \rangle} \left| f^{(n)}(t) \right| < \infty \quad \text{and } M_r = \sup_{t \in \langle c_0,b \rangle} \left| f^{(n)}(t) \right| < \infty.$$

Then the following inequality holds:

$$|T_w f(a,x,a+b-x,b)| \leqslant \begin{cases} \left[M_l^p(c_0-a) + M_r^p(b-c_0) \right]^{1/p} \cdot \|W_{n,w}(\cdot,\sigma)\|_q, \ 1 \leqslant p < \infty \\ \\ \max\{M_l, M_r\} \cdot \|W_{n,w}(\cdot,\sigma)\|_1, \hspace{3.3cm} p = \infty. \end{cases}$$

Proof. The proof follows from the Theorem 1 for the special case of subdivision $\sigma := \{x_0 = a, x_1 = x, x_2 = a+b-x < x_3 = b\}$. Therefore $W_{n,w}(t,\sigma) = W_{n,w}(t,x)$ and $Tf(x_0,x_1,\ldots,x_m) = T_w f(a,x,a+b-x,b)$. $\square$

COROLLARY 1. *Let* $f : [a,b] \to \mathbb{R}$ *be a function such that* $f^{(n-1)} \in D(c_0)$, *for some* $n \in \mathbb{N}$ *and*

$$M_l = \sup_{t \in \langle a, c_0 \rangle} \left| f^{(n)}(t) \right| < \infty \quad \text{and } M_r = \sup_{t \in \langle c_0, b \rangle} \left| f^{(n)}(t) \right| < \infty.$$

If $\{Q_k(t)\}_{k \in \mathbb{N}}$ *is a sequence of polynomials such that* $\deg Q_k = k$, $Q_k(t) = (-1)^k Q_k(a + b - t)$ *and* $\{k! \cdot Q_k\}_{k \in \mathbb{N}}$ *is a sequence of harmonic polynomials, then the following inequality holds:*

$$\left| T_{\frac{1}{b-a}} f(a, x, a+b-x, b) \right| \leqslant
\begin{cases}
\frac{2^{1/q}}{n!(b-a)} \left[\frac{(x-a)^{nq+1}}{nq+1} + \int_x^{\frac{a+b}{2}} |Q_n(t)|^q \, dt \right]^{1/q} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{1}{p} + \frac{1}{q} = 1 \\
\cdot \left[M_l^p(c_0 - a) + M_r^p(b - c_0) \right]^{1/p}, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 < p, q < \infty \\[2mm]
\frac{1}{n!(b-a)} \cdot \max\{ (x-a)^n, \sup_{t \in \langle x, \frac{a+b}{2} \rangle} |Q_n(t)| \} \\
\cdot \left[M_l(c_0 - a) + M_r(b - c_0) \right], \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad p = 1, \; q = \infty \\[2mm]
\frac{2}{n!(b-a)} \left[\frac{(x-a)^{n+1}}{n+1} + \int_x^{\frac{a+b}{2}} |Q_n(t)| \, dt \right] \cdot \max\{ M_l, M_r \}, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad p = \infty, \; q = 1.
\end{cases}$$

Proof. We apply Theorem 5 with $w(t) = \frac{1}{b-a}$ and

$$W_{n, \frac{1}{b-a}}(t, x) =
\begin{cases}
\frac{(t-a)^n}{n!(b-a)} & \text{for } t \in [a, x], \\[2mm]
\frac{Q_n(t)}{n!(b-a)} & \text{for } t \in (x, a+b-x], \\[2mm]
\frac{(t-b)^n}{n!(b-a)} & \text{for } t \in (a+b-x, b].
\end{cases} \tag{2.1}$$

$\square$

Let us give the upper bound for

$$T_w f(a, x, a+b-x, b) - \frac{(-1)^n}{b-a} \cdot \int_a^b W_{n,w}(t, x) dt \cdot \int_a^b f^{(n)}(t) dt,$$

for functions $f : [a, b] \to \mathbb{R}$ such that $f^{(n)} \in D(c_0)$. First, let us compute

$$\eta_w := \frac{1}{b-a} \int_a^b W_{n,w}(t, x) dt.$$

LEMMA 1.   (i) *For any weight function* $w : [a, b] \to [0, \infty)$ *we have*

$$\eta_w = (-1)^n \frac{A_{n+1}(x) + B_{n+1}(x)}{b-a}.$$

(ii) For the uniform weight function $w(t) = \frac{1}{b-a}$ we have

$$\eta_{\frac{1}{b-a}} = \begin{cases} 0 & \text{for odd } n \\[2mm] \frac{2A_{k+1}(x)}{b-a} & \text{for even } n. \end{cases}$$

Proof.

(i) Since w_{kn} are $w-$ harmonic sequences of functions, we have

$$\eta_w = \frac{1}{b-a}\left[\int_a^x w_{1n}(t)dt + \int_x^{a+b-x} w_{2n}(t)dt + \int_{a+b-x}^b w_{3n}(t)dt\right]$$

$$= \frac{1}{b-a}\left[w_{1,n+1}(x) + w_{2,n+1}(a+b-x) - w_{2,n+1}(x) - w_{3,n+1}(a+b-x)\right]$$

$$= (-1)^n\frac{A_{n+1}(x) + B_{n+1}(x)}{b-a}.$$

(ii) Specially, for $w(t) = \frac{1}{b-a}$ we have

$$A_{n+1}(x) = (-1)^n\left[\frac{(x-a)^{n+1}}{(n+1)!(b-a)} - \frac{Q_{n+1}(x)}{(n+1)!(b-a)}\right]$$

and

$$B_{n+1}(x) = (-1)^n\left[\frac{Q_{n+1}(a+b-x)}{(n+1)!(b-a)} - \frac{(a+b-x-b)^{n+1}}{(n+1)!(b-a)}\right]$$

$$= (-1)^n\left[\frac{(-1)^{n+1}Q_{n+1}(x)}{(n+1)!(b-a)} - \frac{(-1)^{n+1}(x-a)^{n+1}}{(n+1)!(b-a)}\right]$$

$$= \frac{(x-a)^{n+1}}{(n+1)!(b-a)} - \frac{Q_{n+1}(x)}{(n+1)!(b-a)} = (-1)^nA_{n+1}(x),$$

so we have

$$\eta_{\frac{1}{b-a}} = \frac{(-1)^n + 1}{b-a}\cdot A_{n+1}(x) = \begin{cases} 0 & \text{for odd } n \\[2mm] \frac{2A_{k+1}(x)}{b-a} & \text{for even } n. \end{cases}$$

$\square$

Now we can set the following theorem.

THEOREM 6. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n)} \in D(c_0)$, for some $n \in \mathbb{N}$ and*

$$M_l = \sup_{t\in\langle a,c_0\rangle}\left|f^{(n+1)}(t)\right| < \infty \quad \text{and } M_r = \sup_{t\in\langle c_0,b\rangle}\left|f^{(n+1)}(t)\right| < \infty.$$

Assume $1 \leqslant p, q \leqslant \infty$ are conjugate exponents ($\frac{1}{p} + \frac{1}{q} = 1$) and $W_{n,w}(\cdot, x)$ defined by (1.6). Then the following inequality holds:

$$\left| T_w f(a, x, a+b-x, b) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} [A_{n+1}(x) + B_{n+1}(x)] \right|$$

$$\leqslant \begin{cases} \frac{1}{(p+1)^{1/p}} \left[M_l^p (c_0-a)^{p+1} + M_r^p (b-c_0)^{p+1} \right]^{1/p} \cdot \| W_{n,w}(\cdot, x) - \eta_w \|_{q, [a,b]}, & 1 \leqslant p < \infty \\ \\ \max\{ M_l(c_0 - a), M_r(b - c_0) \} \cdot \| W_{n,w}(\cdot, x) - \eta_w \|_{1, [a,b]}, & p = \infty, \ q = 1. \end{cases}$$

Proof. The proof follows from the Theorem 2 applied to the two-point integral formula. According to Lemma 1 we have the lefthand side equals to

$$\left| T_w f(a, x, a+b-x, b) - (-1)^n \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \int_a^b W_{n,w}(t, x) dt \right|$$

$$= \left| T_w f(a, x, a+b-x, b) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} (b-a) \cdot \eta_w \right|$$

$$= \left| T_w f(a, x, a+b-x, b) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} [A_{n+1}(x) + B_{n+1}(x)] \right|,$$

while in the righthand side we put $W_{n,w}(t, \sigma) = W_{n,w}(t, x)$ and the proof is finished. □

COROLLARY 2. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $f^{(n)} \in D(c_0)$, for some $n \in \mathbb{N}$ and*

$$M_l = \sup_{t \in \langle a, c_0 \rangle} \left| f^{(n+1)}(t) \right| < \infty \quad \text{and } M_r = \sup_{t \in \langle c_0, b \rangle} \left| f^{(n+1)}(t) \right| < \infty.$$

Assume $1 \leqslant p, q \leqslant \infty$ are conjugate exponents ($\frac{1}{p} + \frac{1}{q} = 1$) and $W_{n, \frac{1}{b-a}}(\cdot, x)$ defined by (2.1). Then the following inequality holds:

$$\left| T_{\frac{1}{b-a}} f(a, x, a+b-x, b) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} [1 + (-1)^n] \cdot A_{n+1}(x) \right|$$

$$\leqslant \begin{cases} \frac{1}{(p+1)^{1/p}} \left[M_l^p (c_0-a)^{p+1} + M_r^p (b-c_0)^{p+1} \right]^{1/p} \\ \cdot \| W_{n, \frac{1}{b-a}}(\cdot, x) - \eta_{\frac{1}{b-a}} \|_{q, [a,b]}, & 1 \leqslant p < \infty \\ \\ \max\{ M_l(c_0-a), M_r(b-c_0) \} \cdot \| W_{n, \frac{1}{b-a}}(\cdot, x) - \eta_{\frac{1}{b-a}} \|_{1, [a,b]}, & p = \infty, \ q = 1. \end{cases}$$

Proof. Apply Theorem 6 with uniform weight function and Lemma 1(ii). □

REMARK 1. If we put odd n into the Corollary 2 we obtain $\eta_{\frac{1}{b-a}} = 0$ and

$$\left| T_{\frac{1}{b-a}} f(a,x,a+b-x,b) \right|$$

$$\leqslant \begin{cases} \frac{1}{(p+1)^{1/p}} \left[M_l^p (c_0-a)^{p+1} + M_r^p (b-c_0)^{p+1} \right]^{1/p} \\ \quad \cdot \frac{2^{1/q}}{n!(b-a)} \left[\frac{(x-a)^{nq+1}}{nq+1} + \int_x^{\frac{a+b}{2}} |Q_n(t)|^q \, dt \right]^{1/q}, \qquad 1 < p, q < \infty \\[2mm] \frac{1}{2} \left[M_l (c_0-a)^2 + M_r (b-c_0)^2 \right] \\ \quad \cdot \frac{1}{n!(b-a)} \cdot \max\{ (x-a)^n, \sup_{t \in \langle x, \frac{a+b}{2} \rangle} |Q_n(t)| \} \quad p=1, \; q=\infty \\[2mm] \max\{ M_l (c_0-a), M_r (b-c_0) \} \\ \quad \cdot \frac{2}{n!(b-a)} \left[\frac{(x-a)^{n+1}}{n+1} + \int_x^{\frac{a+b}{2}} |Q_n(t)| \, dt \right], \qquad p=\infty, \; q=1. \end{cases}$$

At the end we shall consider the upper bound for

$$T_w f(a,x,a+b-x,b) - (-1)^n f^{(n)}(c_0) \cdot \int_a^b W_{n,w}(t,x) dt.$$

THEOREM 7. *Let* $f : [a,b] \to \mathbb{R}$ *be a function such that* $f^{(n)} \in D(c_0)$, *for some* $n \in \mathbb{N}$ *and*

$$M_l = \sup_{t \in \langle a,c_0 \rangle} \left| f^{(n+1)}(t) \right| < \infty \quad \text{and } M_r = \sup_{t \in \langle c_0,b \rangle} \left| f^{(n+1)}(t) \right| < \infty.$$

Assume $1 \leqslant p, q \leqslant \infty$ *are conjugate exponents* ($\frac{1}{p} + \frac{1}{q} = 1$) *and* $W_{n,w}(\cdot, x)$ *defined by* (*1.6*). *Then the following inequality holds:*

$$\left| T_w f(a,x,a+b-x,b) - [A_{n+1}(x) + B_{n+1}(x)] \cdot f^{(n)}(c_0) \right|$$

$$\leqslant \begin{cases} \frac{1}{(p+1)^{1/p}} \left[M_l^p (c_0-a)^{p+1} + M_r^p (b-c_0)^{p+1} \right]^{1/p} \cdot \| W_{n,w}(\cdot,x) \|_{q,[a,b]}, & 1 \leqslant p < \infty \\[2mm] \max\{ M_l (c_0-a), M_r (b-c_0) \} \cdot \| W_{n,w}(\cdot,x) \|_{1,[a,b]}, & p=\infty, \; q=1. \end{cases}$$

Proof. We apply Theorem 3 and Lemma 1 to obtain the lefthand side:

$$\left| T_w f(a,x,a+b-x,b) - (-1)^n \cdot f^{(n)}(c_0) \int_a^b W_{n,w}(t,x) dt \right|$$

$$= \left| T_w f(a,x,a+b-x,b) - (-1)^n (b-a) \eta_w \cdot f^{(n)}(c_0) \right|$$

$$= \left| T_w f(a,x,a+b-x,b) - [A_{n+1}(x) + B_{n+1}(x)] \cdot f^{(n)}(c_0) \right|,$$

while in the righthand side we put $W_{n,w}(t,\sigma) = W_{n,w}(t,x)$ and the proof is finished. $\square$

COROLLARY 3. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n)} \in D(c_0)$, for some $n \in \mathbb{N}$ and*

$$M_l = \sup_{t \in \langle a,c_0 \rangle} \left| f^{(n+1)}(t) \right| < \infty \quad and \; M_r = \sup_{t \in \langle c_0,b \rangle} \left| f^{(n+1)}(t) \right| < \infty.$$

Assume $1 \leqslant p, q \leqslant \infty$ are conjugate exponents ($\frac{1}{p} + \frac{1}{q} = 1$) and $W_{n, \frac{1}{b-a}}(\cdot, x)$ defined by (2.1). Then the following inequality holds:

$$\left| T_{\frac{1}{b-a}} f(a,x,a+b-x,b) - A_{n+1}(x) \cdot ((-1)^n + 1) \cdot f^{(n)}(c_0) \right|$$

$$\leqslant \begin{cases} \frac{1}{(p+1)^{1/p}} \left[M_l^p (c_0-a)^{p+1} + M_r^p (b-c_0)^{p+1} \right]^{1/p} \\ \cdot \frac{2^{1/q}}{n!(b-a)} \left[\frac{(x-a)^{nq+1}}{nq+1} + \int_x^{\frac{a+b}{2}} |Q_n(t)|^q \, dt \right]^{1/q}, \qquad 1 < p,q < \infty \\[2mm] \frac{1}{2} \left[M_l (c_0-a)^2 + M_r (b-c_0)^2 \right] \\ \cdot \frac{1}{n!(b-a)} \cdot \max\{(x-a)^n, \sup_{t \in \langle x, \frac{a+b}{2} \rangle} |Q_n(t)|\} \quad p = 1, \; q = \infty \\[2mm] \max\{M_l(c_0-a), M_r(b-c_0)\} \\ \cdot \frac{2}{n!(b-a)} \left[\frac{(x-a)^{n+1}}{n+1} + \int_x^{\frac{a+b}{2}} |Q_n(t)| \, dt \right], \qquad p = \infty, \; q = 1. \end{cases}$$

Proof. Apply Theorem 6 with uniform weight function and Lemma 1(ii). $\square$

REMARK 2. If we put odd n into the Corollary 3 we obtain

$$\left| T_{\frac{1}{b-a}} f(a,x,a+b-x,b) \right|$$

$$\leqslant \begin{cases} \frac{1}{(p+1)^{1/p}} \left[M_l^p (c_0-a)^{p+1} + M_r^p (b-c_0)^{p+1} \right]^{1/p} \\ \cdot \frac{2^{1/q}}{n!(b-a)} \left[\frac{(x-a)^{nq+1}}{nq+1} + \int_x^{\frac{a+b}{2}} |Q_n(t)|^q \, dt \right]^{1/q}, \qquad 1 < p,q < \infty \\[2mm] \frac{1}{2} \left[M_l (c_0-a)^2 + M_r (b-c_0)^2 \right] \\ \cdot \frac{1}{n!(b-a)} \cdot \max\{(x-a)^n, \sup_{t \in \langle x, \frac{a+b}{2} \rangle} |Q_n(t)|\} \quad p = 1, \; q = \infty \\[2mm] \max\{M_l(c_0-a), M_r(b-c_0)\} \\ \cdot \frac{2}{n!(b-a)} \left[\frac{(x-a)^{n+1}}{n+1} + \int_x^{\frac{a+b}{2}} |Q_n(t)| \, dt \right], \qquad p = \infty, \; q = 1. \end{cases}$$

3. Application to two-point quadrature formulas

For $n = 1$ and $Q_1(t) = t - \frac{a+b}{2}$ we have the kernel

$$W_{1, \frac{1}{b-a}}(t,x) = \begin{cases} \frac{t-a}{b-a} & \text{for } t \in [a,x], \\[2mm] \frac{t - \frac{a+b}{2}}{b-a} & \text{for } t \in (x, a+b-x], \\[2mm] \frac{t-b}{b-a} & \text{for } t \in (a+b-x, b]. \end{cases}$$

In this case we have $A_1(x) = B_1(x) = \frac{1}{2}$ and

$$T_{\frac{1}{b-a}} f(a, x, a+b-x, b) = \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2}[f(x) + f(a+b-x)].$$

If $f : [a,b] \to \mathbb{R}$ be a function such that $f \in D(c_0)$ and

$$M_l = \sup_{t \in \langle a, c_0 \rangle} |f'(t)| < \infty \quad \text{and} \quad M_r = \sup_{t \in \langle c_0, b \rangle} |f'(t)| < \infty,$$

then by Corollary 1 we have

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2}[f(x) + f(a+b-x)] \right|$$

$$\leqslant \begin{cases} \frac{2^{1/q}}{(b-a)(q+1)^{1/q}} \left[(x-a)^{q+1} + \left(\frac{a+b}{2} - x\right)^{q+1} \right]^{1/q} \frac{1}{p} + \frac{1}{q} = 1 \\ \cdot \left[M_l^p(c_0 - a) + M_r^p(b - c_0) \right]^{1/p}, & 1 < p, q < \infty \\[2mm] \frac{1}{b-a} \cdot \max\{x-a, \frac{a+b}{2} - x\} \\ \cdot [M_l(c_0 - a) + M_r(b - c_0)], & p = 1, q = \infty \\[2mm] \frac{1}{b-a} \left[(x-a)^2 + \left(\frac{a+b}{2} - x\right)^2 \right] \cdot \max\{M_l, M_r\}, & p = \infty, q = 1. \end{cases}$$

It is easy to show that functions $S_q(x) = (x-a)^{q+1} + \left(\frac{a+b}{2} - x\right)^{q+1}$, $x \in [a, \frac{a+b}{2}]$ and $S_\infty(x) = \max\{x-a, \frac{a+b}{2} - x\}$ attain minimum value for $x = \frac{3a+b}{4}$. In that case we get the inequality related to the two-point Maclaurin quadrature formula:

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] \right|$$

$$\leqslant \begin{cases} \frac{(b-a)^{1/q}}{4(q+1)^{1/q}} \cdot \left[M_l^p(c_0 - a) + M_r^p(b - c_0) \right]^{1/p}, & 1 < p, q < \infty \\[2mm] \frac{1}{4} \cdot [M_l(c_0 - a) + M_r(b - c_0)], & p = 1, q = \infty \\[2mm] \frac{b-a}{8} \cdot \max\{M_l, M_r\}, & p = \infty, q = 1. \end{cases}$$

For $x = a$ we get the inequality related to trapezoidal quadrature formula:

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2}[f(a) + f(b)] \right|$$

$$\leqslant \begin{cases} \frac{(b-a)^{1/q}}{2(q+1)^{1/q}} \cdot \left[M_l^p(c_0 - a) + M_r^p(b - c_0) \right]^{1/p}, & 1 < p, q < \infty \\[2mm] \frac{1}{2} \cdot [M_l(c_0 - a) + M_r(b - c_0)], & p = 1, q = \infty \\[2mm] \frac{b-a}{4} \cdot \max\{M_l, M_r\}, & p = \infty, q = 1. \end{cases}$$

For $x = \frac{2a+b}{3}$ we get the inequality related to the two-point Newton-Cotes quadrature formula:

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right) \right] \right|$$

$$\leqslant \begin{cases} (\frac{2}{3})^{1/q} \cdot \frac{(b-a)^{1/q}}{(q+1)^{1/q}} \cdot \left[M_l^p(c_0 - a) + M_r^p(b - c_0) \right]^{1/p}, \ 1 < p,q < \infty \\[2mm] \frac{1}{3} \cdot \left[M_l(c_0 - a) + M_r(b - c_0) \right], \qquad\qquad p = 1, \ q = \infty \\[2mm] \frac{5(b-a)}{36} \cdot \max\{M_l, M_r\}, \qquad\qquad\quad p = \infty, \ q = 1. \end{cases}$$

For $x = \frac{a+b}{2}$ we get the inequality related to midpoint quadrature formula:

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \right|$$

$$\leqslant \begin{cases} \frac{(b-a)^{1/q}}{2(q+1)^{1/q}} \cdot \left[M_l^p(c_0 - a) + M_r^p(b - c_0) \right]^{1/p}, \ 1 < p,q < \infty \\[2mm] \frac{1}{2} \cdot \left[M_l(c_0 - a) + M_r(b - c_0) \right], \qquad\quad p = 1, \ q = \infty \\[2mm] \frac{b-a}{4} \cdot \max\{M_l, M_r\}, \qquad\qquad\quad p = \infty, \ q = 1. \end{cases}$$

Now, let us assume that $f' \in D(c_0)$, and

$$M_l = \sup_{t \in \langle a, c_0 \rangle} \left| f''(t) \right| < \infty \quad \text{and} \quad M_r = \sup_{t \in \langle c_0, b \rangle} \left| f''(t) \right| < \infty.$$

Then by Corollary 2 we have

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2}\left[f(x) + f(a+b-x) \right] \right|$$

$$\leqslant \begin{cases} \frac{1}{(p+1)^{1/p}} \left[M_l^p(c_0 - a)^{p+1} + M_r^p(b - c_0)^{p+1} \right]^{1/p} \\ \cdot \frac{2^{1/q}}{(b-a)(q+1)^{q+1}} \left[(x-a)^{q+1} + (\frac{a+b}{2} - x)^{q+1} \right]^{1/q}, \ 1 < p,q < \infty \\[2mm] \frac{1}{2} \left[M_l(c_0 - a)^2 + M_r(b - c_0)^2 \right] \\ \cdot \frac{1}{(b-a)} \cdot \max\{x - a, \frac{a+b}{2} - x\} \qquad\qquad p = 1, \ q = \infty \\[2mm] \max\{M_l(c_0 - a), M_r(b - c_0)\} \\ \cdot \frac{2}{(b-a)} \left[\frac{(x-a)^2}{2} + (\frac{a+b}{2} - x)^2 \right], \qquad\quad p = \infty, \ q = 1. \end{cases}$$

Similar as before we conclude that the minimum constant is obtained for $x = \frac{3a+b}{4}$ and therefore we get inequalities related to Maclaurin two-point quadrature formula:

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] \right|$$

$$\leqslant \begin{cases} \frac{(b-a)^{1/q}}{4(q+1)^{1/q}(p+1)^{1/p}} \cdot \left[M_l^p(c_0-a)^{p+1} + M_r^p(b-c_0)^{p+1} \right]^{1/p}, \ 1 < p,q < \infty \\[3mm] \frac{1}{4}\left[M_l(c_0-a)^2 + M_r(b-c_0)^2 \right] \hspace{3.2cm} p=1, \ q=\infty \\[3mm] \frac{b-a}{8} \cdot \max\{M_l(c_0-a), M_r(b-c_0)\}, \hspace{2.2cm} p=\infty, \ q=1. \end{cases}$$

For $x = a$ we get the inequality related to trapezoidal quadrature formula:

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2}[f(a) + f(b)] \right|$$

$$\leqslant \begin{cases} \frac{(b-a)^{1/q}}{2(q+1)^{1/q}} \cdot \left[M_l^p(c_0-a)^{p+1} + M_r^p(b-c_0)^{p+1} \right]^{1/p}, \ 1 < p,q < \infty \\[3mm] \frac{1}{2} \cdot \max\{M_l(c_0-a)^2, M_r(b-c_0)^2\}, \hspace{2.4cm} p=1, \ q=\infty \\[3mm] \frac{b-a}{4} \cdot \max\{M_l(c_0-a), M_r(b-c_0)\}, \hspace{2.2cm} p=\infty, \ q=1. \end{cases}$$

For $x = \frac{2a+b}{3}$ we get the inequality related to the two-point Newton-Cotes quadrature formula:

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right) \right] \right|$$

$$\leqslant \begin{cases} \left(\frac{2}{3}\right)^{1/q} \cdot \frac{(b-a)^{1/q}}{(q+1)^{1/q}} \cdot \left[M_l^p(c_0-a)^{p+1} + M_r^p(b-c_0)^{p+1} \right]^{1/p}, \ 1 < p,q < \infty \\[3mm] \frac{1}{3} \cdot \max\{M_l(c_0-a)^2, M_r(b-c_0)^2\}, \hspace{2.4cm} p=1, \ q=\infty \\[3mm] \frac{5(b-a)}{36} \max\{M_l(c_0-a), M_r(b-c_0)\}, \hspace{2.0cm} p=\infty, \ q=1. \end{cases}$$

For $x = \frac{a+b}{2}$ we get the inequality related to midpoint quadrature formula:

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \right|$$

$$\leqslant \begin{cases} \frac{(b-a)^{1/q}}{2(q+1)^{1/q}} \cdot \left[M_l^p(c_0-a)^{p+1} + M_r^p(b-c_0)^{p+1} \right]^{1/p}, \ 1 < p,q < \infty \\[3mm] \frac{1}{2} \cdot \max\{M_l(c_0-a)^2, M_r(b-c_0)^2\}, \hspace{2.4cm} p=1, \ q=\infty \\[3mm] \frac{b-a}{4} \cdot \max\{M_l(c_0-a), M_r(b-c_0)\}, \hspace{2.2cm} p=\infty, \ q=1. \end{cases}$$

Conclusion 1. *The optimal constant in whole class of two-point quadrature formulas is obtained for Maclaurin two-point quadrature formula. It can be observed that constant in this case are better then constants achieved in* [2].

REFERENCES

[1] S. KOVAČ, J. PEČARIĆ, S. TIPURIĆ-SPUŽEVI'C, *Weighted Ostrowski Type Inequalities with Application to Onepoint Integral Formula*, Mediterreanian Journal of Mathematics, **11** (2014), 13–30

[2] M. NIEZGODA, *Grüss and Ostrowski Type Inequalities*, Applied Mathematics and Computation, **217** (23) (2011), 9779–9789.

[3] S. KOVAČ, J. PEČARIĆ, *Generalization of an Integral Formula of Guessab and Schmeisser*, Banach Journal of Mathematical Analysis **5** (2011), no. 1, 1–18.

[4] S. KOVAČ, J. PEČARIĆ, A. VUKELIĆ, *A Generalization of General Two-point Formula with Applications in Numerical Integration*, Nonlinear Analysis Series A: Theory, Methods and Applications **68** (8) (2008), 2445–2463.

[5] A. GUESSAB, G. SCHMEISSER, *Sharp Integral Inequalities of the Hermite-Hadamard Type*, Journal of Approximation Theory **115** (2) (2002), 260–288.

(Received November 7, 2014)

S. Kovač
Faculty of Geotechnical Engineering
University of Zagreb
Hallerova aleja 7, 42 000 Varaždin, Croatia
e-mail: skovac@gfv.hr

J. Pečarić
Faculty of Textile Technology
University of Zagreb
Prilaz baruna Filipovića 28a, 10 000 Zagreb, Croatia
e-mail: pecaric@hazu.hr

S. Tipurić-Spuževič
Faculty of Science and Education
University of Mostar
Matice Hrvatske bb, 88 000 Mostar, Bosnia and Herzegovina
e-mail: sanja.spuzevic@tel.net.ba

Journal of Mathematical Inequalities
www.ele-math.com
jmi@ele-math.com