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INEQUALITIES FOR THE MIXED DISCRIMINANTS

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Abstract. In the paper, some new inequalities for the mixed discriminants are established, which are the matrix analogues of inequalities of the well-known mixed volumes function.

1. Introduction

Let $x_1,\dots,x_n$ be a set of nonnegative quantities and by $E_i(x)$ the i-th elementary symmetric function of an n-tuple $x = x(x_1,\dots,x_n)$ of positive reals is defined by $E_0(x) = 1$ and

$$E_i(x) = \sum_{1<j_1<\cdots<j_i\leqslant n} x_{j_1}x_{j_2}\cdots x_{j_i},\ 1\leqslant i\leqslant n.$$

An interesting inequality for the symmetric function was established ([1], also see [2], p. 33) as follows.

$$\frac{E_i(x+y)}{E_{i-1}(x+y)} \geqslant \frac{E_i(x)}{E_{i-1}(x)} + \frac{E_i(y)}{E_{i-1}(y)}. \tag{1.1}$$

A matrix analogue of (1.1) is the following result of Bergstrom [3].

Let K and L be positive definite matrices, and let K_i and L_i denote the submatrices obtained by deleting the i-th row and column. Then

$$\frac{\det(K+L)}{\det(K_i+L_i)} \geqslant \frac{\det(K)}{\det(K_i)} + \frac{\det(L)}{\det(L_i)}. \tag{1.2}$$

An interesting proof is due to Bellman [4] (also see [2], p. 67). A generalization of (1.2) was established by Ky Fan [5] (also see [6–7]).

There is a remarkable similarity between inequalities about symmetric functions (or determinants of symmetric matrices) and inequalities about the mixed volumes of convex bodies. In 1991, V. Milman asked if there is version of (1.1) or (1.2) in the theory of mixed volumes and it was stated as the following open question (see [8]):

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QUESTION 1.1. For which values of i and every pair of convex bodies K and L in $\mathbb{R}^n$, is it true that

$$\frac{W_i(K+L)}{W_{i+1}(K+L)} \geqslant \frac{W_i(K)}{W_{i+1}(K)} + \frac{W_i(L)}{W_{i+1}(L)}? \tag{1.3}$$

The convex body is the compact and convex subsets with non-empty interiors in $\mathbb{R}^n$. $W_i(K)$ denotes the quermassintegral of convex body K and $W_{i+1}(K)$ denotes the mixed volumes $V(\underbrace{K,\ldots,K}_{n-i-1},\underbrace{B,\ldots,B}_{i+1})$. The sum $+$ is the usual Minkowski vector sum and B denotes the unit ball.

Minkowski's Theorem provides a fundamental relation between volume and operations of addition and multiplication of convex bodies by nonnegative reals: If $K_1,\ldots,K_m$ are convex bodies, $m \in \mathbb{N}$, then the volume of $t_1K_1+\cdots+t_mK_m$ is a homogeneous polynomial of degree n in $t_i > 0$ (see [14]). That is

$$V(t_1K_1+\cdots+t_mK_m) = \sum_{1\leqslant i_1,\ldots,i_n\leqslant m} V(K_{i_1},\ldots,K_{i_n})t_{i_1}\cdots t_{i_n},$$

where the coefficients $V(K_{i_1},\ldots,K_{i_n})$ is chosen to be invariant under permutations of their arguments. The coefficient $V(K_{i_1},\ldots,K_{i_n})$ is called the *mixed volume* of the n-tupe $(K_{i_1},\ldots,K_{i_n})$. Steiner's formula is a special case of Minkowski's theorem; the volume of $K+tB$, $t>0$, can be expanded as a polynomial in t:

$$V(K+tB) = \sum_{i=0}^{n}\binom{n}{i}W_i(K)t^i,$$

where $W_i(K) := V(\underbrace{K,\ldots,K}_{n-i},\underbrace{B,\ldots,B}_{i})$ is the *quermassintegral* of convex body K.

A part answer (L must be a ball) of (1.3) was established by Gianopoulos, Hartzoulaki and Paouris [9]).

If K is a convex body and D is a ball in $\mathbb{R}^n$, then for $i=0,\ldots,n-1$

$$\frac{W_i(K+D)}{W_{i+1}(K+D)} \geqslant \frac{W_i(K)}{W_{i+1}(K)} + \frac{W_i(D)}{W_{i+1}(D)}. \tag{1.4}$$

The answer to the above question is negative; it can be proved that (1.3) is true in full generality only when $i=n-1$ or $i=n-2$ (the details see [10]). Moreover, a dual inequality of (1.4) for the dual quermassintegral of star bodies was proved by Li and Leng [11].

In the paper, we establish some new matrix analogues of the mixed volumes inequalities. Our main results are given in Theorems 3.1–3.3 (see Section 3).

2. Mixed discriminants and Aleksandrov's inequality

Recall that for positive definite $n\times n$ matrices $K_1,\dots,K_N$ and $\lambda_1,\dots,\lambda_N\geqslant 0$, the determinant of the linear combination $\lambda_1K_1+\cdots+\lambda_NK_N$ is a homogeneous polynomial of degree n in the λ_i (see e.g. [12]),

$$\det(\lambda_1K_1+\cdots+\lambda_NK_N)=\sum_{1\leqslant i_1,\dots\dots,i_n\leqslant N} D(K_{i_1},\dots,K_{i_n})\lambda_{i_1}\cdots\lambda_{i_n}, \tag{2.1}$$

where the coefficient $D(K_{i_1},\dots,K_{i_n})$ are chosen to be invariant under permutations of their arguments. The coefficient $D(K_{i_1},\dots,K_{i_n})$ is called the *mixed discriminant* of $K_{i_1},\dots,K_{i_n}$.

The mixed discriminant $D(K,\dots,K,I,\dots,I)$, with $n-k$ copies of K and k copies of the identity matrix, I, will be abbreviated by $D_k(K)$. From (2.1), we have

$$D_i(K+\lambda I)=\sum_{j=0}^{n-i}\binom{n-i}{j}\lambda^jD_{i+j}(K). \tag{2.2}$$

Note that the elementary mixed discriminants $D_0(K),\dots,D_n(K)$ are thus defined as the coefficients of the polynomial

$$\det(K+\lambda I)=\sum_{i=0}^{n}\binom{n}{i}\lambda^iD_i(K). \tag{2.3}$$

Obviously, $D_0(K)=\det(K)$ while $nD_{n-1}(K)$ is the trace of K.

The well-known Aleksandrov's inequality for mixed discriminants can be stated as follows (see [13], also see [14], p. 383 or [15], p. 35):

Let $K_1,K_2,\dots,K_n$ be real symmetric positively definite $n\times n$ matrices. Then

$$D(K_1,K_2,K_3\dots,K_n)^2\geqslant D(K_1,K_1,K_3\dots,K_n)D(K_2,K_2,K_3\dots,K_n), \tag{2.4}$$

with equality if and only if $K_1=\lambda K_2$ with positive number λ.

3. Some matrix analogues of the mixed volumes inequalities

THEOREM 3.1. *Let K be symmetric positively definite matrix and I stand for the identity matrix and $t\geqslant 0$. If $0\leqslant i<n$, then*

$$\frac{D_i(K+tI)}{D_{i+1}(K+tI)}\geqslant\frac{D_i(K)+tD_{i+1}(K)}{D_{i+1}(K)}, \tag{3.1}$$

with equality if and only if $K=\mu I$. Here $D_i(K)=D(\underbrace{K,\dots,K}_{n-i},\underbrace{I,\dots,I}_{i})$ is the mixed discriminant.

Proof. If $f_i(s)=D_i(K+sI)$, then by the linearity of the mixed discriminant we see that

$$\begin{aligned}f_i(s+\varepsilon)&=\sum_{j=0}^{n-i}\binom{n-i}{j}\varepsilon^jD_{i+j}(K+sI)\\&=f_i(s)+\varepsilon(n-i)f_{i+1}(s)+o(\varepsilon^2).\end{aligned}$$

Hence

$$f_i'(s) = \lim_{\varepsilon \to 0} \frac{f(s+\varepsilon) - f(s)}{\varepsilon} = (n-i) f_{i+1}(s). \tag{3.2}$$

Similarly, we obtain

$$f_{i+1}'(s) = (n-i-1) f_{i+2}(s). \tag{3.3}$$

From (2.4), we obtain for all $0 \leqslant i < n$

$$f_{i+1}^2(s) - f_i(s) f_{i+2}(s) \geqslant 0, \tag{3.4}$$

with equality if and only if $K = \mu I$.

From (3.2), (3.3) and (3.4), we have

$$\begin{aligned} f_i'(s) f_{i+1}(s) - f_i(s) f_{i+1}'(s) &= f_{i+1}^2(s) + (n-i-1)(f_{i+1}^2(s) - f_i(s) f_{i+2}(s)) \\ &\geqslant f_{i+1}^2(s). \end{aligned}$$

This implies that the function $m_i : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $m_i(s) = f_i(s)/f_{i+1}(s)$ satisfies

$$m_i'(s) = \left(\frac{f_i(s)}{f_{i+1}(s)} \right)' \geqslant 1.$$

Hence, for every $t \geqslant 0$

$$m_i(t) \geqslant m_i(0) + t, \tag{3.5}$$

with equality if and only if $K = \mu I$.

Inequality (3.1) easy follows from (3.5).

The proof is completed. □

THEOREM 3.2. *Let K be symmetric positively definite matrix and I stand for the identity matrix. If $0 \leqslant j < i \leqslant n-1$ and $t \in [0, +\infty)$, then*

$$f(t) = \frac{D_j(K+tI)}{D_i(K+tI)}$$

is an increasing function.

Proof. If $m_i(t) = D_i(K+tI)/D_{i+1}(K+tI)$, then for $0 \leqslant j < i \leqslant n-1$

$$\begin{aligned} f(t) &= \frac{D_j(K+tI)}{D_{j+1}(K+tI)} \cdot \frac{D_{j+1}(K+tI)}{D_{j+2}(K+tI)} \cdot \frac{D_{j+2}(K+tI)}{D_{j+3}(K+tI)} \cdots \frac{D_{i-2}(K+tI)}{D_{i-1}(K+tI)} \cdot \frac{D_{i-1}(K+tI)}{D_i(K+tI)} \\ &= m_j(t) \cdot m_{j+1}(t) \cdot m_{j+2}(t) \cdots m_{i-2}(t) \cdot m_{i-1}(t). \end{aligned}$$

Hence

$$f'(t) = \sum_{i=j}^{i-1} \prod_{\substack{k=1 \\ k \neq i}}^{i-1} m_i'(t) m_k(t).$$

Notice that

$$m_j'(t), m_{j+1}'(t), \ldots, m_{i-1}'(t) \geqslant 1,$$

we have $f'(t) \geqslant 0$.

This completes the proof. □

Similarly, from the proof of Theorem 3.2, for the mixed volumes of convex bodies, we obtain:

COROLLARY 3.1. *Let K be a convex body in $\mathbb{R}^n$. If $0 \leqslant j < i \leqslant n-1$, then*

$$f(t) = \frac{W_j(K+tB)}{W_i(K+tB)}, \quad t \in [0,+\infty).$$

is an increasing function.

THEOREM 3.3. *Let K be symmetric positively definite matrix and I stand for the identity matrix. If $s,t \in [0,+\infty)$ and $0 \leqslant i \leqslant n-1$, then*

$$\frac{D_i(K+(s+t)/2\cdot I)}{D_{i+1}(K+(s+t)/2\cdot I)} \leqslant \frac{D_i((K+sI)/2)}{D_{i+1}((K+sI)/2)} + \frac{D_i((K+tI)/2)}{D_{i+1}((K+tI)/2)}.$$

Proof. From (3.1), it easy follows that the function $g(s) = D_i(K+sI)/D_{i+1}(K+sI)$ is a concave function on $[0+\infty)$. Hence

$$\begin{aligned}
\frac{D_i(K+(s+t)/2\cdot I)}{D_{i+1}(K+(s+t)/2\cdot I)} &= \frac{D_i\Big((K+tI)/2+(K+sI)/2\Big)}{D_{i+1}\Big((K+tI)/2+(K+sI)/2\Big)} \\
&= g\left(\frac{s+t}{2}\right) \leqslant \frac{1}{2}(g(s)+g(t)) \\
&= \frac{(1/2)^{n-i}D_i(K+sI)}{(1/2)^{n-i-1}D_{i+1}(K+sI)} + \frac{(1/2)^{n-i}D_i(K+tI)}{(1/2)^{n-i-1}D_{i+1}(K+tI)} \\
&= \frac{D_i((K+sI)/2)}{D_{i+1}((K+sI)/2)} + \frac{D_i((K+tI)/2)}{D_{i+1}((K+tI)/2)}.
\end{aligned}$$

The proof is completed. □

From the proof of Theorem 3.3, for the mixed volumes of convex bodies, we obtain:

COROLLARY 3.2. *Let K be a convex body in $\mathbb{R}^n$. If $s,t \in [0,+\infty)$ and $0 \leqslant i \leqslant n-1$, then*

$$\frac{W_i(K+(s+t)/2\cdot B)}{W_{i+1}(K+(s+t)/2\cdot B)} \leqslant \frac{W_i((K+sB)/2)}{W_{i+1}((K+sB)/2)} + \frac{W_i((K+tB)/2)}{W_{i+1}((K+tB)/2)}.$$

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