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SHARP INEQUALITIES FOR TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

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Abstract. We establish several sharp inequalities for trigonometric and hyperbolic functions. Our results sharpen some known inequalities.

1. Introduction

The main object of this paper is to present several sharp inequalities for trigonometric and hyperbolic functions. We also indicate relevant connections of the results presented here with those derived in earlier works.

We begin by listing some preliminaries. The following elementary power series expansions are useful in the sequel.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty, \tag{1.1}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \qquad |x| < \infty, \tag{1.2}$$

$$\ln\left(\frac{\tan x}{x}\right) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n-1}-1)|B_{2n}|}{n(2n)!} x^{2n}, \qquad 0 < |x| < \frac{\pi}{2}, \tag{1.3}$$

$$\ln(\sec x) = \sum_{n=1}^{\infty} \frac{2^{2n-1}(2^{2n}-1)|B_{2n}|}{n(2n)!} x^{2n}, \qquad |x| < \frac{\pi}{2}, \tag{1.4}$$

where $B_n \ (n = 0, 1, 2, \ldots)$ are Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

It is known [1, p. 805] that

$$\frac{2(2n)!}{(2\pi)^{2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}(1-2^{1-2n})}, \qquad n \geqslant 1. \tag{1.5}$$

The following lemmas are also needed in the sequel.

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LEMMA 1.1. ([2, 3, 4]) *Let* $-\infty < a < b < \infty$*, and* $f,g:[a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable in* (a,b)*. Suppose* $g' \neq 0$ *on* (a,b)*. If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b)*, then so are*

$$[f(x)-f(a)]/[g(x)-g(a)] \text{ and } [f(x)-f(b)]/[g(x)-g(b)].$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

LEMMA 1.2. ([13]) *Let* $a_n \in \mathbb{R}$ *and* $b_n > 0$*,* $n = 0,1,2,\ldots$ *be real numbers with* $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ *being strictly increasing (respectively, decreasing). If the power series* $A(x) := \sum_{n=0}^{\infty} a_n x^n$ *and* $B(x) := \sum_{n=0}^{\infty} b_n x^n$ *are convergent for* $|x| < R$*, then the function* $A(x)/B(x)$ *is strictly increasing (respectively, decreasing) on* $(0,R)$*.*

2. Becker-Stark inequality

Becker and Stark [5] proved the following inequality:

$$\frac{8}{\pi^2-4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2-4x^2}, \qquad 0 < x < \frac{\pi}{2}. \tag{2.1}$$

The constant 8 and π^2 are the best possible.

Zhu and Hua [17] established a general refinement of the Becker-Stark inequalities by using the power series expansion of the tangent function via Bernoulli numbers and the property of a function involving Riemann's zeta one. Zhu [18] extended the tangent function to Bessel functions.

In view of the second inequality in (2.1), the following question can be posed: What are the largest number α and the smallest number β such that the inequalities

$$\left(\frac{\pi^2}{\pi^2-4x^2}\right)^{\alpha} < \frac{\tan x}{x} < \left(\frac{\pi^2}{\pi^2-4x^2}\right)^{\beta} \tag{2.2}$$

are valid for all $0 < |x| < \pi/2$? Theorem 2.1 below answers this question. We remark that, in fact, Chen and Chueng [6] have proved Theorem 2.1 by using the method given by Malešević [9]. Very recently, Malešević *et al.* [10] presented a similar proof of this theorem based on finite Taylor approximations. Here we provide a new proof.

THEOREM 2.1. *For* $0 < |x| < \pi/2$*, inequality* (2.2) *holds with best possible constants*

$$\alpha = \frac{\pi^2}{12} = 0.822467033\ldots \quad \text{and} \quad \beta = 1. \tag{2.3}$$

Proof. Without a loss of generality, we may assume that $0 < x < \pi/2$. Clearly, the second inequality in (2.2) holds for $\beta = 1$. Now we are in a position to prove the first inequality in (2.2) for $\alpha = \pi^2/12$. To this end, we consider the function $f(x)$ defined by

$$f(x) = \ln\left(\frac{\tan x}{x}\right) - \frac{\pi^2}{12}\ln\left(\frac{\pi^2}{\pi^2-4x^2}\right), \qquad 0 < x < \frac{\pi}{2}.$$

By using (1.3), we obtain

$$
\begin{aligned}
f(x) &= \ln\left(\frac{\tan x}{x}\right) + \frac{\pi^2}{12}\ln\left(1-\left(\frac{2x}{\pi}\right)^2\right) \\
&= \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n-1}-1)\,|B_{2n}|}{n(2n)!}x^{2n} - \frac{\pi^2}{12}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{2}{\pi}\right)^{2n}x^{2n} \\
&= \sum_{n=2}^{\infty}\left(\frac{(2^{2n-1}-1)\,|B_{2n}|}{(2n)!} - \frac{\pi^2}{12}\left(\frac{1}{\pi}\right)^{2n}\right)\frac{(2x)^{2n}}{n}.
\end{aligned}
$$

By (1.5), we find that for $n \geqslant 2$,

$$
\begin{aligned}
\frac{(2^{2n-1}-1)\,|B_{2n}|}{(2n)!} - \frac{\pi^2}{12}\left(\frac{1}{\pi}\right)^{2n} &> \frac{(2^{2n-1}-1)}{(2n)!}\frac{2(2n)!}{(2\pi)^{2n}} - \frac{\pi^2}{12}\left(\frac{1}{\pi}\right)^{2n} \\
&= \left(1-\frac{1}{2^{2n-1}}-\frac{\pi^2}{12}\right)\frac{1}{\pi^{2n}} > 0.
\end{aligned}
$$

Hence, $f(x) > 0$ for $0 < x < \pi/2$, and thus, the first inequality in (2.2) holds.

The inequality (2.2) can be rewritten as

$$
\alpha < \frac{\ln\left(\frac{\tan x}{x}\right)}{\ln\left(\frac{\pi^2}{\pi^2-4x^2}\right)} < \beta, \qquad 0 < x < \frac{\pi}{2}.
$$

Elementary calculations reveal that

$$
\lim_{x\to 0^+} \frac{\ln\left(\frac{\tan x}{x}\right)}{\ln\left(\frac{\pi^2}{\pi^2-4x^2}\right)} = \frac{\pi^2}{12} \quad \text{and} \quad \lim_{x\to(\pi/2)^-} \frac{\ln\left(\frac{\tan x}{x}\right)}{\ln\left(\frac{\pi^2}{\pi^2-4x^2}\right)} = 1.
$$

Hence, inequality (2.2) holds with best possible constants given in (2.3). □

REMARK 2.2. There is no strict comparison between the two lower bounds $\frac{8}{\pi^2-4x^2}$ and $\left(\frac{\pi^2}{\pi^2-4x^2}\right)^{\pi^2/12}$ in (2.1) and (2.2).

REMARK 2.3. By using the Maple software, we find that

$$
\frac{\tan x}{x} - \left(\frac{\pi^2}{\pi^2-4x^2}\right)^{\pi^2/12} = \frac{7\pi^2-60}{90\pi^2}x^4 + O(x^6), \qquad x \to 0
$$

and

$$
\frac{\tan x}{x} - \frac{\pi^2}{\pi^2-4x^2} = \frac{\pi^2-12}{3\pi^2}x^2 + O(x^4), \qquad x \to 0.
$$

This shows that as $x \to 0$, the lower-approximation $\left(\frac{\pi^2}{\pi^2-4x^2}\right)^{\pi^2/12}$ is better than the upper-approximation $\frac{\pi^2}{\pi^2-4x^2}$ in (2.2).

3. Sharp bounds for the secant function

Theorem 3.1 establishes sharp bounds of the secant function. The proof of Theorem 3.1 makes use of the first inequality in (2.1).

THEOREM 3.1. (i) *For* $0 < |x| < \pi/2$,

$$\frac{\pi^2}{\pi^2-4x^2} < \sec x < \frac{4\pi}{\pi^2-4x^2}. \tag{3.1}$$

The constant π^2 *and* 4π *are best possible.*

(ii) *For* $0 < |x| < \pi/2$,

$$\left(\frac{\pi^2}{\pi^2-4x^2}\right)^{\lambda} < \sec x < \left(\frac{\pi^2}{\pi^2-4x^2}\right)^{\mu} \tag{3.2}$$

with best possible constants

$$\lambda = 1 \quad \textit{and} \quad \mu = \frac{\pi^2}{8}. \tag{3.3}$$

Proof. Without a loss of generality, we may assume that $0 < x < \pi/2$. Let

$$f_1(x) = (\pi^2 - 4x^2)\sec x, \qquad 0 < x < \frac{\pi}{2}.$$

By differentiating and using the first inequality in (2.1), we obtain

$$\frac{\cos x}{x(\pi^2-4x^2)} f_1'(x) = \frac{\tan x}{x} - \frac{8}{\pi^2-4x^2} > 0.$$

Therefore, the function $f_1(x)$ is strictly increasing on $(0, \pi/2)$. Noting that

$$\lim_{t\to 0^+} f_1(t) = \pi^2 \quad \text{and} \quad \lim_{t\to(\pi/2)^-} f_1(t) = 4\pi,$$

we have

$$\pi^2 = f_1(0) < f_1(x) = (\pi^2-4x^2)\sec x < \lim_{x\to(\pi/2)^-} f_1(x) = 4\pi$$

for all $x \in (0, \pi/2)$, with the constants π^2 and 4π being best possible.

Clearly, the first inequality in (3.2) holds for $\lambda = 1$. Now we are in a position to prove the second inequality in (3.2) for $\mu = \pi^2/8$. To this end, we consider the function $f_2(x)$ defined by

$$f_2(x) = \ln(\sec x) - \frac{\pi^2}{8}\ln\left(\frac{\pi^2}{\pi^2-4x^2}\right), \qquad 0 < x < \frac{\pi}{2}.$$

By using (1.4), we obtain

$$\begin{aligned} f_2(x) &= \ln(\sec x) + \frac{\pi^2}{8}\ln\left(1-\left(\frac{2x}{\pi}\right)^2\right) \\ &= \sum_{n=1}^{\infty} \frac{2^{2n-1}(2^{2n}-1)\,|B_{2n}|}{n(2n)!}x^{2n} - \frac{\pi^2}{8}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{2}{\pi}\right)^{2n}x^{2n} \\ &= -\left(\frac{12-\pi^2}{12\pi^2}\right)x^4 + \sum_{n=2}^{\infty}\left(\frac{(2^{2n}-1)\,|B_{2n}|}{(2n)!} - \frac{\pi^2}{4}\left(\frac{1}{\pi}\right)^{2n}\right)\frac{(2x)^{2n}}{2n}. \end{aligned}$$

By (1.5), we find that for $n \geqslant 2$,

$$\begin{aligned} \frac{(2^{2n}-1)\,|B_{2n}|}{(2n)!} - \frac{\pi^2}{4}\left(\frac{1}{\pi}\right)^{2n} &< \frac{(2^{2n}-1)}{(2n)!}\frac{2(2n)!}{(2\pi)^{2n}(1-2^{1-2n})} - \frac{\pi^2}{4}\left(\frac{1}{\pi}\right)^{2n} \\ &= 2\left(\frac{4^n-1}{4^n-2} - \frac{\pi^2}{8}\right)\frac{1}{\pi^{2n}} < 0. \end{aligned}$$

Hence, $f_2(x) < 0$ for $0 < x < \pi/2$, and thus, the second inequality in (3.2) holds for $\mu = \pi^2/8$.

The inequality (3.2) can be rewritten as

$$\lambda < \frac{\ln(\sec x)}{\ln\left(\frac{\pi^2}{\pi^2-4x^2}\right)} < \mu, \qquad 0 < x < \frac{\pi}{2}.$$

Elementary calculations reveal that

$$\lim_{x\to 0^+}\frac{\ln(\sec x)}{\ln\left(\frac{\pi^2}{\pi^2-4x^2}\right)} = \frac{\pi^2}{8} \quad \text{and} \quad \lim_{x\to(\pi/2)^-}\frac{\ln(\sec x)}{\ln\left(\frac{\pi^2}{\pi^2-4x^2}\right)} = 1.$$

Hence, inequality (3.2) holds with best possible constants given in (3.3). □

REMARK 3.1. There is no strict comparison between the two upper bounds $\frac{4\pi}{\pi^2-4x^2}$ and $\left(\frac{\pi^2}{\pi^2-4x^2}\right)^{\pi^2/8}$ in (3.1) and (3.2).

REMARK 3.2. By using the Maple software, we find that

$$\sec x - \frac{\pi^2}{\pi^2-4x^2} = \frac{\pi^2-8}{2\pi^2}x^2 + O(x^4), \qquad x \to 0$$

and

$$\sec x - \left(\frac{\pi^2}{\pi^2-4x^2}\right)^{\pi^2/8} = \frac{\pi^2-12}{12\pi^2}x^4 + O(x^6), \qquad x \to 0.$$

This shows that as $x \to 0$, the upper-approximation $\left(\frac{\pi^2}{\pi^2-4x^2}\right)^{\pi^2/8}$ is better than the lower-approximation $\frac{\pi^2}{\pi^2-4x^2}$ in (3.2).

4. Inequalities involving the sine and cosine functions

Another inequality which is of interest to us is Huygens inequality [8], which asserts that

$$2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}>3 \qquad \text{for all} \qquad 0<x<\frac{\pi}{2}. \tag{4.1}$$

Zhu [16] showed some new inequalities of the Huygens type for trigonometric and hyperbolic functions. A simple algebra shows that (4.1) can be written as follows

$$\frac{3\cos x}{1+2\cos x}<\frac{\sin x}{x} \qquad \text{for all} \qquad 0<x<\frac{\pi}{2}. \tag{4.2}$$

In view of inequality (4.2), we now define the function $P(x)$ by

$$P(x)=\frac{a+b\cos x}{1+c\cos x}, \qquad 0<x<\frac{\pi}{2}. \tag{4.3}$$

We are interested in finding the values of the parameters a, b and c such that $P(x)$ approximates as fast as possible to $\sin x/x$ as $x\to 0$. This is addressed in Theorem 4.1. Motivated by the result of Theorem 4.1, we establish sharp bounds for $\sin x/x$ in Theorem 4.2.

THEOREM 4.1. *Let $P(x)$ be defined by (4.3). Then for*

$$a=\frac{9}{14}, \quad b=\frac{3}{7} \qquad \textit{and} \qquad c=\frac{1}{14}, \tag{4.4}$$

we have

$$\lim_{x\to 0}\frac{\frac{\sin x}{x}-P(x)}{x^6}=-\frac{1}{2100}. \tag{4.5}$$

In particular, the speed of the function $P(x)$ approximating $\sin x/x$ is given by the order estimate $O\left(x^6\right)$ as $x\to 0$.

Proof. The power series expansion of $\frac{\sin x}{x}-P(x)$ near 0 is

$$\begin{aligned}\frac{\sin x}{x}-P(x)=&\frac{1+c-a-b}{1+c}+\frac{3b-3ca-1-2c-c^2}{6(1+c)^2}x^2\\&+\frac{-5b+25bc+5ca-25c^2a+3c+3c^2+c^3+1}{120(1+c)^3}x^4\\&+\frac{7b-196bc+427bc^2-7ca+196c^2a-427c^3a-4c-6c^2-4c^3-c^4-1}{5040(1+c)^4}x^6\\&+O(x^8).\end{aligned} \tag{4.6}$$

It is easy to check that for a, b, c as defined in (4.4), we have

$$\begin{cases}1+c-a-b=0\\3b-3ca-1-2c-c^2=0\\-5b+25bc+5ca-25c^2a+3c+3c^2+c^3+1=0,\end{cases}$$

and so

$$\frac{\sin x}{x} - P(x) = \frac{\sin x}{x} - \frac{9+6\cos x}{14+\cos x} = -\frac{1}{2100}x^6 + O(x^8), \qquad x \to 0.$$

This completes the proof of Theorem 4.1. □

THEOREM 4.2. *Let* $0 < |x| < \pi/2$. *Then*
(i) *the inequality*

$$\frac{p+6\cos x}{14+\cos x} < \frac{\sin x}{x} < \frac{q+6\cos x}{14+\cos x} \tag{4.7}$$

holds with best possible constants

$$p = \frac{28}{\pi} = 8.91267681\ldots \quad \textit{and} \quad q = 9; \tag{4.8}$$

(ii) *the inequality*

$$\left(\frac{9+6\cos x}{14+\cos x}\right)^r < \frac{\sin x}{x} < \left(\frac{9+6\cos x}{14+\cos x}\right)^s \tag{4.9}$$

holds with best possible constants

$$r = \frac{\ln(\pi/2)}{\ln(14/9)} = 1.02206706\ldots \quad \textit{and} \quad s = 1. \tag{4.10}$$

Proof. Without a loss of generality, we may assume that $0 < x < \pi/2$. Let

$$g(x) = \frac{\sin x(14+\cos x)}{x} - 6\cos x, \qquad 0 < x < \frac{\pi}{2}.$$

Differentiating and using (1.1) and (1.2), we obtain

$$\begin{aligned} -x^2 g'(x) &= (14-6x^2)\sin x + \frac{1}{2}\sin(2x) - x\cos(2x) - 14x\cos x \\ &= \sum_{n=3}^{\infty}(-1)^n \frac{2n(4^n-12n+8)}{(2n+1)!}x^{2n+1} = \sum_{n=3}^{\infty}(-1)^n u_n(x), \end{aligned}$$

where

$$u_n(x) = \frac{2n(4^n-12n+8)}{(2n+1)!}x^{2n+1}.$$

Elementary calculations reveal that for $0 < x < \pi/2$ and $n \geqslant 3$,

$$\begin{aligned} \frac{u_{n+1}(x)}{u_n(x)} &= \frac{x^2(4^{n+1}-12n-4)}{2n(2n+3)(4^n-12n+8)} < \frac{(\pi/2)^2(4^{n+1}-12n-4)}{2n(2n+3)(4^n-12n+8)} \\ &= \frac{\pi^2(4^{n+1}-12n-4)}{8n(2n+3)(4^n-12n+8)} < \frac{\pi^2}{8n} < 1. \end{aligned}$$

Therefore, for fixed $x \in (0, \pi/2)$, the sequence $n \longmapsto u_n(x)$ is strictly decreasing with regard to $n \geqslant 3$. Hence, for $0 < x < \pi/2$,

$$-x^2 g'(x) > u_3(x) - u_4(x) = \frac{3}{70}x^7 - \frac{1}{210}x^9 > 0\,,$$

and then $g(x)$ is strictly decreasing on $(0, \pi/2)$. Noting that

$$\lim_{x \to 0^+} g(x) = 9 \quad \text{and} \quad \lim_{x \to (\pi/2)^-} g(x) = \frac{28}{\pi},$$

we have

$$\frac{28}{\pi} = \lim_{x \to (\pi/2)^-} g(x) < g(x) = \frac{\sin x(14 + \cos x)}{x} - 6\cos x < \lim_{x \to 0^+} g(x) = 9$$

for all $x \in \left(0, \frac{\pi}{2}\right)$, with the constants $\frac{28}{\pi}$ and 9 being best possible.

Let

$$h(x) = \frac{\ln\left(\frac{\sin x}{x}\right)}{\ln\left(\frac{9+6\cos x}{14+\cos x}\right)}, \quad 0 < x < \frac{\pi}{2} \quad \text{and} \quad h(0) = 1, \tag{4.11}$$

and let

$$h_1(x) = \ln\left(\frac{\sin x}{x}\right), \quad 0 < x < \frac{\pi}{2} \quad \text{and} \quad h_1(0) = 0,$$
$$h_2(x) = \ln\left(\frac{9 + 6\cos x}{14 + \cos x}\right), \qquad 0 \leqslant x < \frac{\pi}{2}.$$

Then, for $0 < x < \pi/2$,

$$\frac{h_1'(x)}{h_2'(x)} = \frac{(42 + 31\cos x + 2\cos^2 x)(\sin x - x\cos x)}{25x\sin^2 x} = h_3(x).$$

Elementary calculations reveal that

$$\begin{aligned}
25x^2 \sin^3 x h_3'(x) &= -23x\sin(2x) - 31x\sin x + x\sin(2x)\cos^2 x \\
&\quad + (40 + 48x^2)\cos^2 x + (2 - 2x^2)\cos^4 x \\
&\quad + (62x^2 - 31)\cos x + 31\cos^3 x + 42x^2 - 42 \\
&= -31x\sin x - \frac{45}{2}x\sin(2x) + \frac{1}{4}x\sin(4x) \\
&\quad + \left(62x^2 - \frac{31}{4}\right)\cos(x) + (21 + 23x^2)\cos(2x) \\
&\quad + \frac{31}{4}\cos(3x) + \left(\frac{1 - x^2}{4}\right)\cos(4x) + \frac{261}{4}x^2 - \frac{85}{4} \\
&= \frac{6}{7}x^8 - \frac{11}{35}x^{10} + \sum_{n=6}^{\infty}(-1)^n v_n(x),
\end{aligned} \tag{4.12}$$

where

$$v_n(x) = \frac{(4n^2-10n+16)16^{n-1}+31\cdot 9^n-(23n^2-34n-21)4^{n+1}-992n^2+744n-31}{4\cdot(2n)!}x^{2n}.$$

Elementary calculations reveal that

$$\frac{v_{n+1}(x)}{v_n(x)} = \frac{4x^2}{2n+1}\alpha_n,$$

where

$$\alpha_n = \frac{(4n^2-2n+10)\cdot 16^n+279\cdot 9^n-(368n^2+192n-512)4^n-992n^2-1240n-279}{(n+1)((2n^2-5n+8)16^n+248\cdot 9^n-(736n^2-1088n-672)4^n-7936n^2+5952n-248)}.$$

It is not difficult to show that

$$0 < \alpha_n < 1 \quad \text{for} \quad n \geqslant 6.$$

Hence, for $0 < x < \pi/2$ and $n \geqslant 6$, we have

$$\frac{v_{n+1}(x)}{v_n(x)} < \frac{4\left(\frac{\pi}{2}\right)^2}{2n+1} = \frac{\pi^2}{2n+1} < 1.$$

Therefore, for fixed $x \in (0,\pi/2)$, the sequence $n \longmapsto v_n(x)$ is strictly decreasing for $n \geqslant 6$. It follows from (4.12) that

$$25x^2\sin^3 x h_3'(x) > \frac{6}{7}x^8 - \frac{11}{35}x^{10} > 0, \quad 0 < x < \frac{\pi}{2},$$

and therefore, $h_3(x) = \frac{h_1'(x)}{h_2'(x)}$ is strictly increasing on $(0,\pi/2)$. By Lemma 1.1, the function

$$h(x) = \frac{h_1(x)}{h_2(x)} = \frac{h_1(x)-h_1(0)}{h_2(x)-h_2(0)}$$

is strictly increasing on $(0,\pi/2)$. Noting that

$$\lim_{x\to 0^+} h(x) = 1 \quad \text{and} \quad \lim_{x\to(\pi/2)^-} h(x) = \frac{\ln(\pi/2)}{\ln(14/9)},$$

we have

$$1 = \lim_{x\to 0^+} h(x) < h(x) = \frac{\ln\left(\frac{\sin x}{x}\right)}{\ln\left(\frac{9+6\cos x}{14+\cos x}\right)} < \lim_{x\to(\pi/2)^-} h(x) = \frac{\ln(\pi/2)}{\ln(14/9)}$$

for all $x \in (0,\pi/2)$, with the constants $\frac{\ln(\pi/2)}{\ln(14/9)}$ and 1 being best possible. The proof of Theorem 4.2 is complete. □

REMARK 4.1. The lower bound $\left(\frac{9+6\cos x}{14+\cos x}\right)^{\frac{\ln(\pi/2)}{\ln(14/9)}}$ in (4.9) is a bit sharper than one $\frac{(28/\pi)+6\cos x}{14+\cos x}$ in (4.7).

REMARK 4.2. By using the Maple software, we find that

$$\frac{\sin x}{x}-\left(\frac{9+6\cos x}{14+\cos x}\right)^{\frac{\ln(\pi/2)}{\ln(14/9)}}=\frac{\ln(9\pi/28)}{6\ln(14/9)}x^2+O(x^4),\qquad x\to 0$$

and

$$\frac{\sin x}{x}-\frac{9+6\cos x}{14+\cos x}=-\frac{1}{2100}x^6+O(x^8),\qquad x\to 0.$$

This shows that as $x\to 0$, the upper-approximation $\frac{9+6\cos x}{14+\cos x}$ is better than the lower-approximation $\left(\frac{9+6\cos x}{14+\cos x}\right)^{\frac{\ln(\pi/2)}{\ln(14/9)}}$ in (4.9).

5. Frame's inequalities

Frame [7] proved that for $0<x<5$,

$$\frac{\left(3+(x^2/11)\right)\sinh x}{2+\cosh x+(x^2/11)}<x<\frac{\left(3+(x^2/10)\right)\sinh x}{2+\cosh x+(x^2/10)},\tag{5.1}$$

which should be called *Frame's inequality*.

In view of inequality (5.1), the following question can be posed: What are the best possible constants ρ_1 and ρ_2 such that the inequalities

$$\frac{(3+\rho_1 x^2)\sinh x}{2+\cosh x+\rho_1 x^2}<x<\frac{(3+\rho_2 x^2)\sinh x}{2+\cosh x+\rho_2 x^2}\tag{5.2}$$

are valid for all $x>0$? The following Theorem 5.1 answers this question.

THEOREM 5.1. *For $x>0$, inequality (5.2) holds with best possible constants*

$$\rho_1=0\quad\textit{and}\quad\rho_2=\frac{1}{10}.\tag{5.3}$$

Proof. Inequality (5.2) can be written for $x>0$ as

$$\rho_1<\frac{2x+x\cosh x-3\sinh x}{x^2(\sinh x-x)}<\rho_2.$$

Let

$$G(x)=\frac{2x+x\cosh x-3\sinh x}{x^2(\sinh x-x)}=\frac{A(x)}{B(x)},$$

where

$$A(x) = 2x + x\cosh x - 3\sinh x = \sum_{n=2}^{\infty} \frac{2(n-1)}{(2n+1)!} x^{2n+1} = \sum_{n=2}^{\infty} a_n x^{2n+1},$$

with

$$a_n = \frac{2(n-1)}{(2n+1)!},$$

and

$$B(x) = x^2(\sinh x - x) = \sum_{n=2}^{\infty} \frac{1}{(2n-1)!} x^{2n+4} = \sum_{n=2}^{\infty} b_n x^{2n+4},$$

with

$$b_n = \frac{1}{(2n-1)!}.$$

Clearly, the sequence

$$\frac{a_n}{b_n} = \frac{n-1}{n(2n+1)}, \qquad n \geqslant 2$$

is strictly decreasing. By Lemma 1.2, the function $G(x)$ is strictly decreasing on $(0,\infty)$. Noting that

$$\lim_{x\to 0^+} G(x) = \frac{1}{10} \quad \text{and} \quad \lim_{x\to\infty} G(x) = 0,$$

we have

$$0 = \lim_{x\to\infty} G(x) < G(x) = \frac{2x + x\cosh x - 3\sinh x}{x^2(\sinh x - x)} < \lim_{x\to 0^+} G(x) = \frac{1}{10}$$

for all $x \in (0,\infty)$, with the constants 0 and $1/10$ being best possible. The proof of Theorem 5.1 is complete. □

Theorem 5.2 establishes a trigonometric version of inequality (5.2).

THEOREM 5.2. *For* $0 < x < \pi/2$,

$$\frac{(3-\rho_1 x^2)\sin x}{2+\cos x - \rho_1 x^2} < x < \frac{(3-\rho_2 x^2)\sin x}{2+\cos x - \rho_2 x^2} \tag{5.4}$$

with best possible constants

$$\rho_1 = \frac{1}{10} \quad \textit{and} \quad \rho_2 = \frac{8\pi - 24}{\pi^3 - 2\pi^2} = 0.100535582\ldots.$$

Proof. Inequality (5.4) can be written for $0 < x < \pi/2$ as

$$\rho_1 < \frac{2x + x\cos x - 3\sin x}{x^2(x - \sin x)} < \rho_2.$$

Let

$$H(x) = \frac{2x + x\cos x - 3\sin x}{x^2(x - \sin x)}, \quad 0 < x < \frac{\pi}{2}.$$

Differentiation yields

$$\begin{aligned} x^3 H'(x) &= \frac{(11x - x^3)\sin x + \frac{1}{2}x\sin(2x) - 3x^2\cos x + 3\cos(2x) - 3x^2 - 3}{x^2 + \sin^2 x - 2x\sin x} \\ &= \frac{C(x)}{D(x)}, \end{aligned}$$

where

$$\begin{aligned} C(x) &= (11x - x^3)\sin x + \frac{1}{2}x\sin(2x) - 3x^2\cos x + 3\cos(2x) - 3x^2 - 3 \\ &= \sum_{n=5}^{\infty} (-1)^{n-1} \frac{32n + 8n^3 - 24n^2 + (n-6)2^{2n-1}}{(2n)!} x^{2n} \\ &= \frac{1}{75600}x^{10} - \frac{1}{453600}x^{12} + \sum_{n=7}^{\infty} (-1)^{n-1} c_n(x), \end{aligned} \tag{5.5}$$

with

$$c_n(x) = \frac{32n + 8n^3 - 24n^2 + (n-6)2^{2n-1}}{(2n)!} x^{2n},$$

and

$$\begin{aligned} D(x) = x^2 + \sin^2 x - 2x\sin x &= \sum_{n=3}^{\infty} (-1)^{n-1} \frac{2^{2n-1} - 4n}{(2n)!} x^{2n} \\ &= \sum_{n=3}^{\infty} (-1)^{n-1} d_n(x), \end{aligned} \tag{5.6}$$

with

$$d_n(x) = \frac{2^{2n-1} - 4n}{(2n)!} x^{2n}.$$

Elementary calculations reveal that for $0 < x < \pi/2$ and $n \geqslant 7$,

$$\begin{aligned}\frac{c_{n+1}(x)}{c_n(x)} &= \frac{2x^2\left(4n+8+4n^3+(n-5)4^n\right)}{(2n+1)(n+1)\left(64n+16n^3-48n^2+(n-6)4^n\right)}\\ &< \frac{2(\pi/2)^2\left(4n+8+4n^3+(n-5)4^n\right)}{(2n+1)(n+1)\left(64n+16n^3-48n^2+(n-6)4^n\right)}\\ &= \frac{\pi^2}{2(2n+1)}\frac{4n+8+4n^3+(n-5)4^n}{(n+1)\left(64n+16n^3-48n^2+(n-6)4^n\right)}\\ &< \frac{\pi^2}{2(2n+1)} < 1.\end{aligned}$$

Therefore, for fixed $x \in (0, \pi/2)$, the sequence $n \longmapsto c_n(x)$ is strictly decreasing with regard to $n \geqslant 7$. It follows from (5.5) that

$$C(x) > \frac{1}{75600}x^{10} - \frac{1}{453600}x^{12} > 0, \qquad 0 < x < \frac{\pi}{2}.$$

Elementary calculations reveal that for $0 < x < \pi/2$ and $n \geqslant 3$,

$$\begin{aligned}\frac{d_{n+1}(x)}{d_n(x)} &= \frac{2x^2(4^n-2n-2)}{(2n+1)(n+1)(4^n-8n)} < \frac{2(\pi/2)^2(4^n-2n-2)}{(2n+1)(n+1)(4^n-8n)}\\ &= \frac{\pi^2(4^n-2n-2)}{2(2n+1)(n+1)(4^n-8n)} < \frac{\pi^2}{2(2n+1)} < 1.\end{aligned}$$

Therefore, for fixed $x \in (0, \pi/2)$, the sequence $n \longmapsto d_n(x)$ is strictly decreasing with regard to $n \geqslant 3$. It follows from (5.6) that

$$D(x) > d_3(x) - d_4(x) = \frac{1}{36}x^6 - \frac{1}{360}x^8 > 0, \qquad 0 < x < \frac{\pi}{2}.$$

Hence,

$$H'(x) = \frac{1}{x^3}\frac{C(x)}{D(x)} > 0, \qquad 0 < x < \frac{\pi}{2},$$

and therefore, $H(x)$ is strictly increasing on $(0, \pi/2)$. Noting that

$$\lim_{x\to 0^+} H(x) = \frac{1}{10} \quad \text{and} \quad \lim_{x\to(\pi/2)^-} H(x) = \frac{8\pi-24}{\pi^3-2\pi^2},$$

we have

$$\begin{aligned}\frac{1}{10} = \lim_{x\to 0^+} H(x) < H(x) &= \frac{2x + x\cos x - 3\sin x}{x^2(x-\sin x)}\\ &< \lim_{x\to(\pi/2)^-} H(x) = \frac{8\pi-24}{\pi^3-2\pi^2}\end{aligned}$$

for all $x \in (0, \pi/2)$, with the constants $1/10$ and $(8\pi - 24)/(\pi^3 - \pi^2)$ being best possible. The proof of Theorem 5.2 is complete. □

REMARK 5.1. (i) For $0 < |x| < \pi/2$, it is known in the literature that

$$\frac{\sin x}{x} < \frac{2 + \cos x}{3}. \tag{5.7}$$

Inequality (5.7) was first mentioned by the German philosopher and theologian Nicolaus de Cusa (1401-1464), by a geometrical method. A rigorous proof of inequality (5.7) was given by Huygens [8], who used (5.7) to estimate the number π. The inequality is now known as Cusa's inequality (see [11, 12, 14, 15]). Further interesting historical facts about inequality (5.7) can be found in [14].

(ii) Taking $\rho_1 = 1/10$ and $\rho_2 = (8\pi - 24)/(\pi^3 - \pi^2)$, inequality (5.4) can be written for $0 < |x| < \pi/2$ as

$$\frac{2 + \cos x - \big((8\pi - 24)/(\pi^3 - \pi^2)\big)x^2}{3 - \big((8\pi - 24)/(\pi^3 - \pi^2)\big)x^2} < \frac{\sin x}{x} < \frac{2 + \cos x - (x^2/10)}{3 - (x^2/10)}. \tag{5.8}$$

The second inequality in (5.8) is sharper than inequality (5.7). In fact, the right side of inequality (5.4) is better than (5.7) for any $0 < \rho_1 < 1$, (not only for $\rho_1 = 1/10$), as this is equivalent to $0 < \cos x < 1$ (simple computations).

(iii) Taking $\rho_1 = 0$ and $\rho_2 = 1/10$, inequality (5.2) can be written for $x \neq 0$ as

$$\frac{2 + \cosh x + (x^2/10)}{3 + (x^2/10)} < \frac{\sinh x}{x} < \frac{2 + \cosh x}{3}. \tag{5.9}$$

The second inequality in (5.9) is hyperbolic version of inequality (5.7). The second inequality in (5.9) appears in [12].

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