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AN OPTIMAL INEQUALITIES CHAIN FOR BIVARIATE MEANS

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Abstract. Let $p \in \mathbb{R}$, M be a bivariate mean, and M_p be defined by $M_p(a,b) = M^{1/p}(a^p,b^p)$ $(p \neq 0)$ and $M_0(a,b) = \lim_{p\to 0} M_p(a,b)$. In this paper, we prove that the sharp inequalities $L_2(a,b) < P(a,b) < NS_{1/2}(a,b) < He(a,b) < A_{2/3}(a,b) < I(a,b) < Z_{1/3}(a,b) < Y_{1/2}(a,b)$ hold for all $a,b > 0$ with $a \neq b$, where $L(a,b) = (a-b)/(\log a - \log b)$, $P(a,b) = (a-b)/[2\arcsin((a-b)/(a+b))]$, $NS(a,b) = (a-b)/[2\operatorname{arcsinh}((a-b)/(a+b))]$, $He(a,b) = (a+\sqrt{ab}+b)/3$, $A(a,b) = (a+b)/2$, $I(a,b) = 1/e(a^a/b^b)^{1/(a-b)}$, $Z(a,b) = a^{a/(a+b)}b^{b/(a+b)}$ and $Y(a,b) = I(a,b)e^{1-ab/L^2(a,b)}$ are respectively the logarithmic, first Seiffert, Neuman-Sándor, Heronian, arithmetic, identric, power-exponential and exponential-geometric means of a and b.

1. Introduction

A bivariate real valued function $M : (0,\infty) \times (0,\infty) \to (0,\infty)$ is said to be a bivariate mean if

$$\min(a,b) \leqslant M(a,b) \leqslant \max(a,b)$$

for all $a,b > 0$. Clearly, each bivariate mean M is reflexive, that is,

$$M(a,a) = a$$

for any $a > 0$. M is symmetric if

$$M(a,b) = M(b,a)$$

for all $a,b > 0$, and M is said to be homogeneous (of degree one) if

$$M(\lambda a, \lambda b) = \lambda M(a,b)$$

for any $\lambda, a, b > 0$.

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There are many symmetric and homogeneous bivariate means, for example,

- the arithmetic mean $A(a,b)$ is defined by

$$A(a,b)=\frac{a+b}{2}; \tag{1.1}$$

- the geometric mean $G(a,b)$ is given by

$$G(a,b)=\sqrt{ab}; \tag{1.2}$$

- the Heronian mean $He(a,b)$ is defined by

$$He(a,b)=\frac{a+b+\sqrt{ab}}{3}; \tag{1.3}$$

- the logarithmic mean $L(a,b)$ is given by

$$L(a,b)=\frac{a-b}{\log a-\log b} \text{ if } a\neq b \quad \text{and} \quad L(a,a)=a; \tag{1.4}$$

- the identric (exponential) mean $I(a,b)$ is defined by

$$I(a,b)=e^{-1}\left(\frac{a^a}{b^b}\right)^{1/(a-b)} \quad \text{if } a\neq b \quad \text{and} \quad I(a,a)=a; \tag{1.5}$$

- the first Seiffert mean $P(a,b)$ is defined in [21] as follows

$$P(a,b)=\frac{a-b}{2\arcsin\frac{a-b}{a+b}} \text{ if } a\neq b \quad \text{and} \quad P(a,a)=a; \tag{1.6}$$

- the second Seiffert mean $T(a,b)$ is defined in [22] and given by

$$T(a,b)=\frac{a-b}{2\arctan\frac{a-b}{a+b}} \quad \text{if} \quad a\neq b \quad \text{and} \quad T(a,a)=a; \tag{1.7}$$

- the Neuman-Sándor mean NS is introduced in [15] and given by

$$NS(a,b)=\frac{a-b}{2\operatorname{arcsinh}\frac{a-b}{a+b}} \quad \text{if} \quad a\neq b \quad \text{and} \quad NS(a,a)=a; \tag{1.8}$$

- the power-exponential mean $Z(a,b)$, or the special case of Gini means [9] is defined by

$$Z(a,b)=a^{\frac{a}{a+b}}b^{\frac{b}{a+b}}=\frac{I(a^2,b^2)}{I(a,b)}; \tag{1.9}$$

- the exponential-geometric mean $Y(a,b)$ is introduced in [24] and given by

$$Y(a,b) = I(a,b)\exp\left(1 - \frac{G^2(a,b)}{L^2(a,b)}\right). \tag{1.10}$$

Let $p \in \mathbb{R}$ be a real number and M be a bivariate mean. Then the function M_p : $(0,\infty)\times(0,\infty)\to(0,\infty)$ defined by

$$M_p(a,b) = M^{1/p}(a^p,b^p) \text{ if } p \neq 0 \quad \text{and} \quad M_0(a,b) = \lim_{p\to 0} M_p(a,b) \tag{1.11}$$

(see [3]) is likely to be a mean of positive numbers a and b. For instants, all $M_p(a,b)$ are means for $M = A$, He, L, I, P, T, NS, Z and Y. It is well known that A_p is the classical power or Hölder mean of a and b, which is strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$.

If $M_p(a,b)$ is proved to be a mean, then it is called "p-order M mean". Since the form of $M_p(a,b)$ is similar to power mean $A_p(a,b)$, so we also call it "power-type mean". Also, we note that

$$M^{\lambda}_{\lambda p}(a,b) = M^{1/p}(a^{\lambda p},b^{\lambda p}) = M_p(a^{\lambda},b^{\lambda}) \tag{1.12}$$

for all $\lambda \in \mathbb{R}$.

Recently, to find the inequalities between different bivariate means have attracted the attention of many researchers. Lin [14] presented the best possible upper and lower power mean bounds for the logarithmic mean as follows

$$A_0(a,b) < L(a,b) < A_{1/3}(a,b). \tag{1.13}$$

for all $a,b > 0$ with $a \neq b$.

Jiao and Cao [13] established that

$$L(a,b) < He_{1/2}(a,b) < A_{1/3}(a,b) \tag{1.14}$$

for all $a,b > 0$ with $a \neq b$, which is equivalent to

$$L_2(a,b) < He(a,b) < A_{2/3}(a,b) \tag{1.15}$$

(also see [2]).

Stolarsky [18] and Pittenger [17] found that the double inequality

$$A_{2/3}(a,b) < I(a,b) < A_{\log 2}(a,b) \tag{1.16}$$

holds for all $a,b > 0$ and $a \neq b$ with the best possible parameters $2/3$ and $\log 2$.

Jagers [12] (also see [10]) proved that

$$A_{1/2}(a,b) < P(a,b) < A_{2/3}(a,b)$$

for all $a,b > 0$ with $a \neq b$. Hästö [11] found that the double inequality

$$A_{\log 2/\log \pi}(a,b) < P(a,b) < A_{2/3}(a,b) \tag{1.17}$$

holds for all $a,b>0$ and $a\neq b$ with the best possible parameters $\log 2/\log\pi$ and $2/3$.

In 1995, Seiffert [22] presented that

$$A(a,b)<T(a,b)<A_2(a,b) \tag{1.18}$$

for all $a,b>0$ with $a\neq b$, it was improved by Yang [27] (also see [7]) as

$$A_{\log 2/(\log\pi-\log 2)}(a,b)<T(a,b)<A_{5/3}(a,b) \tag{1.19}$$

with the best possible parameters $\log 2/(\log\pi-\log 2)$ and $5/3$. Utilizing (1.12) the second inequality of 1.19 can be written as

$$T_{2/5}(a,b)<A_{2/3}(a,b). \tag{1.20}$$

Chu et al. [4] found an optimal double inequality

$$He_{\log 3/(\log\pi-\log 2)}(a,b)<T(a,b)<He_{5/2}(a,b) \tag{1.21}$$

for all $a,b>0$ with $a\neq b$. The second inequality of (1.21) is equivalent to

$$T_{2/5}(a,b)<He(a,b). \tag{1.22}$$

Yang [28] (also see [7], [5], [29]) presented the sharp bounds for the Neuman-Sándor mean in terms of power means as follows

$$A_{\log 2/[\log(\log(3+2\sqrt{2}))]}(a,b)<NS(a,b)<A_{4/3}(a,b) \tag{1.23}$$

for all $a,b>0$ with $a\neq b$. The second inequality of (1.23) implies that

$$NS_{1/2}(a,b)<A_{2/3}(a,b). \tag{1.24}$$

For the power-exponential mean Z, it follows from the comparison theorem for Gini means given by Páles [16] (also see [1], [20], [25]) that the optimal inequality

$$Z(a,b)>A_2(a,b) \tag{1.25}$$

holds for all $a,b>0$ with $a\neq b$, which can be rewritten as

$$Z_{1/3}(a,b)>A_{2/3}(a,b). \tag{1.26}$$

Sándor [19] showed that

$$L(a,b)<P(a,b)<I(a,b) \tag{1.27}$$

for all $a,b>0$ with $a\neq b$.

Neuman and Sándor [15] found that the inequalities

$$G(a,b)<L(a,b)<P(a,b)<A(a,b)<NS(a,b)<T(a,b)<A_2(a,b)$$

hold for all $a,b>0$ with $a\neq b$.

For all $a,b>0$ with $a\neq b$, the following chain of inequalities

$$L_2(a,b)<He(a,b)<A_{2/3}(a,b)<I(a,b)<Z_{1/3}(a,b)<Y_{1/2}(a,b) \tag{1.28}$$

is due to Yang [25, (5.17)]. Recently, Costin and Toader [8] proved that

$$\begin{aligned} G(a,b)<L(a,b)<A_{1/2}(a,b)<P(a,b)<A(a,b) \\ <NS(a,b)<A_{3/2}(a,b)<T(a,b)<A_2(a,b), \end{aligned} \tag{1.29}$$

which was improved independently in [28] and [7] as

$$\begin{aligned} A_0(a,b)<L(a,b)<A_{1/3}(a,b)<A_{\log 2/\log\pi}(a,b)<P(a,b) \\ <A_{2/3}(a,b)<I(a,b)<A_{\log 2}(a,b)<A_{\log 2/\log[\log(3+2\sqrt{2})]}(a,b) \\ <NS(a,b)<A_{4/3}(a,b)<A_{\log 2/(\log\pi-\log 2)}(a,b)<T(a,b)<A_{5/3}(a,b). \end{aligned} \tag{1.30}$$

The main purpose of this paper is to prove that the means $P_p(a,b)$, $T_p(a,b)$, $NS_p(a,b)$ and $Z_p(a,b)$ are strictly increasing with respect to $p\in\mathbb{R}$ for fixed $a,b>0$ with $a\neq b$, and the sharp inequalities $L_2(a,b)<P(a,b)<NS_{1/2}(a,b)<He(a,b)<A_{2/3}(a,b)<I(a,b)<Z_{1/3}(a,b)<Y_{1/2}(a,b)$ hold for all $a,b>0$ with $a\neq b$.

2. Monotonicity properties of power-type means

The following theorem shows that M_p defined by (1.11) is indeed a mean, that is, "power-type mean".

THEOREM 1. *Let M be a differentiable bivariate mean. Then M_p defined by (1.11) is also a bivariate mean. In particular, $M_0=G$ if M is symmetric.*

Proof. We divide the proof into two cases.

Case 1: $p\neq 0$. Without loss of generality, we assume that $p>0$ and $b>a>0$. Since M is a mean, we have $a^p\leqslant M(a^p,b^p)\leqslant b^p$, which implies that $a\leqslant M^{1/p}(a^p,b^p)\leqslant b$. Therefore, $M_p(a,b)$ is a mean.

Case 2: $p=0$. A simple calculation yields

$$M_0(a,b)=\lim_{p\to 0}M^{1/p}(a^p,b^p)=\exp\left(\frac{\partial M(1,1)}{\partial x}\log a+\frac{\partial M(1,1)}{\partial y}\log b\right).$$

It has shown in [23] that

$$\frac{\partial M(x,x)}{\partial x},\frac{\partial M(x,x)}{\partial y}\in[0,1] \quad \text{and} \quad \frac{\partial M(x,x)}{\partial x}+\frac{\partial M(x,x)}{\partial y}=1, \tag{2.1}$$

which implies that $M_0(a,b)$ is also a mean.

In particular, if M is symmetric, that is, $M(x,y)=M(y,x)$, then we clearly see that $\partial M(x,y)/\partial x=\partial M(y,x)/\partial y$, and so $\partial M(x,x)/\partial x=\partial M(x,x)/\partial y$. It follows from (2.1) that $\partial M(x,x)/\partial x=\partial M(x,x)/\partial y=1/2$. Therefore, $M_0=G$.

This completes the proof. □

The following sufficient condition for the monotonicity of p-order M mean M_p can be found in the literature [24].

LEMMA 1. *Let M be a homogeneous and differentiable bivariate mean. Then the function $M_p(x,y)$ defined by (1.11) is strictly increasing (decreasing) with respect to $p\in\mathbb{R}$ for fixed $x,y>0$ with $x\neq y$ if $\mathscr{I}(x,y)=(\log M(x,y))_{xy}<(>)0$ for all $x,y>0$ with $x\neq y$.*

Yang [26] proved that $\mathscr{I}(x,y)=(\log M(x,y))_{xy}<0$ for all $x,y>0$ with $x\neq y$ in the case of $M=A,He,L,I,Y$. Therefore, the p-order arithmetic mean (i.e., power mean) $A_p(x,y)$, p-order Heroian mean $He_p(x,y)$, p-order logarithmic mean $L_p(x,y)$, p-order identric (exponential) mean $I_p(x,y)$ and p-order exponential-geometric mean $Y_p(x,y)$ are strictly increasing with respect to $p\in\mathbb{R}$ for fixed $x,y>0$ with $x\neq y$. Next, we prove that the p-order first Seiffert mean $P_p(x,y)$, p-order second Seiffert mean $T_p(x,y)$, p-order Neuman-Sándor mean $NS_p(x,y)$ and p-order power-exponential mean $Z_p(x,y)$ are also strictly increasing with respect to $p\in\mathbb{R}$ for fixed $x,y>0$ with $x\neq y$.

THEOREM 2. *The p-order first Seiffert mean $P_p(x,y)$, p-order second Seiffert mean $T_p(x,y)$, p-order Neuman-Sándor mean $NS_p(x,y)$ and p-order power-exponential mean $Z_p(x,y)$ are strictly increasing with respect to $p\in\mathbb{R}$ for fixed $x,y>0$ with $x\neq y$.*

Proof. By Lemma 1, it suffices to show that for all $x,y>0$ with $x\neq y$, $\mathscr{I}(x,y)=(\log M(x,y))_{xy}<0$ in the case of $M=P$, T and NS, and $\partial\left(\log(Z_p(x,y)\right)/\partial p>0$. We divide the proof into four cases.

Case 1: $M=P$. Then elaborated computations lead to

$$\begin{aligned}\mathscr{I}(x,y)=(\log P(x,y))_{xy}&=\frac{1}{(x-y)^2}-\frac{1}{\arcsin^2\frac{x-y}{x+y}}\frac{1}{(x+y)^2}-\frac{1}{2}\frac{1}{\arcsin\frac{x-y}{x+y}}\frac{x-y}{\sqrt{xy}(x+y)^2}\\&=\frac{1}{(x-y)^2}-\frac{4}{(x+y)^2(x-y)^2}P^2(x,y)-\frac{1}{(x+y)^2\sqrt{xy}}P(x,y).\end{aligned}$$

Making use of the well known inequality $P(x,y)>G(x,y)=\sqrt{xy}$, we get

$$\mathscr{I}(x,y)<\frac{1}{(x-y)^2}-\frac{4}{(x+y)^2(x-y)^2}xy-\frac{1}{(x+y)^2\sqrt{xy}}\sqrt{xy}=0.$$

Case 2: $M=T$. Then elaborated computations yield

$$\begin{aligned}\mathscr{I}(x,y)=(\log T(x,y))_{xy}&=\frac{1}{(x-y)^2}-\frac{y}{\arctan^2\frac{x-y}{x+y}}\frac{x}{(x^2+y^2)^2}-\frac{1}{\arctan\frac{x-y}{x+y}}\frac{x^2-y^2}{(x^2+y^2)^2}\\&=\frac{1}{(x-y)^2}-\frac{4xy}{(x^2+y^2)^2(x-y)^2}T^2(x,y)-\frac{2(x+y)}{(x^2+y^2)^2}T(x,y).\end{aligned}$$

Using the inequality $T(x,y)>A(x,y)=(x+y)/2$, one has

$$\begin{aligned}\mathscr{I}(x,y)&<\frac{1}{(x-y)^2}-\frac{4xy}{(x^2+y^2)^2(x-y)^2}\left(\frac{x+y}{2}\right)^2-\frac{2(x+y)}{(x^2+y^2)^2}\left(\frac{x+y}{2}\right)\\&=-\frac{xy}{(x^2+y^2)^2}<0.\end{aligned}$$

Case 3: $M = NS$. Then elaborated computations give

$$\begin{aligned}\mathscr{I}(x,y) &= (\log NS(x,y))_{xy} = \frac{1}{(x-y)^2} - \frac{1}{\operatorname{arcsinh}^2\frac{x-y}{x+y}}\frac{2xy}{(x^2+y^2)(x+y)^2}\\ &\quad - \frac{\sqrt{2}(x^2+y^2+xy)}{(x+y)^2\left(\sqrt{x^2+y^2}\right)^3}\frac{x-y}{\operatorname{arcsinh}\frac{x-y}{x+y}}\\ &= \frac{1}{(x-y)^2} - \frac{8xy}{(x-y)^2(x^2+y^2)(x+y)^2}NS^2(x,y) - \frac{2\sqrt{2}(x^2+y^2+xy)}{(x+y)^2\left(\sqrt{x^2+y^2}\right)^3}NS(x,y).\end{aligned}$$

Applying the inequality

$$NS(x,y) > \frac{A^2(x,y)}{A_2(x,y)} = \frac{\left(\frac{x+y}{2}\right)^2}{\sqrt{\frac{x^2+y^2}{2}}}$$

given in [28] leads to

$$\mathscr{I}(x,y) < \frac{1}{(x-y)^2} - \frac{8xy}{(x-y)^2(x^2+y^2)(x+y)^2}\left(\frac{\left(\frac{x+y}{2}\right)^2}{\sqrt{\frac{x^2+y^2}{2}}}\right)^2 - \frac{2\sqrt{2}(x^2+y^2+xy)}{(x+y)^2\left(\sqrt{x^2+y^2}\right)^3}\frac{\left(\frac{x+y}{2}\right)^2}{\sqrt{\frac{x^2+y^2}{2}}} = 0.$$

Case 4: It follows from (1.9) that

$$\log Z_p(x,y) = \frac{x^p}{x^p+y^p}\log x + \frac{y^p}{x^p+y^p}\log y$$

and

$$\frac{\partial}{\partial p}(\log Z_p(x,y)) = x^p y^p \frac{(\log x - \log y)^2}{(x^p+y^p)^2} > 0. \quad \square$$

3. Inequalities for certain power-type means

THEOREM 3. *The inequality $L_p(a,b) < P(a,b)$ holds for all $a,b > 0$ with $a \neq b$ if and only if $p \leqslant 2$.*

Proof. Since both the means L_p and P are symmetric, without loss of generality, we assume that $a < b$. Let $x = a/b \in (0,1)$, then inequality $L_p(a,b) < P(a,b)$ is equivalent to

$$\left(\frac{x^p-1}{p\log x}\right)^{1/p} < \frac{x-1}{2\arcsin\frac{x-1}{x+1}}. \tag{3.1}$$

Necessity. If $L_p(a,b) < P(a,b)$ for all $a,b > 0$ with $a \neq b$, then inequality (3.1) leads to

$$\lim_{x\to 1}\frac{\left(\frac{x^p-1}{p\log x}\right)^{1/p} - \frac{x-1}{2\arcsin\frac{x-1}{x+1}}}{(x-1)^2} = \frac{1}{24}p - \frac{1}{12} \leqslant 0,$$

which implies that $p \leqslant 2$.

Sufficiency. We prove that the inequality (3.1) holds if $p \leqslant 2$. By Theorem 2, it suffices to show that the inequality (3.1) holds if $p = 2$. Let the function f_1 be defined on $(0,1)$ by

$$f_1(x) = 2\frac{(x+1)}{x-1}\arcsin^2\frac{x-1}{x+1} - \log x.$$

Differentiating $f_1(x)$ yields

$$\begin{aligned} f_1'(x) &= -4\frac{\arcsin^2\frac{x-1}{x+1}}{(x-1)^2} + 4\left(\arcsin\frac{x-1}{x+1}\right)\frac{1}{\sqrt{x}(x-1)} - \frac{1}{x} \\ &= -\frac{\left(x-1-2\sqrt{x}\arcsin\frac{x-1}{x+1}\right)^2}{x(x-1)^2} < 0 \end{aligned}$$

for all $x \in (0,1)$, which implies that f_1 is strictly decreasing on $(0,1)$. Therefore, $f_1(x) > \lim_{x\to 1^-} f_1(x) = 0$, and

$$\frac{x^2-1}{2\log x} < \frac{(x-1)^2}{4\left(\arcsin\frac{x-1}{x+1}\right)^2}$$

for all $x \in (0,1)$. $\square$

REMARK 1. Recently, Chu et al. [6] gave a different proof of Theorem 3, but our proof seems to be more simple.

THEOREM 4. *The inequality $P_p(a,b) < NS(a,b)$ holds for $a,b > 0$ with $a \neq b$ if and only if $p \leqslant 2$.*

Proof. Without loss of generality, we assume that $a < b$. Let $x = a/b \in (0,1)$, then inequality $P_p(a,b) < NS(a,b)$ is equivalent to

$$\left(\frac{x^p-1}{2\arcsin\frac{x^p-1}{x^p+1}}\right)^{1/p} < \frac{x-1}{2\log\frac{x-1+\sqrt{2(x^2+1)}}{x+1}}. \tag{3.2}$$

Necessity. If $P_p(a,b) < NS(a,b)$ for $a,b > 0$ with $a \neq b$, then inequality (3.2) leads to

$$\lim_{x\to 1}\frac{\left(\frac{x^p-1}{2\arcsin\frac{x^p-1}{x^p+1}}\right)^{1/p} - \frac{x-1}{2\log\frac{x-1+\sqrt{2(x^2+1)}}{x+1}}}{(x-1)^2} = \frac{1}{12}p - \frac{1}{6} \leqslant 0,$$

which implies that $p \leqslant 2$.

Sufficiency. We prove that the inequality (3.2) holds if $p \leqslant 2$. By Theorem 2, it suffices to show that the inequality (3.2) holds if $p = 2$. We define the function f_2 by

$$f_2(x) = 2\frac{\log^2 \frac{x-1+\sqrt{2(x^2+1)}}{x+1}}{x-1}(x+1) - \arcsin\frac{x^2-1}{x^2+1}, \quad x \in (0,1).$$

Differentiating $f_2(x)$ gives

$$\begin{aligned} f_2'(x) &= -4\frac{\log^2 \frac{x-1+\sqrt{2(x^2+1)}}{x+1}}{(x-1)^2} + 4\frac{\log \frac{x-1+\sqrt{2(x^2+1)}}{x+1}}{x-1}\frac{\sqrt{2}}{\sqrt{x^2+1}} - \frac{2}{x^2+1} \\ &= -\left(2\frac{\log \frac{x-1+\sqrt{2(x^2+1)}}{x+1}}{x-1} - \frac{\sqrt{2}}{\sqrt{x^2+1}}\right)^2 < 0 \end{aligned}$$

for all $x \in (0,1)$, which implies that f_2 is strictly decreasing on $(0,1)$. Therefore, $f_2(x) > \lim_{x\to 1^-} f_2(x) = 0$, and

$$\frac{x^2-1}{2\arcsin\frac{x^2-1}{x^2+1}} < \left(\frac{x-1}{2\log\frac{x-1+\sqrt{2(x^2+1)}}{x+1}}\right)^2$$

for all $x \in (0,1)$. □

THEOREM 5. *The inequality $NS(a,b) < He_p(a,b)$ holds for $a,b > 0$ with $a \neq b$ if and only if $p \geqslant 2$.*

Proof. We assume that $a < b$, let $x = a/b \in (0,1)$. Then inequality $NS(a,b) < He_p(a,b)$ is equivalent to

$$\frac{x-1}{2\log\frac{x-1+\sqrt{2(x^2+1)}}{x+1}} < \left(\frac{x^p + x^{p/2} + 1}{3}\right)^{1/p}. \tag{3.3}$$

Necessity. If $NS(a,b) < He_p(a,b)$, then inequality (3.3) leads to the conclusion that

$$\lim_{x\to 1}\frac{\frac{x-1}{2\log\frac{x-1+\sqrt{2(x^2+1)}}{x+1}} - \left(\frac{x^p+x^{p/2}+1}{3}\right)^{1/p}}{(x-1)^2} = \frac{1}{6} - \frac{1}{12}p \leqslant 0,$$

which gives $p \geqslant 2$.

Sufficiency. We prove the inequality (3.3) holds if $p \geqslant 2$. By Theorem 2, it suffices to show that the inequality (3.3) holds if $p = 2$. To this end, we define the function f_3 by

$$f_3(x) = \frac{x-1}{\sqrt{\frac{x^2+x+1}{3}}} - 2\log\frac{x-1+\sqrt{2(x^2+1)}}{x+1}.$$

Differentiating $f_3(x)$ yields

$$f_3'(x) = \frac{1}{\sqrt{\frac{x^2+x+1}{3}}} - \frac{2x+1}{6}\frac{x-1}{\left(\frac{x^2+x+1}{3}\right)^{\frac{3}{2}}} - \frac{2\sqrt{2}}{(x+1)\sqrt{x^2+1}}$$
$$= \sqrt{2}\frac{x\sqrt{\frac{x^2+1}{2}} - 2\left(\frac{x^2+x+1}{3}\right)^{3/2} + \left(\frac{x^2+1}{2}\right)^{3/2}}{(x+1)\sqrt{x^2+1}\left(\sqrt{\frac{x^2+x+1}{3}}\right)^3}$$
$$= -\sqrt{2}\frac{\left(\sqrt{\frac{x^2+1}{2}}-\sqrt{\frac{x^2+x+1}{3}}\right)^2\left(\sqrt{\frac{x^2+1}{2}}+2\sqrt{\frac{x^2+x+1}{3}}\right)}{(x+1)\sqrt{x^2+1}\left(\sqrt{\frac{x^2+x+1}{3}}\right)^3} < 0$$

for all $x \in (0,1)$, which shows that f_3 is strictly decreasing on $(0,1)$. Hence $f_3(x) > \lim_{x\to 1^-} f_3(x) = 0$, and

$$\frac{x-1}{\sqrt{\frac{x^2+x+1}{3}}} > 2\log\frac{x-1+\sqrt{2(x^2+1)}}{x+1},$$

which implies that the inequality (3.3) holds if $p = 2$. □

THEOREM 6. *The inequality $I(a,b) < Z_p(a,b)$ holds for all $a,b > 0$ with $a \neq b$ if and only if $p \geqslant 1/3$.*

Proof. We assume that $a < b$, let $x = a/b \in (0,1)$. Then inequality $I(a,b) < Z_p(a,b)$ is equivalent to

$$e^{-1}x^{x/(x-1)} < x^{x^p/(x^p+1)}. \tag{3.4}$$

Necessity. If $I(a,b) < Z_p(a,b)$, then inequality (3.4) gives

$$\lim_{x\to 1}\frac{e^{-1}x^{x/(x-1)} - x^{x^p/(x^p+1)}}{(x-1)^2} = \frac{1}{12} - \frac{1}{4}p \leqslant 0,$$

which yields $p \geqslant 1/3$.

Sufficiency. The inequality $I(a,b) < Z_{1/3}(a,b)$ can be found in the literature [25, (5.7)], then from the monotonicity of Theorem 2 we clearly see that $I(a,b) < Z_{1/3}(a,b) \leqslant Z_p(a,b)$ if $p \geqslant 1/3$. □

THEOREM 7. *The inequality $Z_p(a,b) < Y(a,b)$ holds for $a,b > 0$ with $a \neq b$ if and only if $p \leqslant 2/3$.*

Proof. We assume that $a < b$ and $x = a/b \in (0,1)$. Then inequality $Z_p(a,b) < Y(a,b)$ is equivalent to

$$x^{x^{x^p/(x^p+1)}} < e^{-1}x^{x/(x-1)}\exp\left(1 - \frac{x\log^2 x}{(x-1)^2}\right). \tag{3.5}$$

Necessity. If $Z_p(a,b) < Y(a,b)$, then inequality (3.5) leads to the conclusion that

$$\lim_{x\to 1}\frac{\log Z_p(x,1)-\log Y(x,1)}{(x-1)^2}=\lim_{x\to 1}\frac{\frac{x^p}{x^p+1}\log x-\frac{x}{x-1}\log x+1-\left(1-\frac{x\log^2 x}{(x-1)^2}\right)}{(x-1)^2}=\frac{1}{4}p-\frac{1}{6}\leqslant 0,$$

which gives $p \leqslant 2/3$.

Sufficiency. The inequality $Y(a,b) > Z_{2/3}(a,b) \geqslant Z_p(a,b)$ for $p \leqslant 2/3$ follows easily from the inequality $Z_{2/3}(a,b) < Y(a,b)$ [25, (5.12)] and the monotonicity of Z_p with respect to p in Theorem 2. □

4. Remarks and a conjecture

REMARK 2. From Lemma 4 we clearly see that the inequalities $P(a,b) < NS_p(a,b)$ holds for all $a,b > 0$ with $a \neq b$ if and only if $p \geqslant 1/2$.

REMARK 3. It follows from Lemma 5 that the inequality $NS_p(a,b) < He(a,b)$ holds for all $a,b > 0$ with $a \neq b$ if and only if $p \leqslant 1/2$.

REMARK 4. From Theorem 3 together with Remarks 2 and 3 we clearly see that the chain of inequalities (1.28) can be refined as

$$\begin{aligned} &L_2(a,b) < P(a,b) < NS_{1/2}(a,b) < He(a,b) \\ &< A_{2/3}(a,b) < I(a,b) < Z_{1/3}(a,b) < Y_{1/2}(a,b) \end{aligned} \tag{4.1}$$

for all $a,b > 0$ and $a \neq b$ with the best possible parameters.

The chain of inequalities (4.1) does not contain the power-type second Seiffert mean T_p. From (1.22) and (1.28) it is easy to obtain that

$$T_{2/5}(a,b) < He(a,b) < A_{2/3}(a,b) < I(a,b) < Z_{1/3}(a,b) < Y_{1/2}(a,b). \tag{4.2}$$

If $NS_{1/2}(a,b) < T_{2/5}(a,b)$ holds, then we get the perfect chain of inequalities for power-type means

$$\begin{aligned} &L_2(a,b) < P(a,b) < NS_{1/2}(a,b) < T_{2/5}(a,b) \\ &< He(a,b) < A_{2/3}(a,b) < I(a,b) < Z_{1/3}(a,b) < Y_{1/2}(a,b). \end{aligned} \tag{4.3}$$

Elaborated computations show that

$$\lim_{x\to 1}\frac{NS(x,1)-T_p(x,1)}{(x-1)^2}=\frac{1}{6}-\frac{5}{24}p$$

and

$$NS\left(0^+,1\right)-T_{4/5}\left(0^+,1\right)=\frac{1}{2\log\left(\sqrt{2}+1\right)}-\left(\frac{2}{\pi}\right)^{5/4}<0.$$

Therefore, we propose a conjecture as follows.

CONJECTURE 1. *The inequality* $NS(a,b) < T_p(a,b)$ *holds for all* $a,b > 0$ *with* $a \neq b$ *if and only if* $p \geqslant 4/5$.

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