Journal of Mathematical Inequalities
Volume 9, Number 2 (2015), 321–330 doi:10.7153/jmi-09-26

INEQUALITIES FOR DUAL QUERMASSINTEGRALS OF THE p–CROSS–SECTION BODIES

WEIDONG WANG AND LI YAN

(*Communicated by J. Pečarić*)

Abstract. Gardner and Giannopoulos defined the p-cross-section body C_pK ($p > -1$) of convex body K in Euclidean space $\mathbb{E}^n$. In this paper, we obtain inequalities for dual quermassintegrals of the p-cross-section body C_pK. Further, two monotonic inequalities concerning the C_pK are given.

1. Introduction

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{E}^n$, for the set of convex bodies containing the origin in their interiors in $\mathbb{E}^n$ by $\mathscr{K}_o^n$. Let $V(K)$ denote the n-dimensional volume of body K. For the unit sphere in $\mathbb{E}^n$, denotes by S^{n-1}.

If K is a compact star-shaped (about the origin) in $\mathbb{E}^n$, its radial function, $\rho_K = \rho(K,\cdot)$, is defined by (see [3, 22])

$$\rho(K,u) = \max\{\lambda \geqslant 0 : \lambda u \in K\},$$

for all $u \in S^{n-1}$. If ρ_K is positive and continuous, K will be called a star body (about the origin). Let $\mathscr{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbb{E}^n$. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

In mid 1990s, Lutwak in [12, 13] showed that the Firey sum (see [2]) of convex bodies led to the Brunn-Minkowski theory for each $p \geqslant 1$, and established an embryonic L_p-Brunn-Minkowski theory. This theory has expanded rapidly (see [4, 5, 7, 8, 9, 14, 15, 16, 17, 18, 19, 20, 23, 24, 25, 26, 27, 29, 30]).

In 1999, Gardner and Giannopoulos in [4] introduced the notion of p-cross-section body as follows: For $K \in \mathscr{K}^n$ and nonzero $p > -1$, the p-cross-section body, C_pK, of K is defined by

$$\rho^p_{C_pK}(u) = \frac{1}{V(K)} \int_K V_{n-1}(K \cap (u^\perp + x))^p dx, \tag{1.1}$$

Mathematics subject classification (2010): 52A40, 52A20, 52A39.
Keywords and phrases: p-cross-section body, dual quermassintegrals, intersection body, monotonic.
Research is supported in part by the Natural Science Foundation of China (Grant No. 11371224).

Paper JMI-09-26

for all $u \in S^{n-1}$. They also defined that for each $u \in S^{n-1}$,

$$\rho_{C_0K}(u) = \exp\left(\frac{1}{V(K)}\int_K \log V_{n-1}(K\cap(u^\perp + x))dx\right)$$

and

$$\rho_{C_\infty K}(u) = \max_{x\in K} V_{n-1}(K\cap(u^\perp + x)).$$

Note that the classical cross-section body CK of $K \in \mathscr{K}^n$ is defined by (see [3])

$$\rho_{CK}(u) = \max_{x\in K} V_{n-1}(K\cap(u^\perp + x)),$$

for each $u \in S^{n-1}$. Compare to above definitions of CK and $C_\infty K$, we know

$$C_\infty K = CK. \tag{1.2}$$

Further, Gardner and Giannopoulos ([4]) proved that for $K \in \mathscr{K}^n$, $-1 < p < q$, then

$$CK \subseteq a_{n,q}C_qK \subseteq a_{n,p}C_pK, \tag{1.3}$$

in each inclusion equality holds if and only if $n = 2$ and K is a triangle. Where

$$a_{n,p} = \left(\frac{np+n-p}{n}\right)^{\frac{1}{p}},$$

for nonzero $p > -1$, and

$$a_{n,0} = \lim_{p\to 0} a_{n,p} = e^{(n-1)/n}.$$

Recall that intersection body IK of $K \in \mathscr{S}_o^n$ is a centered body defined by (see [3])

$$\rho_{IK}(u) = V_{n-1}(K\cap u^\perp) = \frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \rho_K^{n-1}(v)dv, \tag{1.4}$$

for each $u \in S^{n-1}$. Here $V_{n-1}(M)$ denotes the $n-1$-dimensional volume of body M.

Because of $K\cap(u^\perp + x) = (K-x)\cap u^\perp$, then by (1.4) and (1.1) we know that for nonzero $p > -1$,

$$\rho^p_{C_pK}(u) = \frac{1}{V(K)}\int_K \rho^p_{I(K-x)}(u)dx. \tag{1.5}$$

For the classical cross-section body, Busemann's theorem shows that if K is centrally symmetric with center x, then CK is convex. Meyer (see [21]) proved that CK is convex when $n = 3$, but Brehm in [1] showed that when $n \geqslant 4$, CK is not convex when K is a simplex.

According to the definition of p-cross-section body, Gardner and Giannopoulos in [4] pointed out that ρ_{C_pK} is continuous for $K \in \mathscr{K}^n$. Further, they (see [4]) showed that C_1K is convex, and C_pK is convex when $n = 2$ and $p > 0$ or $n = 3$ and $p = \infty$.

For the p-cross-section bodies, Wang and Zhou in [28] gave the following result:

THEOREM 1.A. *If* $K \in \mathscr{K}^n$, $p > -1$, *then exists* $x_0 \in K$ *such that for* $-1 < p < n$,

$$V(C_pK) \leqslant V(I(K - x_0)), \tag{1.6}$$

for $p > n$,

$$V(C_pK) \geqslant V(I(K - x_0)). \tag{1.7}$$

In every inequality with equality if and only if $C_pK = I(K - x_0)$. *For* $p = n$, *(1.6) (or (1.7)) is identic.*

Further, they ([28]) established the following monotony inequalities of C_pK.

THEOREM 1.B. *For* $K, L \in \mathscr{K}^n$, $p > 0$, *if* $K \subseteq L$, *then*

$$V(K)^{\frac{n}{p}} V(C_pK) \leqslant V(L)^{\frac{n}{p}} V(C_pL),$$

with equality if and only if $K = L$.

THEOREM 1.C. *For* $K, L \in \mathscr{K}^n$, *nonzero* $p > -1$, *if* $C_pK \subseteq C_pL$, *then there exist* $x_0 \in K$ *and* $y_0 \in L$ *such that*

$$V(I(K - x_0)) \leqslant V(I(L - y_0)),$$

with equality if and only if $p = n$ *and* $C_pK = C_pL$ *or* $p \neq n$ *and* $C_pK = C_pL$ *and* $I(K - x_0) = I(L - y_0)$.

Except [28], the reports of p-cross-section bodies are few since this notion was introduced. In this paper, we continue to research the p-cross-section bodies. First, we extend Theorem 1.A from volume to dual quermassintegrals form.

THEOREM 1.1. *If* $K \in \mathscr{K}^n$, $p > 0$, *real* $i \neq n$, *then exists* $x_0 \in K$ *such that for* $i < n - p$ *or* $i > n$,

$$\widetilde{W}_i(C_pK) \leqslant \widetilde{W}_i(I(K - x_0)); \tag{1.8}$$

for $n - p < i < n$,

$$\widetilde{W}_i(C_pK) \geqslant \widetilde{W}_i(I(K - x_0)). \tag{1.9}$$

In every case with equality if and only if $C_pK = I(K - x_0)$. *For* $i = n - p$, *(1.8) (or (1.9)) is identic.*

Here, $\widetilde{W}_i(K)$ denotes the dual quermassintegrals of star body K. Obviously, let $i = 0$ in Theorem 1.1 and notice $\widetilde{W}_0(K) = V(K)$, we easily get Theorem 1.A.

As the application of inequality (1.8), and notice that $CK \subseteq a_{n,p}C_pK$ by (1.3), we have that

COROLLARY 1.1. *If* $K \in \mathscr{K}^n$, $p > 0$, *real* $i < n - p$ *or* $i > n$, *then there exists* $x_0 \in K$ *such that*

$$\widetilde{W}_i(CK) \leqslant a_{n,p}^{n-i} \widetilde{W}_i(I(K - x_0)),$$

with equality if and only if $n = 2$ *and* K *is a triangle and* $CK = a_{n,p}I(K - x_0)$.

Let $p \rightarrow +\infty$ in inequality (1.9) and use (1.2), we obtain the following result.

COROLLARY 1.2. *If $K \in \mathscr{K}^n$, real $i < n$, then there exists $x_0 \in K$ such that*

$$\widetilde{W}_i(CK) \geqslant \widetilde{W}_i(I(K-x_0)),$$

with equality if and only if $CK = I(K-x_0)$.

Next, we obtain the following stronger results than Theorem 1.B and Theorem 1.C.

THEOREM 1.2. *Let $K,L \in \mathscr{K}^n$, $p > -1$. If $K \subseteq L$, then*

$$V(K)^{1/p}C_pK \subseteq V(L)^{1/p}C_pL. \tag{1.10}$$

Equality hold if and only if $K = L$ and there exist $x_0 \in K$ and $y_0 \in L$ such that $I(K-x_0) = I(L-y_0)$.

THEOREM 1.3. *Let $K,L \in \mathscr{K}^n$, $p > -1$. If $C_pK \subseteq C_pL$, then there exist $x_0 \in K$ and $y_0 \in L$ such that*

$$I(K-x_0) \subseteq I(L-y_0), \tag{1.11}$$

with equality if and only if $C_pK = C_pL$ and $I(K-x_0) = I(L-y_0)$.

Let $p \longrightarrow +\infty$ in Theorem 1.3 and use (1.2), we get that

COROLLARY 1.3. *Let $K,L \in \mathscr{K}^n$, if $CK \subseteq CL$, then there exist $x_0 \in K$ and $y_0 \in L$ such that*

$$I(K-x_0) \subseteq I(L-y_0),$$

with equality if and only if $CK = CL$ and $I(K-x_0) = I(L-y_0)$.

2. L_p-dual mixed quermassintegrals

The notion of dual quermassintegrals was given by Lutwak (see [11]). For $K \in \mathscr{S}_o^n$, i is any real, the dual quermassintegrals, $\widetilde{W}_i(K)$, of K are defined by

$$\widetilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i}du. \tag{2.1}$$

Obviously, let $i = 0$ in (2.1), then

$$\widetilde{W}_0(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n}du = V(K).$$

If $K,L \in \mathscr{S}_o^n$, $p > 0$, $\lambda,\mu \geqslant 0$ (not both zero), the L_p-radial combination, $\lambda \cdot K \tilde{+}_p \mu \cdot L \in \mathscr{S}_o^n$, of K and L is defined by (see [6, 7])

$$\rho(\lambda \cdot K \tilde{+}_p \mu \cdot L, \cdot)^p = \lambda\rho(K,\cdot)^p + \mu\rho(L,\cdot)^p. \tag{2.2}$$

Associated with (2.1) and (2.2), we define a kind of L_p-dual mixed quermassintegrals as follows: For $K,L \in \mathscr{S}_o^n$ and real $i \neq n$, the L_p-dual mixed quermassintegrals, $\widetilde{W}_{p,i}(K,L)$, of K and L are defined by

$$\frac{n-i}{p}\widetilde{W}_{p,i}(K,L) = \lim_{\varepsilon \longrightarrow 0^+} \frac{\widetilde{W}_i(K \tilde{+}_p \varepsilon \cdot L) - \widetilde{W}_i(K)}{\varepsilon}. \tag{2.3}$$

From definition (2.3), the integral representation of L_p-dual mixed quermassintegrals can be established as follows:

THEOREM 2.1. *If $K,L \in \mathscr{S}_o^n$, $p>0$, and real $i \neq n$, then*

$$\widetilde{W}_{p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-p-i}(u)\rho_L^p(u)du. \tag{2.4}$$

Proof. From (2.2) and (2.3), for $i \neq n$, we have that

$$\begin{aligned}
&\lim_{\varepsilon \longrightarrow 0^+} \frac{\widetilde{W}_i(K \tilde{+}_p \varepsilon \cdot L) - \widetilde{W}_i(K)}{\varepsilon} \\
&= \lim_{\varepsilon \longrightarrow 0^+} \frac{1}{n}\int_{S^{n-1}} \frac{\rho(K \tilde{+}_p \varepsilon \cdot L,u)^{n-i} - \rho(K,u)^{n-i}}{\varepsilon} du \\
&= \lim_{\varepsilon \longrightarrow 0^+} \frac{1}{n}\int_{S^{n-1}} \frac{[\rho(K,u)^p + \varepsilon\rho(L,u)^p]^{\frac{n-i}{p}} - \rho(K,u)^{n-i}}{\varepsilon} du.
\end{aligned}$$

By Hospital's rule we see that

$$\begin{aligned}
&\lim_{\varepsilon \longrightarrow 0^+} \frac{[\rho(K,\cdot)^p + \varepsilon\rho(L,\cdot)^p]^{\frac{n-i}{p}} - \rho(K,\cdot)^{n-i}}{\varepsilon} \\
&= \lim_{\varepsilon \longrightarrow 0^+} \rho_K^{n-i} \frac{[1 + \varepsilon(\rho_K/\rho_L)^p]^{\frac{n-i}{p}} - 1}{\varepsilon} \\
&= \frac{n-i}{p}\rho_K^{n-p-i}\rho_L^p,
\end{aligned}$$

thus we get formula (2.4) by definition (2.3). □

From (2.4), we easily know that

$$\widetilde{W}_{p,i}(K,K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i}du = \widetilde{W}_i(K), \tag{2.5}$$

$$\widetilde{W}_{p,n-p}(K,L) = \widetilde{W}_{n-p}(L). \tag{2.6}$$

The Minkowski's inequality for the L_p-dual mixed quermassintegrals is given that

THEOREM 2.2. *Let $K,L \in \mathscr{S}_o^n$, $p>0$, and real $i \neq n$. If $i < n-p$, then*

$$\widetilde{W}_{p,i}(K,L) \leqslant \widetilde{W}_i(K)^{(n-p-i)/(n-i)}\widetilde{W}_i(L)^{p/(n-i)}; \tag{2.7}$$

if $n-p<i<n$ or $i>n$, then

$$\widetilde{W}_{p,i}(K,L) \geqslant \widetilde{W}_i(K)^{(n-p-i)/(n-i)}\widetilde{W}_i(L)^{p/(n-i)}. \tag{2.8}$$

In every case, equality holds in every inequality if and only if K and L are dilates. For $i=n-p$, (2.7) (or (2.8)) is identic.

Proof. For $i<n-p$, from (2.4) and together with Hölder inequality (see [10]), we have that

$$\begin{aligned}\widetilde{W}_{p,i}(K,L) &= \frac{1}{n}\int_{S^{n-1}}\rho_K^{n-p-i}(u)\rho_L^p(u)du\\ &\leqslant \left[\frac{1}{n}\int_{S^{n-1}}[\rho_K^{n-p-i}(u)]^{\frac{n-i}{n-p-i}}du\right]^{\frac{n-p-i}{n-i}}\left[\frac{1}{n}\int_{S^{n-1}}[\rho_L^p(u)]^{\frac{n-i}{p}}du\right]^{\frac{p}{n-i}}\\ &= \left[\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)du\right]^{\frac{n-p-i}{n-i}}\left[\frac{1}{n}\int_{S^{n-1}}\rho_L^{n-i}(u)du\right]^{\frac{p}{n-i}}\\ &= \widetilde{W}_i(K)^{\frac{n-p-i}{n-i}}\widetilde{W}_i(L)^{\frac{p}{n-i}},\end{aligned}$$

this give inequality (2.7) when $i<n-p$. According to the condition of equality holds in the Hölder inequality, we know the equality holds in inequality (2.7) if and only if K and L are dilates.

Similarly, we can prove for $n-p<i<n$ or $i>n$, inequality (2.8) is true.

For $i=n-p$, by (2.1) and (2.6) then

$$\widetilde{W}_{p,i}(K,L)^{n-i} = \widetilde{W}_{p,n-p}(K,L)^p = \widetilde{W}_{n-p}(L)^p,$$

and

$$\widetilde{W}_i(K)^{n-p-i}\widetilde{W}_i(L)^p = \widetilde{W}_{n-p}(K)^{n-p-(n-p)}\widetilde{W}_{n-p}(L)^p = \widetilde{W}_{n-p}(L)^p,$$

thus (2.7) (or (2.8)) is identic when $i=n-p$. □

3. Proofs of the Theorems

LEMMA 3.1. *If $K\in\mathscr{K}^n$, $p>0$ and real $i\neq n$, then for any $Q\in\mathscr{S}_o^n$,*

$$\widetilde{W}_{p,i}(Q,C_pK) = \frac{1}{V(K)}\int_K \widetilde{W}_{p,i}(Q,I(K-x))dx. \tag{3.1}$$

Proof. From (1.5) and (2.4), then for any $Q\in\mathscr{S}_o^n$ and $p>0$, we have that

$$\begin{aligned}\widetilde{W}_{p,i}(Q,C_pK) &= \frac{1}{n}\int_{S^{n-1}}\rho_Q(u)^{n-p-i}\rho_{C_pK}(u)^p du\\ &= \frac{1}{nV(K)}\int_{S^{n-1}}\int_K \rho_Q(u)^{n-p-i}\rho_{I(K-x)}(u)^p dxdu\\ &= \frac{1}{V(K)}\int_K\left[\frac{1}{n}\int_{S^{n-1}}\rho_Q(u)^{n-p-i}\rho_{I(K-x)}(u)^p du\right]dx\\ &= \frac{1}{V(K)}\int_K \widetilde{W}_{p,i}(Q,I(K-x))dx. \quad \square\end{aligned}$$

Proof of Theorem 1.1. Let $Q=C_pK$ in (3.1) and use (2.5), we have that

$$\widetilde{W}_i(C_pK) = \frac{1}{V(K)}\int_K \widetilde{W}_{p,i}(C_pK,I(K-x))dx.$$

This together with the integral mean value theorem, then there exists $x_o \in K$ such that

$$\widetilde{W}_i(C_pK) = \frac{1}{V(K)}\widetilde{W}_{p,i}(C_pK, I(K-x_o))\int_K dx = \widetilde{W}_{p,i}(C_pK, I(K-x_o)). \tag{3.2}$$

Hence for $i < n-p$, by (3.2) and inequality (2.7), we get that

$$\widetilde{W}_i(C_pK) \leqslant \widetilde{W}_i(C_pK)^{\frac{n-p-i}{n-i}}\widetilde{W}_i(I(K-x_o))^{\frac{p}{n-i}},$$

i.e.,

$$\widetilde{W}_i(C_pK)^{\frac{p}{n-i}} \leqslant \widetilde{W}_i(I(K-x_o))^{\frac{p}{n-i}},$$

this yields inequality (1.8) when $i < n-p$.

For $n-p < i < n$ or $i > n$, associated with (3.2) and inequality (2.8), then

$$\widetilde{W}_i(C_pK) \geqslant \widetilde{W}_i(C_pK)^{\frac{n-p-i}{n-i}}\widetilde{W}_i(I(K-x_o))^{\frac{p}{n-i}},$$

i.e.,

$$\widetilde{W}_i(C_pK)^{\frac{p}{n-i}} \geqslant \widetilde{W}_i(I(K-x_o))^{\frac{p}{n-i}}.$$

Thus for $i > n$,

$$\widetilde{W}_i(C_pK) \leqslant \widetilde{W}_i(I(K-x_o));$$

for $n-p < i < n$,

$$\widetilde{W}_i(C_pK) \geqslant \widetilde{W}_i(I(K-x_o)).$$

From this, inequalities (1.8) and (1.9) are obtained, respectively.

According to the conditions of equality hold in inequalities (2.7) and (2.8), we see that equality hold in (1.8) and (1.9) if and only if C_pK and $I(K-x_o)$ are dilates. But $\widetilde{W}_i(C_pK) = \widetilde{W}_i(I(K-x_o))$, this means that $C_pK = I(K-x_o)$. Hence equality hold in (1.8) and (1.9) if and only if $C_pK = I(K-x_o)$.

For $i = n-p$, by (2.6) we see (1.8) (or (1.9)) is identic. □

Proof of Theorem 1.2. Since $K \subseteq L$, then $K-x \subseteq L-x$ for all $x \in K$. According to definition (2.3), we have that

$$\rho_{I(K-x)}(u) = V_{n-1}((K-x)\cap u^{\perp}) \leqslant V_{n-1}((L-x)\cap u^{\perp}) = \rho_{I(L-x)}(u) \tag{3.3}$$

for all $u \in S^{n-1}$. This together with (1.5), we know that for $p > 0$ and all $u \in S^{n-1}$,

$$V(K)\rho^p_{C_pK}(u) \leqslant V(L)\rho^p_{C_pL}(u).$$

Because of $p > 0$, hence (1.10) is given.

For $-1 < p < 0$, according to (3.3) and (1.5), we get

$$V(K)\rho^p_{C_pK}(u) \geqslant V(L)\rho^p_{C_pL}(u),$$

for all $u \in S^{n-1}$. Therefore, if $-1 < p < 0$, then for all $u \in S^{n-1}$,

$$V(K)^{1/p}\rho_{C_pK}(u) \leqslant V(L)^{1/p}\rho_{C_pL}(u).$$

This still gives (1.10).

Now we give the equality conditions of Theorem 1.2. Obviously, if $K = L$, then equality holds in (1.10). Conversely, equality holds in (1.10) equivalent to

$$\int_K \rho^p_{I(K-x)}(u)dx = \int_L \rho^p_{I(L-y)}(u)dy.$$

Thus, by the integral mean value theorem, there exist $x_0 \in K$ and $y_0 \in L$ such that for all $u \in S^{n-1}$,

$$\rho^p_{I(K-x_0)}(u)\int_K dx = \rho^p_{I(L-y_0)}(u)\int_L dy,$$

i.e.,

$$V(K)^{\frac{1}{p}} I(K-x_0) = V(L)^{\frac{1}{p}} I(L-y_0).$$

Due to $K = L$ implies $V(K) = V(L)$. From this, we see that equality holds in Theorem 1.2 if and only if $K = L$ and there exist $x_0 \in K$ and $y_0 \in L$ such that $I(K-x_0) = I(L-y_0)$. □

Proof of Theorem 1.3. Since $C_pK \subseteq C_pL$, thus by (1.5), we have that for $p > 0$,

$$\frac{1}{V(K)}\int_K \rho^p_{I(K-x)}(u)dx \leqslant \frac{1}{V(L)}\int_L \rho^p_{I(L-y)}(u)dy.$$

According to the integral mean value theorem, there exist $x_0 \in K$ and $y_0 \in L$ such that

$$\frac{1}{V(K)}\rho^p_{I(K-x_0)}(u)\int_K dx \leqslant \frac{1}{V(L)}\rho^p_{I(L-y_0)}(u)\int_L dy,$$

i.e.

$$\rho^p_{I(K-x_0)}(u) \leqslant \rho^p_{I(L-y_0)}(u),$$

for all $u \in S^{n-1}$. Hence, for $p > 0$,

$$I(K-x_0) \subseteq I(L-y_0).$$

For $-1 < p < 0$, due to $C_pK \subseteq C_pL$, then by (1.5) and above proof, we know that there exist $x_0 \in K$ and $y_0 \in L$ such that

$$\rho^p_{I(K-x_0)}(u) \geqslant \rho^p_{I(L-y_0)}(u),$$

for all $u \in S^{n-1}$. Therefore, for $-1 < p < 0$, we still get

$$I(K-x_0) \subseteq I(L-y_0).$$

To sum up, we obtain (1.11) whether $p > 0$ or $-1 < p < 0$. Clearly, equality holds in Theorem 1.3 if and only if $C_pK = C_pL$ and $I(K-x_0) = I(L-y_0)$. □

REFERENCES

[1] U. BREHM, *Convex bodies with nonconvex cross-section bodies*, Mathematika **46**, 1 (1999), 127–129.

[2] W. J. FIREY, *p-means of convex bodies*, Math Scand. **10** (1962), 17–24.

[3] R. J. GARDNER, *Geometric Tomography*, Second ed., Cambridge Univ. Press, Cambridge, 2006.

[4] R. J. GARDNER AND A. A. GIANNOPOULOS, *p-cross-section bodies,* Indiana U. Math. J. **48** (1999), 593–613.

[5] R. J. GARDNER AND G. Y. ZHANG, *Affine inequalities and radial mean bodies,* Amer. J. Math. **120** (1998), 505–528.

[6] E. GRINBERG AND G. Y. ZHANG, *Convolutions, transforms, and convex bodies*, Proc. London Math. Soc. **78** (1999), 77–115.

[7] C. HABERL, *L_p intersection bodies*, Adv. Math. **217** (2008), 2599–2624.

[8] C. HABERL AND M. LUDWIG, *A characterization of L_p intersection bodies*, International Mathematics Research Notices (Int. Math. Res. Not.), 2006, Art ID 10548, 29 pages.

[9] C. HABERL AND F. SCHUSTER, *General L_p affine isoperimetric inequalities*, J. Differential Geom. **83** (2009), 1–26.

[10] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities,* Cambridge University Press, Cambridge, 1959.

[11] E. LUTWAK, *Dual mixed volumes*, Pacific J. Math. **58** (1975), 531–538.

[12] E. LUTWAK, *The Brunn-Minkowski-Firey theory I: mixed volumes and the minkowski problem*, J. Differential Geom. **38** (1993), 131–150.

[13] E. LUTWAK, *The Brunn-Minkowski-Firey theory II: affine and geominimal surface areas*, Adv. Math. **118** (1996), 244–294.

[14] E. LUTWAK, D. YANG AND G. Y. ZHANG, *L_p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111–132.

[15] E. LUTWAK, D. YANG AND G. Y. ZHANG, *Sharp affine L_p Sobolev inequalities*, J. Differential Geom. **52** (2002), 17–38.

[16] E. LUTWAK, D. YANG AND G. Y. ZHANG, *On the L_p-Minkowski problem*, Trans. Amer. Math. Soc. **356** (2004), 4359–4370.

[17] E. LUTWAK, D. YANG AND G. Y. ZHANG, *L_p John ellipsoids*, Proc. London Math. Soc. **90** (2005), 497–520.

[18] E. LUTWAK, D. YANG AND G. Y. ZHANG, *Volume inequalities for subspace of L_p*, J. Differential Geom. **68** (2004), 159–184.

[19] E. LUTWAK AND G. Y. ZHANG, *Blaschke-Santalò inequalities*, J. Differential Geom. **47** (1997), 1–16.

[20] T. Y. MA AND W. D. WANG, *On the analog of Shephard problem for the L_p-projection body*, Math. Ineq. Appl. **14** (2011), 181–192.

[21] M. MEYER, *Maximal hyperplane sections of convex bodies*, Mathematika **46**, 1 (1999), 131–136.

[22] R. SCHNEIDER, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge Univ. Press, Cambridge, Second Expanded Edition, 2014.

[23] W. WANG AND B. W. HE, *L_p-dual affine surface area*, J. Math. Anal. Appl. **348** (2008), 746–751.

[24] W. WANG AND B. W. HE, *Inequalities for L_p-dual affine surface area*, Math. Ineq. Appl. **13** (2010), 319–327.

[25] W. D. WANG AND G. S. LENG, *L_p-dual mixed quermassintegrals*, Indian J. Pure Appl. Math. **36** (2005), 177–188.

[26] W. D. WANG, F. H. LU AND G. S. LENG, *A type of monotonicity on the L_p centroid body and L_p projection body*, Math. Ineq. Appl. **8** (2005), 735–742.

[27] W. D. WANG AND C. QI, *L_p-dual geominimal surface area*, J. Inequal. Appl. **2011**, 6 (2011), 10 pages.

[28] W. D. WANG AND Y. P. ZHOU, *Inequalities for the p-cross-section bodies*, Math. Ineq. Appl. **17**, 3 (2014), 1005–1013.

[29] E. WERNER AND D. YE, *New L_p-affine isoperimetric inequalities*, Adv. Math. **218** (2008), 762–780.
[30] J. YUAN AND C. W. SUM, *L_p-intersection bodies*, J. Math. Anal. Appl. **339** (2008), 1431–1439.

(Received February 3, 2014)

Weidong Wang
Department of Mathematics, China Three Gorges University
Yichang, 443002, P. R. China
e-mail: wdwxh722@163.com

Li Yan
Department of Mathematics, China Three Gorges University
Yichang, 443002, P. R. China