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JENSEN'S INEQUALITY FOR FUNCTIONS SUPERQUADRATIC ON THE COORDINATES

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Abstract. Jensen's type inequalities for functions superquadratic on the coordinates are given. Obtained results are used to prove several Hölder-type inequalities.

1. Introduction

A function $\varphi : C \to \mathbb{R}$ is said to be convex on a convex subset C of a real linear space X if

$$\varphi\left(t\boldsymbol{x}+(1-t)\boldsymbol{y}\right) \leqslant t\varphi\left(\boldsymbol{x}\right)+(1-t)\varphi\left(\boldsymbol{y}\right) \tag{1.1}$$

holds for all $\boldsymbol{x},\boldsymbol{y} \in C$ and $0 \leqslant t \leqslant 1$. A function φ is said to be strictly convex if the inequality in (1.1) is strict whenever $\boldsymbol{x} \neq \boldsymbol{y}$ and $0 < t < 1$.

A function $\varphi : I \times J \to \mathbb{R}$, $I \times J \subset \mathbb{R}^2$, is called *convex on the coordinates* if the partial mappings $\varphi_y : I \to \mathbb{R}$ defined by $\varphi_y(u) := \varphi(u,y)$, and $\varphi_x : J \to \mathbb{R}$ defined by $\varphi_x(v) := \varphi(x,v)$, are convex for all $y \in J$ and $x \in I$. Analogously we define functions which are concave on the coordinates.

Therefore, for the function φ convex on the coordinates, we have (by the well known characterizations of convex functions) the following.

For each $u \in I$ there exists $C_u \in \mathbb{R}$ such that

$$\varphi_y(z) \geqslant \varphi_y(u) + C_u(z-u), \qquad \forall z \in I, \tag{1.2}$$

and for each $v \in J$ there exists $D_v \in \mathbb{R}$ such that

$$\varphi_x(w) \geqslant \varphi_x(v) + D_v(w-v), \qquad \forall w \in J. \tag{1.3}$$

for all $y \in J$ and $x \in I$.

Obviously, if the function $\varphi : I \times J \to \mathbb{R}$ is convex (concave), then it is also convex (concave) on the coordinates, but functions which are convex (concave) on the coordinates need not to be convex (concave) in the standard sense. For instance, the function $\varphi : [0,\infty\rangle^2 \to \mathbb{R}$ defined by

$$\varphi(x,y) = x^p y^q,$$

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where $p,q \geqslant 1,$ is convex on the coordinates, but it is not convex in the standard sense. This means that the class of convex functions is a proper subclass of the class of functions which are convex on the coordinates. Analogously, for $0<p<1$ and $0<q<1$ the function φ is concave on the coordinates, but not concave in the standard sense unless $p+q \leqslant 1$.

Jensen's inequality has many integral analogues. Here we recall the simplest one expressed in the language of the abstract Lebesgue integral (see [6, p. 45] or [5, p. 10]): if $(\Omega, \mathscr{A}, \mu)$ is a measure space with $0<\mu(\Omega)<\infty$ and if $f: \Omega \rightarrow I$ is from $L^1(\mu)$ then

$$\varphi\left(\frac{1}{\mu(\Omega)} \int_{\Omega} f d\mu\right) \leqslant \frac{1}{\mu(\Omega)} \int_{\Omega}(\varphi \circ f) d\mu \tag{1.4}$$

holds for any convex function $\varphi: I \rightarrow \mathbb{R}$. In the case when φ is strictly convex we have equality in (1.4) if and only if φ is constant. Of course, if set I is bounded then function f needs to be measurable only. This inequality is well known as the integral Jensen's inequality.

The following inequality, proved by M. L. Slater in the paper [4], is also related to Jensen's inequality (1.4).

If $(\Omega, \mathscr{A}, \mu)$ is probability measure space and the function $\varphi: I \rightarrow \mathbb{R}$ is a convex and increasing (decreasing) on interval $I \subseteq \mathbb{R}$, then for any measurable function $f: \Omega \rightarrow I$ such that $\varphi \circ f,\ \varphi_{+}^{\prime} \circ f,\ \left(\varphi_{+}^{\prime} \circ f\right) f \in L^1(\mu)$ and $\int_{\Omega}\left(\varphi_{+}^{\prime} \circ f\right) d\mu \neq 0$ the inequality

$$\int_{\Omega}(\varphi \circ f) d\mu \leqslant \varphi\left(\frac{\int_{\Omega}\left(\varphi_{+}^{\prime} \circ f\right) f d\mu}{\int_{\Omega}\left(\varphi_{+}^{\prime} \circ f\right) d\mu}\right) \tag{1.5}$$

holds. If, in addition, φ is strictly convex, then (1.5) becomes equality if and only if $\varphi = const.$ μ-a.e. on Ω. This inequality remains valid if instead of $\varphi_{+}^{\prime}(x)$ we take any value $C_x \in\left[\varphi_{-}^{\prime}(x), \varphi_{+}^{\prime}(x)\right]$. Of course, if φ is differentiable then we take $\varphi^{\prime}$. In the rest of the paper we call this inequality – Slater's inequality.

Jensen-type inequalities for functions which are convex on the coordinates were investigated in [3]. In the same paper the following two theorems were proved:

THEOREM A. *Suppose that*

(i) $(\Omega_1, \mathscr{A}, \mu)$ *and* $(\Omega_2, \mathscr{B}, \nu)$ *are measure spaces;*

(ii) $p: \Omega_1 \rightarrow \mathbb{R},\ p \in L^1(\mu)$, *and* $w: \Omega_2 \rightarrow \mathbb{R},\ w \in L^1(\nu)$, *are nonnegative functions such that* $\int_{\Omega_1} p d\mu \neq 0$ *and* $\int_{\Omega_2} w d\nu \neq 0$*;*

(iii) $g: \Omega_1 \rightarrow I,\ g \in L^{\infty}(\mu)$, *and* $h: \Omega_2 \rightarrow J,\ h \in L^{\infty}(\nu),\ I, J \subset \mathbb{R}$;

(iv) $\varphi: I \times J \rightarrow \mathbb{R}$ *is convex on the coordinates on* $I \times J$.

Then the following inequalities hold:

$$\begin{aligned} \varphi(\bar{g}, \bar{h}) & \leqslant \frac{1}{2}\left\{\frac{1}{P} \int_{\Omega_1} p \varphi(g, \bar{h}) d\mu+\frac{1}{W} \int_{\Omega_2} w \varphi(\bar{g}, h) d\nu\right\} \\ & \leqslant \frac{1}{PW} \int_{\Omega_1} \int_{\Omega_2} p w \varphi(g, h) d\mu d\nu, \end{aligned} \tag{1.6}$$

where

$$P=\int_{\Omega_1} pd\mu, \qquad W=\int_{\Omega_2} wd\nu$$
$$\overline{g}=\frac{1}{P}\int_{\Omega_1} pgd\mu, \qquad \overline{h}=\frac{1}{W}\int_{\Omega_2} whd\nu.$$

If the function φ *is concave on the coordinates inequalities* (1.6) *are reversed. The above inequalities are sharp.*

THEOREM B. *Let* $I=[m,M]$ *and* $J=[n,N]$, *where* $-\infty<m<M<\infty$ *and* $-\infty<n<N<\infty$. *Let the functions* p *and* w *be as in Theorem A, and let the functions* $g:\Omega_1\to I,\ h:\Omega_2\to J$ *be measurable. If the function* $\varphi:I\times J\to\mathbb{R}$ *is continuous and convex on the coordinates on* $I\times J$, *then the following inequalities hold:*

$$\begin{aligned}
&\frac{1}{PW}\int_{\Omega_1}\int_{\Omega_2} pw\varphi(g,h)\,d\mu d\nu\\
&\leqslant \frac{1}{2}\left\{\frac{N-\overline{h}}{N-n}\frac{1}{P}\int_{\Omega_1} p\varphi(g,n)\,d\mu+\frac{\overline{h}-n}{N-n}\frac{1}{P}\int_{\Omega_1} p\varphi(g,N)\,d\mu\right.\\
&\quad\left.+\frac{M-\overline{g}}{M-m}\frac{1}{W}\int_{\Omega_2} w\varphi(m,h)\,d\nu+\frac{\overline{g}-m}{M-m}\frac{1}{W}\int_{\Omega_2} w\varphi(M,h)\,d\nu\right\}\\
&\leqslant \frac{M-\overline{g}}{M-m}\frac{N-\overline{h}}{N-n}\varphi(m,n)+\frac{\overline{g}-m}{M-m}\frac{N-\overline{h}}{N-n}\varphi(M,n)\\
&\quad+\frac{M-\overline{g}}{M-m}\frac{\overline{h}-n}{N-n}\varphi(m,N)+\frac{\overline{g}-m}{M-m}\frac{\overline{h}-n}{N-n}\varphi(M,N).
\end{aligned}\tag{1.7}$$

If the function φ *is concave on the coordinates inequalities* (1.7) *are reversed. The above inequalities are sharp.*

Now, we give a version of Slater's inequality (1.5) for functions which are convex on the coordinates.

THEOREM 1. *Suppose that:*

(i) $(\Omega_1,\mathscr{A},\mu)$ *and* $(\Omega_2,\mathscr{B},\nu)$ *are probability measure spaces;*

(ii) $I,J\subset\mathbb{R},\ \varphi:I\times J\to\mathbb{R}$ *is convex on the coordinates on* $I\times J$;

(iii) C *and* D *are as in* (1.2) *and* (1.3);

(iv) $g:\Omega_1\to I$ *and* $h:\Omega_2\to J$ *are measurable functions such that*

$$\varphi(g,h)\in L^1(\mu\times\nu),$$

$C_g,\ gC_g\in L^1(\mu),\ D_h,\ hD_h\in L^1(\nu),\ \int_{\Omega_1} C_g d\mu\neq 0$ *and* $\int_{\Omega_2} C_h d\nu\neq 0.$

If

$$G=\frac{\int_{\Omega_1} gC_g d\mu}{\int_{\Omega_1} C_g d\mu}\in I, \qquad H=\frac{\int_{\Omega_2} hD_h d\nu}{\int_{\Omega_2} D_h d\nu}\in J,$$

then the following inequalities

$$\int_{\Omega_1}\int_{\Omega_2}\varphi(g,h)\,d\mu d\nu \leqslant \frac{1}{2}\left[\int_{\Omega_1}\varphi(g,H)\,d\mu + \int_{\Omega_2}\varphi(G,h)\,d\nu\right] \leqslant \varphi(G,H). \tag{1.8}$$

hold.

Proof. Applying inequality (1.5) twice to the first and the second variable of the function φ respectively (with C_f and D_f instead of $\varphi'_+ \circ f$, respectively), we get:

$$\int_{\Omega_1}\int_{\Omega_2}\varphi(g,h)\,d\mu d\nu \leqslant \int_{\Omega_1}\varphi(g,H)\,d\mu \leqslant \varphi(G,H),$$
$$\int_{\Omega_1}\int_{\Omega_2}\varphi(g,h)\,d\mu d\nu \leqslant \int_{\Omega_2}\varphi(G,h)\,d\nu \leqslant \varphi(G,H),$$

and hence the inequalities in (1.8) immediately follow. □

In paper [1], the concept of superquadratic functions of one variable was introduced.

DEFINITION 1. A function $\varphi : [0,\infty\rangle \to \mathbb{R}$ is said to be *superquadratic* if for any $t \geqslant 0$ there exist a constant $C_t \in \mathbb{R}$ such that

$$\varphi(s) \geqslant \varphi(t) + C_t(s-t) + \varphi(|s-t|) \tag{1.9}$$

holds for all $s \geqslant 0$.

Note that if $\varphi(x) = x^2$ the condition above reduces to the identity

$$s^2 - t^2 - (s-t)^2 = 2t(s-t).$$

At first glance the condition (1.9) seems to be much stronger than the convexity condition, but if φ takes negative values it may be considerably weaker. In [1] it was proved that if $\varphi \geqslant 0$, then φ is convex.

Recently, in paper [2] authors have investigated superquadratic functions of m variables, which were defined in the following way.

DEFINITION 2. A function $\varphi : [0,\infty\rangle^m \to \mathbb{R}$ is said to be superquadratic if for every $\boldsymbol{x} \in [0,\infty\rangle^m$ there exists a vector $\boldsymbol{c}(\boldsymbol{x}) \in \mathbb{R}^m$ such that

$$\varphi(\boldsymbol{y}) \geqslant \varphi(\boldsymbol{x}) + \langle \boldsymbol{c}(\boldsymbol{x}), \boldsymbol{y}-\boldsymbol{x}\rangle + \varphi(|\boldsymbol{y}-\boldsymbol{x}|) \tag{1.10}$$

holds for all $\boldsymbol{y} \in [0,\infty\rangle^m$, where $|\boldsymbol{y}-\boldsymbol{x}| = (|y_1-x_1|,\ldots,|y_m-x_m|)$. φ is said to be strictly superquadratic if (1.10) is strict for all $\boldsymbol{x} \neq \boldsymbol{y}$.

Analogously as it was done for convex functions, we define functions which are superquadratic on the coordinates in the following way:

DEFINITION 3. We say that a function $\varphi : [0,\infty\rangle^2 \to \mathbb{R}$ is *superquadratic on the coordinates* if the partial mappings $\varphi_y : [0,\infty\rangle \to \mathbb{R}$ defined by $\varphi_y(u) := \varphi(u,y)$, and $\varphi_x : [0,\infty\rangle \to \mathbb{R}$ defined by $\varphi_x(v) := \varphi(x,v)$, are superquadratic for all $y \in [0,\infty\rangle$ and $x \in [0,\infty\rangle$.

Therefore, by Definition 1, for the function φ which is superquadratic on the coordinates (for arbitrary $x,y \geqslant 0$) the following holds: for each $u \geqslant 0$, there exists $C_u \in \mathbb{R}$ such that

$$\varphi_y(z) \geqslant \varphi_y(u) + C_u(z-u) + \varphi_y(|z-u|), \qquad \forall z \geqslant 0, \tag{1.11}$$

and for each $v \geqslant 0$, there exists $D_v \in \mathbb{R}$ such that

$$\varphi_x(w) \geqslant \varphi_x(v) + D_v(w-v) + \varphi_x(|w-v|), \qquad \forall w \geqslant 0. \tag{1.12}$$

Obviously, we do not have to restrict ourselves to the case $n = 2$ (that is, we can do everything with n variables) but for the sake of the simplicity we proceed with two variables only.

According to (1.2) and (1.3) it is obvious that any nonnegative function which is superquadratic on the coordinates is exactly convex on the coordinates. This fact allows us to get refinements of the inequalities which are valid for the functions convex on the coordinates.

In this class we can also find functions which are superquadratic on the coordinates, but not superquadratic in the standard sense. For instance, the function $\varphi : [0,\infty\rangle^2 \to \mathbb{R}$ defined by

$$\varphi(x,y) = x^p y^q,$$

where $p,q \geqslant 2$, is superquadratic on the coordinates, but not superquadratic. We must emphasize here that functions of several variables which are superqadratic are not necessary superqadratic on the coordinates.

In paper [1] the following two theorems were proved.

THEOREM C. *The inequality*

$$\varphi\left(\int f d\mu\right) \leqslant \int \varphi \circ f - \varphi\left(\left|f - \int f d\mu\right|\right) d\mu \tag{1.13}$$

holds for all probability measures μ and all nonnegative, μ-integrable functions f if and only if a function φ is superquadratic.

THEOREM D. *Suppose that φ is superquadratic and that C is given as in (1.9). If μ is a probability measure, f is a nonnegative μ-measurable function, $\int C_f d\mu \neq 0$, and m and M are defined by*

$$m = \int f d\mu, \quad M = \frac{\int f C_f d\mu}{\int C_f d\mu},$$

then

$$\varphi(m) + \int \varphi(|f-m|) d\mu \leqslant \int \varphi \circ f d\mu \leqslant \varphi(M) - \int \varphi(|f-M|) d\mu. \tag{1.14}$$

It is obvious that if function $-\varphi$ is superquadratic inequalities (1.13) and (1.14) are reversed.

In Section 2 we use Theorem C and Theorem D to establish results analogous to those given in Theorem A, Theorem B and Theorem 1, but now for functions which are superquadratic on the coordinates. In Section 3 we show how we can use those results to obtain several Hölder-type inequalities.

2. Main result

Throughout the rest of this section we assume that:

(i) $(\Omega_1, \mathscr{A}, \mu)$ and $(\Omega_2, \mathscr{B}, \nu)$ are probability measure spaces;

(ii) $g : \Omega_1 \to [0,\infty\rangle$, $g \in L^1(\mu)$, and $h : \Omega_2 \to [0,\infty\rangle$, $h \in L^1(\nu)$;

(iii) $\varphi : [0,\infty\rangle^2 \to \mathbb{R}$ is superquadratic on the coordinates on $[0,\infty\rangle^2$ and $\varphi(g,h) \in L^1(\mu \times \nu)$.

THEOREM 2. *Let φ, g and h be as the above. Then the following inequalities hold:*

$$\begin{aligned}
\varphi\left(\overline{g},\overline{h}\right) &\leqslant \frac{1}{2}\left[\int_{\Omega_2} \varphi\left(\overline{g},h\right) - \varphi\left(\overline{g},\left|h-\overline{h}\right|\right) d\nu + \int_{\Omega_1} \varphi\left(g,\overline{h}\right) - \varphi\left(\left|g-\overline{g}\right|,\overline{h}\right) d\mu\right] \\
&\leqslant \frac{1}{2}\int_{\Omega_1}\int_{\Omega_2} 2\varphi(g,h) - \varphi\left(g,\left|h-\overline{h}\right|\right) - \varphi\left(\left|g-\overline{g}\right|,h\right) d\mu d\nu \\
&\quad - \frac{1}{2}\left[\int_{\Omega_1} \varphi\left(\left|g-\overline{g}\right|,\overline{h}\right) d\mu + \int_{\Omega_2} \varphi\left(\overline{g},\left|h-\overline{h}\right|\right) d\nu\right] \qquad (2.1)
\end{aligned}$$

where

$$\overline{g} = \int_{\Omega_1} g d\mu, \quad \overline{h} = \int_{\Omega_2} h d\nu.$$

Proof. Since the function φ is superquadratic on the coordinates, we can use (1.13) on the first and on the second variable to obtain

$$\begin{aligned}
\varphi\left(\overline{g},\overline{h}\right) &\leqslant \int_{\Omega_1} \varphi\left(g,\overline{h}\right) - \varphi\left(\left|g-\overline{g}\right|,\overline{h}\right) d\mu, \\
\varphi\left(\overline{g},\overline{h}\right) &\leqslant \int_{\Omega_2} \varphi\left(\overline{g},h\right) - \varphi\left(\overline{g},\left|h-\overline{h}\right|\right) d\nu,
\end{aligned}$$

from which we can easily obtain the first inequality in (2.1). Applying (1.13) twice again (i.e. on $\varphi(\overline{g},h)$ and $\varphi\left(g,\overline{h}\right)$) we obtain

$$\begin{aligned}
\int_{\Omega_1} \varphi\left(g,\overline{h}\right) d\mu &\leqslant \int_{\Omega_1}\int_{\Omega_2} \varphi(g,h) - \varphi\left(g,\left|h-\overline{h}\right|\right) d\mu d\nu, \\
\int_{\Omega_2} \varphi\left(\overline{g},h\right) d\nu &\leqslant \int_{\Omega_1}\int_{\Omega_2} \varphi(g,h) - \varphi\left(\left|g-\overline{g}\right|,h\right) d\mu d\nu,
\end{aligned}$$

and the second inequality in (2.1) immediately follows. □

REMARK 1. It can be easily seen that if the function φ is also nonnegative (and therefore convex on the coordinates) Theorem 2 represents a refinement of Theorem A (with $p = w = 1$).

THEOREM 3. *Let φ, g and h be as in the previous theorem. Let C and D be as in (1.11) and (1.12) and suppose that $C_g,\ gC_g \in L^1(\mu),\ D_h,\ hD_h \in L^1(\nu),\ \int_{\Omega_1} C_g d\mu \neq 0$ and $\int_{\Omega_2} D_h d\nu \neq 0$. If*

$$G = \frac{\int_{\Omega_1} gC_g d\mu}{\int_{\Omega_1} C_g d\mu} \geqslant 0, \qquad H = \frac{\int_{\Omega_2} hD_h d\nu}{\int_{\Omega_2} D_h d\nu} \geqslant 0,$$

then the following inequalities hold:

$$\begin{aligned}
&\int_{\Omega_1}\int_{\Omega_2} \varphi(g,h)\, d\mu d\nu \\
&\leqslant \frac{1}{2}\left[\int_{\Omega_1} \varphi(g,H)\, d\mu + \int_{\Omega_2} \varphi(G,h)\, d\nu \right. \\
&\quad \left. - \int_{\Omega_1}\int_{\Omega_2} \varphi(g,|h-H|) + \varphi(|g-G|,h)\, d\nu d\mu \right] \\
&\leqslant \varphi(G,H) - \frac{1}{2}\left[\int_{\Omega_1} \varphi(|g-G|,H)\, d\mu + \int_{\Omega_2} \varphi(G,|h-H|)\, d\nu \right] \\
&\quad - \frac{1}{2}\int_{\Omega_1}\int_{\Omega_2} \varphi(|g-G|,h) + \varphi(g,|h-H|)\, d\mu d\nu.
\end{aligned} \tag{2.2}$$

Proof. From (1.14) we have

$$\int_{\Omega_1}\int_{\Omega_2} \varphi(g,h)\, d\mu d\nu \leqslant \int_{\Omega_2}\left[\varphi(G,h) - \int_{\Omega_1} \varphi(|g-G|,h)\, d\mu\right] d\nu,$$
$$\int_{\Omega_1}\int_{\Omega_2} \varphi(g,h)\, d\mu d\nu \leqslant \int_{\Omega_1}\left[\varphi(g,H) - \int_{\Omega_2} \varphi(g,|h-H|)\, d\nu\right] d\mu,$$

and from these we can easily obtain the first inequality in (2.2). Applying again (1.14) on the right sides of the above inequalities, we obtain

$$\begin{aligned}
&\int_{\Omega_2}\left[\varphi(G,h) - \int_{\Omega_1} \varphi(|g-G|,h)\, d\mu\right] d\nu \\
&\leqslant \varphi(G,H) - \int_{\Omega_2} \varphi(G,|h-H|)\, d\nu - \int_{\Omega_1}\int_{\Omega_2} \varphi(|g-G|,h)\, d\mu d\nu,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Omega_1}\left[\varphi(g,H) - \int_{\Omega_2} \varphi(g,|h-H|)\, d\nu\right] d\mu \\
&\leqslant \varphi(G,H) - \int_{\Omega_1} \varphi(|g-G|,H)\, d\mu - \int_{\Omega_1}\int_{\Omega_2} \varphi(g,|h-H|)\, d\mu d\nu,
\end{aligned}$$

from which the second inequality in (2.2) immediately follows. $\square$

REMARK 2. If the function φ is also nonnegative, then it is also convex on the coordinates, so the conditions of the both Theorem 3 and Teorem 1 are satisfied. Since in this case all the integrals in (2.2) are nonnegative it is obvious that the result of Theorem 3 represents a refinement of the result of Theorem 1.

3. Applications

In this section we show how Theorem A, Theorem B, Theorem 1, Theorem 2 and Theorem 3 can be used to obtain some Hölder-type inequalities.

EXAMPLE 1. Let $\varphi : \langle 0,\infty\rangle^2 \to \mathbb{R}$ be defined by $\varphi(x,y) = x^{\frac{1}{p}} y^{\frac{1}{q}}$, where $p,q \in \langle -\infty, 1] \setminus \{0\}$, and let $(\Omega_1, \mathscr{A}, \mu)$ and $(\Omega_2, \mathscr{B}, \nu)$ be probability measure spaces. Let the functions $g : \Omega_1 \to \langle 0,\infty\rangle$ and $h : \Omega_2 \to \langle 0,\infty\rangle$ be such that $g^p \in L^1(\mu)$ and $h^q \in L^1(\nu)$. Since the function φ is convex on the coordinates, from Theorem A we have:

$$\begin{aligned}
&\left(\int_{\Omega_1} g^p d\mu\right)^{\frac{1}{p}} \left(\int_{\Omega_2} h^q d\nu\right)^{\frac{1}{q}} \\
&\leqslant \frac{1}{2}\left\{\left(\int_{\Omega_1} g^p d\mu\right)^{\frac{1}{p}} \int_{\Omega_2} h d\nu + \left(\int_{\Omega_2} h^q d\nu\right)^{\frac{1}{q}} \int_{\Omega_1} g d\mu\right\} \\
&\leqslant \int_{\Omega_1}\int_{\Omega_2} gh d\mu d\nu. \qquad (3.1)
\end{aligned}$$

If the functions g^p and h^q are bounded, i.e., $g^p : \Omega_1 \to [m,M]$ and $h^q : \Omega_2 \to [n,N]$, where $0 \leqslant m < M < \infty$ and $0 \leqslant n < N < \infty$, then from Theorem B we also have

$$\begin{aligned}
&\int_{\Omega_1}\int_{\Omega_2} gh d\mu d\nu \\
&\leqslant \frac{1}{2}\left\{\left(\frac{N-\overline{h^q}}{N-n} n^{\frac{1}{q}} + \frac{\overline{h^q}-n}{N-n} N^{\frac{1}{q}}\right)\int_{\Omega_1} g d\mu \right. \\
&\quad \left. + \left(\frac{M-\overline{g^p}}{M-m} m^{\frac{1}{p}} + \frac{\overline{g^p}-m}{M-m} M^{\frac{1}{p}}\right)\int_{\Omega_2} h d\nu\right\} \qquad (3.2) \\
&\leqslant \frac{M-\overline{g^p}}{M-m}\frac{N-\overline{h^q}}{N-n} m^{\frac{1}{p}} n^{\frac{1}{q}} + \frac{\overline{g^p}-m}{M-m}\frac{N-\overline{h^q}}{N-n} M^{\frac{1}{p}} n^{\frac{1}{q}} \\
&\quad + \frac{M-\overline{g^p}}{M-m}\frac{\overline{h^q}-n}{N-n} m^{\frac{1}{p}} N^{\frac{1}{q}} + \frac{\overline{g^p}-m}{M-m}\frac{\overline{h^q}-n}{N-n} M^{\frac{1}{p}} N^{\frac{1}{q}},
\end{aligned}$$

where

$$\overline{g^p} = \int_{\Omega_1} g^p d\mu, \quad \overline{h^q} = \int_{\Omega_2} h^q d\nu.$$

If $p,q \in [1,+\infty\rangle$ inequalities (3.1) and (3.2) are reversed.

EXAMPLE 2. Let φ, g, h, μ and ν be defined as in the previous example, and let $p,q \in \left\langle 0, \frac{1}{2}\right]$. In this case the function φ is superquadratic on the coordinates, so from Theorem 2 we get:

$$\begin{aligned}
&\left(\int_{\Omega_1} g^p d\mu\right)^{\frac{1}{p}} \left(\int_{\Omega_2} h^q d\nu\right)^{\frac{1}{q}} \\
&\leqslant \frac{1}{2}\left[\left(\int_{\Omega_1} g^p d\mu\right)^{\frac{1}{p}} \int_{\Omega_2} h - \left|h^q - \int_{\Omega_2} h^q d\nu\right|^{\frac{1}{q}} d\nu \right. \\
&\quad \left. + \left(\int_{\Omega_2} h^q d\nu\right)^{\frac{1}{q}} \int_{\Omega_1} g - \left|g^p - \int_{\Omega_1} g^p d\mu\right|^{\frac{1}{p}} d\mu\right] \\
&\leqslant \frac{1}{2}\int_{\Omega_1}\int_{\Omega_2} 2gh - g\left|h^q - \int_{\Omega_2} h^q d\nu\right|^{\frac{1}{q}} - h\left|g^p - \int_{\Omega_1} g^p d\mu\right|^{\frac{1}{p}} d\mu d\nu \\
&\quad - \frac{1}{2}\left[\left(\int_{\Omega_2} h^q d\nu\right)^{\frac{1}{q}} \int_{\Omega_1}\left|g^p - \int_{\Omega_1} g^p d\mu\right|^{\frac{1}{p}} d\mu \right. \\
&\quad \left. + \left(\int_{\Omega_1} g^p d\mu\right)^{\frac{1}{p}} \int_{\Omega_2}\left|h^q - \int_{\Omega_2} h^q d\nu\right|^{\frac{1}{q}} d\nu\right]. \qquad (3.3)
\end{aligned}$$

Also, if $C_{g^p},\ g^p C_{g^p} \in L^1(\mu),\ D_{h^q},\ h^q D_{h^q} \in L^1(\nu),\ \int_{\Omega_1} C_{g^p} d\mu \neq 0$ and $\int_{\Omega_2} D_{h^q} d\nu \neq 0$, then for

$$G = \frac{\int_{\Omega_1} g^p C_{g^p} d\mu}{\int_{\Omega_1} C_{g^p} d\mu} \geqslant 0, \quad H = \frac{\int_{\Omega_2} h^q D_{h^q} d\nu}{\int_{\Omega_2} D_{h^q} d\nu} \geqslant 0,$$

applying Theorem 3, we get:

$$\begin{aligned}
&\int_{\Omega_1}\int_{\Omega_2} gh d\mu d\nu \\
&\leqslant \frac{1}{2}\left[\int_{\Omega_1} g H^{\frac{1}{q}} d\mu + \int_{\Omega_2} G^{\frac{1}{p}} h d\nu \right. \\
&\quad \left. - \int_{\Omega_1}\int_{\Omega_2} g\left|h^q - H\right|^{\frac{1}{q}} + \left|g^p - G\right|^{\frac{1}{p}} h d\nu d\mu\right] \\
&\leqslant G^{\frac{1}{p}} H^{\frac{1}{q}} - \frac{1}{2}\left[\int_{\Omega_1}\left|g^p - G\right|^{\frac{1}{p}} H^{\frac{1}{q}} d\mu + \int_{\Omega_2} G^{\frac{1}{p}}\left|h^q - H\right|^{\frac{1}{q}} d\nu\right] \\
&\quad - \frac{1}{2}\int_{\Omega_1}\int_{\Omega_2}\left|g^p - G\right|^{\frac{1}{p}} h + g\left|h^q - H\right|^{\frac{1}{q}} d\mu d\nu. \qquad (3.4)
\end{aligned}$$

If $p,q \in \left[\frac{1}{2}, +\infty\right\rangle$ inequalities (3.3) and (3.4) are reversed.

REMARK 3. As we can see, in the case $p,q \in \left\langle 0, \frac{1}{2}\right]$ inequalities in (3.3) are the refinements of inequalities in (3.1). If $p,q \in \left\langle \frac{1}{2}, 1\right\rangle$ combining (3.3) and (3.1) we obtain the following sequence of four inequalities with $\left(\int_{\Omega_1} g^p d\mu\right)^{\frac{1}{p}} \left(\int_{\Omega_2} h^q d\nu\right)^{\frac{1}{q}}$ in the middle of it:

$$\begin{aligned}
&\int_{\Omega_1}\int_{\Omega_2} ghd\mu d\nu \\
&\geqslant \frac{1}{2}\left\{\left(\int_{\Omega_1} g^p d\mu\right)^{\frac{1}{p}}\int_{\Omega_2} hd\nu+\left(\int_{\Omega_2} h^q d\nu\right)^{\frac{1}{q}}\int_{\Omega_1} gd\mu\right\} \\
&\geqslant \left(\int_{\Omega_1} g^p d\mu\right)^{\frac{1}{p}}\left(\int_{\Omega_2} h^q d\nu\right)^{\frac{1}{q}} \\
&\geqslant \frac{1}{2}\left[\left(\int_{\Omega_1} g^p d\mu\right)^{\frac{1}{p}}\int_{\Omega_2} h-\left|h^q-\int_{\Omega_2} h^q d\nu\right|^{\frac{1}{q}} d\nu\right. \\
&\qquad \left.+\left(\int_{\Omega_2} h^q d\nu\right)^{\frac{1}{q}}\int_{\Omega_1} g-\left|g^p-\int_{\Omega_1} g^p d\mu\right|^{\frac{1}{p}} d\mu\right] \\
&\geqslant \frac{1}{2}\int_{\Omega_1}\int_{\Omega_2} 2gh-g\left|h^q-\int_{\Omega_2} h^q d\nu\right|^{\frac{1}{q}}-h\left|g^p-\int_{\Omega_1} g^p d\mu\right|^{\frac{1}{p}} d\mu d\nu \\
&\qquad -\frac{1}{2}\left[\left(\int_{\Omega_2} h^q d\nu\right)^{\frac{1}{q}}\int_{\Omega_1}\left|g^p-\int_{\Omega_1} g^p d\mu\right|^{\frac{1}{p}} d\mu\right. \\
&\qquad \left.+\left(\int_{\Omega_1} g^p d\mu\right)^{\frac{1}{p}}\int_{\Omega_2}\left|h^q-\int_{\Omega_2} h^q d\nu\right|^{\frac{1}{q}} d\nu\right].
\end{aligned}$$

REMARK 4. In the case $p,q\in\left\langle 0,\frac{1}{2}\right]$ inequalities (3.4) refine

$$\int_{\Omega_1}\int_{\Omega_2} ghd\mu d\nu \leqslant \frac{1}{2}\left[\int_{\Omega_1} gH^{\frac{1}{q}}d\mu+\int_{\Omega_2} G^{\frac{1}{p}}hd\nu\right]\leqslant G^{\frac{1}{p}}H^{\frac{1}{q}},$$

which are obtained applying Theorem 1 for the function $\varphi:\langle 0,\infty\rangle^2\to\mathbb{R}$ defined by $\varphi(x,y)=x^{\frac{1}{p}}y^{\frac{1}{q}}$ for $p,q\in\langle-\infty,1]\setminus\{0\}$ under the above conditions (for (3.4)) and with the substitutions

$$g\leftrightarrow g^p \qquad \text{and} \qquad h\leftrightarrow h^q.$$

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