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TWO INEQUALITIES FOR VOLUMES OF SIMPLEXES WITH APPLICATIONS

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Abstract. In this paper, we study the problems of volumes of n-dimensional simplexes in the Euclidean space E^n by the method of barycenter coordinate. An inequality for volumes of n-dimensional simplex and its escenters simplex is established,and two inequalities for volumes of two simplexes are established. As the application of the inequalities, we obtain generalization of the n-dimensional Euler inequality.

1. Introduction

In this paper, let Ω be an n-dimensional simplex in the Euclidean space E^n with vertex set $\Omega(A) = \{A_0, A_1, \cdots, A_n\}$ and $V(\Omega)$ its volume. The ith face f_i of Ω is an $(n-1)$-dimensional simplex spanned by the vertex set $A \setminus \{A_i\}$ $(i = 0, 1, \cdots, n)$. Let F_i be content of the ith face f_i of Ω, $a_{ij} = |A_iA_j|$ $(0 \leqslant i < j \leqslant n)$ denote the edge-lengths of Ω, O and G denote the circum-center and the bary-center of Ω respectively. Let R and r be the circum-radius and the in-radius of Ω respectively, m_i be the median of Ω from vertex for $i = 0, 1, \cdots, n$.

Let P be an interior point of simplex Ω and H_i $(i = 0, 1, \cdots, n)$ the foot of the perpendicular drawn from point P to the ith face f_i of Ω. We call simplex $\Omega_H = \operatorname{conv}\{H_0, H_1, \cdots, H_n\}$ the orthocentric simplex for point P and simplex Ω. Let $V(\Omega_H)$ denote the volume of Ω_H. In [1], Y. Zhang obtained an inequality as follows

$$V(\Omega_H) \leqslant \frac{1}{n^n} V(\Omega), \tag{1.1}$$

with equality if Ω is regular simplex and point P is the incenter of Ω.

Let I_i be the incenter of ith face f_i of Ω for $i = 0, 1, \cdots, n$. We call simplex $\Omega_I = \operatorname{conv}\{I_0, I_1, \cdots, I_n\}$ the incenter simplex of Ω (see [2]). In [2], T. Y. Ma obtained an inequality as follows

$$V(\Omega_I) \leqslant \frac{1}{n^n} V(\Omega), \tag{1.2}$$

with equality if and only if there exist $x_k \geqslant 0$ $(k = 0, 1, \cdots, n)$ such that $a_{ij} = |x_i x_j|$ $(0 \leqslant i < j \leqslant n)$. Here $V(\Omega_I)$ denotes the volume of Ω_I.

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For $i = 0, 1, \cdots, n$, let A_i' be the second point of intersection of the line A_iG with the circumscribed hypersphere S_{n-1} of Ω. Let $V(\Omega')$ denote the volume of simplex $\Omega' = \text{conv}\{A_0', A_1', \cdots, A_n'\}$. An inequality is established in [3, 4] as follows

$$V(\Omega') \geqslant V(\Omega), \tag{1.3}$$

with equality if Ω is regular simplex.

Let $B_0, B_1, \cdots, B_n$ denote the escenters of simplex Ω, we call simplex $\Omega_B = \text{conv}\{B_0, B_1, \cdots, B_n\}$ the escenter simplex of Ω. Let $V(\Omega_B)$ denote the volume of simplex Ω_B. In this paper, we obtain following two inequalities for volumes of two simplexes.

THEOREM 1. *For an n-dimensional simplex Ω and its escenters simplex Ω_B, we have*

$$V(\Omega_B) \leqslant \left(\frac{2}{n-1}\right)^n \left(\frac{R}{nr}\right)^{2(n+1)} V(\Omega), \tag{1.4}$$

with equality if Ω is regular simplex.

THEOREM 2. *Let Ω be an n-dimensional simplex and Ω' an n-simplex defined above, then we have*

$$V(\Omega') \leqslant \left(\frac{R^2 - \overline{OG}^2}{n^2 r^2}\right)^{\frac{n^2-1}{n}} V(\Omega), \tag{1.5}$$

with equality if Ω is regular simplex.

From inequalities (1.5) and (1.3), we obtain generalization of the n-dimensional Euler inequality as follows (see [5]).

COROLLARY 1. *For an n-dimensional simplex Ω, we have*

$$R^2 \geqslant n^2 r^2 + \overline{OG}^2. \tag{1.6}$$

Equality holds if Ω is regular simplex.

From inequality (1.5), we get an inequality as follows.

COROLLARY 2. *Let Ω be an n-dimensional simplex and Ω' an n-simplex defined above, then we have*

$$V(\Omega') \leqslant \left(\frac{R}{nr}\right)^{\frac{2(n^2-1)}{n}} V(\Omega). \tag{1.7}$$

Equality holds if Ω is regular simplex.

2. Some Lemmas and Proof of Theorems

To prove theorems above, we need some lemmas as follows. Let θ_{ij} be the dihedral angle formed by two faces f_i and f_j of the n-dimensional simplex Ω, and V_{ij} the $(n-2)$-dimensional volume of the $(n-2)$-dimensional simplex $\Omega_{ij} = f_i \bigcap f_j$ whose vertexes are A_k $(k = 0, 1, \cdots, n,\ k \neq i, j)$. For $i = 0, 1, \cdots, n$, let r_i denote the inradius of the ith face f_i of Ω. Put $F = \sum_{i=0}^{n} F_i$.

LEMMA 1. ([2, 6]) *Let* Ω *be the coordinate simplex and* $\Omega_C = \operatorname{conv}\{C_0, C_1, \cdots, C_n\}$ *be an arbitrary simplex in* E^n. *Let* $(\lambda_{i0}, \lambda_{i1}, \cdots, \lambda_{in})$ *be the gauge barycenter coordinate of point* C_i *about* Ω *for* $i = 0, 1, \cdots, n$. *Then*

$$\frac{V(\Omega_C)}{V(\Omega)} = |\det(\lambda_{ij})_{i,j=0}^{n}|. \tag{2.1}$$

Here $V(\Omega_C)$ *denotes the volume of simplex* Ω_C.

LEMMA 2. ([7]) *For an* n*-dimensional simplex* Ω, *we have*

$$V(\Omega) = \frac{n-1}{n} \cdot \frac{F_i F_j}{V_{ij}} \sin\theta_{ij} \quad (0 \leqslant i < j \leqslant n). \tag{2.2}$$

LEMMA 3. ([8]) *Let* B_i *be the ith escenter of the* n*-dimensional simplex* Ω, *then the gauge barycenter coordinate of point* B_i *about coordinate simplex* Ω *is*

$$B_i = \frac{1}{F - 2F_i}(F_0, F_1, \cdots, F_{i-1}, -F_i, F_{i+1}, \cdots, F_n). \tag{2.3}$$

LEMMA 4. ([4]) *For an* n*-dimensional simplex* Ω, *we have*

$$r \leqslant \left[\frac{(n!)^2}{n^n(n+1)^{n+1}}\right]^{\frac{1}{2n}} \cdot V(\Omega)^{\frac{1}{n}}, \tag{2.4}$$

with equality if and only if Ω *is regular.*

LEMMA 5. ([4, 8]) *For an* n*-dimensional simplex* Ω, *we have*

$$F_i = \sum_{j=0, j\neq i}^{n} F_j \cos\theta_{ij} \quad (i = 0, 1, \cdots, n). \tag{2.5}$$

LEMMA 6. *Let* Ω *be an* n*-dimensional simplex, then*

$$\prod_{i=0}^{n}(F - 2F_i) \geqslant (n-1)^{n+1}\left(\frac{n^{3n}}{(n!)^2}\right)^{\frac{n+1}{n-1}}\left(\frac{n+1}{n^2}\right)^{(n+1)} r^{2(n+1)}\left(\prod_{i=0}^{n} F_i\right)^{\frac{n-3}{n-1}}, \tag{2.6}$$

with equality if Ω *is regular.*

Proof. Using Cauchy's inequality and equality (2.5), we get

$$\begin{aligned} \sum_{j=0, j\neq i}^{n} F_j \sin\theta_{ij} &= \sum_{j=0, j\neq i}^{n} [(F_j + F_j\cos\theta_{ij})(F_j - F_j\cos\theta_{ij})]^{\frac{1}{2}} \\ &\leqslant \Big[\sum_{j=0, j\neq i}^{n} [(F_j + F_j\cos\theta_{ij})\Big]^{\frac{1}{2}} \Big[\sum_{j=0, j\neq i}^{n} (F_j - F_j\cos\theta_{ij})\Big]^{\frac{1}{2}} \\ &= F^{\frac{1}{2}}(F - 2F_i)^{\frac{1}{2}} \end{aligned} \tag{2.7}$$

By the formula for the volume of an n-simplex in [4], we have

$$\frac{nV(\Omega)}{\sum\limits_{j=0}^{n} F_j} = \frac{nV(\Omega)}{F} = r. \tag{2.8}$$

Using the formula (2.8) to $(n-1)$-dimensional simplex f_i, we get

$$\frac{(n-1)F_i}{\sum\limits_{j=0,j\neq i}^{n} V_{ij}} = r_i \quad (i=0,1,\cdots,n). \tag{2.9}$$

By Lemma 2, we have

$$\sum_{j=0,j\neq i}^{n} V_{ij} = \frac{n-1}{nV(\Omega)} \cdot F_i \sum_{j=0,j\neq i}^{n} F_j \sin\theta_{ij}. \tag{2.10}$$

From (2.8), (2.9) and (2.10), we get

$$\frac{r}{r_i} = \frac{1}{F} \sum_{j=0,j\neq i}^{n} F_j \sin\theta_{ij}. \tag{2.11}$$

Substituting (2.7) into (2.11), we get

$$\frac{r}{r_i} \leqslant \frac{(F-2F_i)^{\frac{1}{2}}}{F^{\frac{1}{2}}} \quad (i=0,1,\cdots,n). \tag{2.12}$$

Using (2.12) and (2.8), we get

$$\prod_{i=0}^{n}(F-2F_i) \geqslant \frac{(r^2F)^{n+1}}{\prod\limits_{i=0}^{n} r_i^2} = \frac{(nV(\Omega))^{2(n+1)}}{F^{n+1}\prod\limits_{i=0}^{n} r_i^2}. \tag{2.13}$$

It easy to know that equality holds in (2.13) if simplex Ω is regular.

By Lemma 4, we have

$$r_i \leqslant \left[\frac{(n-1)!^2}{(n-1)^{n-1}n^n}\right]^{\frac{1}{2(n-1)}} F_i^{\frac{1}{n-1}} \quad (i=0,1,\cdots,n). \tag{2.14}$$

Using (2.13), (2.14), (2.8) and the arithmetic-geometric mean inequality, we get

$$\begin{aligned} \prod_{i=0}^{n}(F-2F_i) &\geqslant (n-1)^{n+1}\left(\frac{n^{3n}}{n!^2}\right)^{\frac{n+1}{n-1}} \cdot \frac{V(\Omega)^{2(n+1)}}{F^{n+1}\prod\limits_{i=0}^{n} F_i^{\frac{2}{n-1}}} \\ &= (n-1)^{n+1}\left(\frac{n^{3n}}{n!^2}\right)^{\frac{n+1}{n-1}} \cdot \frac{r^{2(n+1)}\left(\sum\limits_{i=0}^{n} F_i\right)^{n+1}}{n^{2(n+1)}\prod\limits_{i=0}^{n} F_i^{\frac{2}{n-1}}} \\ &\geqslant (n-1)^{n+1}\left(\frac{n^{3n}}{n!^2}\right)^{\frac{n+1}{n-1}} \cdot \left(\frac{n+1}{n^2}\right)^{n+1} \cdot r^{2(n+1)} \cdot \prod_{i=0}^{n} F_i^{\frac{n-3}{n-1}} \end{aligned} \tag{2.15}$$

It is easy to see that equality holds in (2.15) if simplex Ω is regular. □

LEMMA 7. *Let m_i $(i=0,1,\cdots,n)$ and $V(\Omega)$ denote the medians and the volume of the n-dimensional simplex Ω respectively, then we have*

$$\frac{\sum\limits_{i=0}^{n} m_i^2}{\prod\limits_{i=0}^{n} m_i^2} \leqslant \frac{n^n}{n!^2(n+1)^{n-2}V(\Omega)^2}, \tag{2.16}$$

with equality if Ω is regular simplex.

Proof. Let G denote the barycenter of Ω, and $V(\Omega_i)$ denote the volume of n-dimensional simplex $\Omega_i = \operatorname{conv}\{A_0,\cdots,A_{i-1},G,A_{i+1},\cdots,A_n\}$ for $i=0,1,\cdots,n$. By the properties of barycenter of the n-dimensional simplex Ω, we have

$$V(\Omega_i) = (n+1)^{-1}V(\Omega), \qquad |GA_i| = n(n+1)^{-1}m_i \quad (i=0,1,\cdots,n). \tag{$*$}$$

Let $\overrightarrow{GA_i} = n(n+1)^{-1}\alpha_i$ $(i=0,1,\cdots,n)$, where α_i is the unit vector of $\overrightarrow{GA_i}$. Using inequality (15) in [7], we have

$$\sum_{i=0}^{n} \det(\alpha_l^T\alpha_k)_{l,k\neq i} \leqslant \left(\frac{n+1}{n}\right)^n, \tag{2.17}$$

with equality if and only if the nonzero eigenvalues of Gram matrix $Q=(\alpha_i^T\alpha_j)_{i,j=0}^n$ are all same. It is easy to know that equality holds in (2.17) if Ω is regular simplex.

By the formula for the volume of a simplex, we have

$$\begin{aligned}(n+1)^{-1}V(\Omega) = V(\Omega_i) &= \frac{1}{n!}[\det(|GA_l|\cdot|GA_k|\alpha_l^T\alpha_k)_{l,k\neq i}]^{\frac{1}{2}}\\ &= \frac{1}{n!}[\det(n^2(n+1)^{-2}m_lm_k\alpha_l^T\alpha_k)_{l,k\neq i}]^{\frac{1}{2}}\\ &= \frac{n^n}{n!(n+1)^n}\Big(\prod_{j=0,j\neq i}^{n} m_j\Big)[\det(\alpha_l^T\alpha_k)_{l,k\neq i}]^{\frac{1}{2}}\end{aligned}$$

i.e.

$$\det(\alpha_l^T\alpha_k)_{l,k\neq i} = \frac{n!^2(n+1)^{2(n-1)}V(\Omega)^2}{n^{2n}}\cdot\frac{m_i^2}{\prod\limits_{j=0}^{n} m_j^2} \quad (i=0,1,\cdots,n). \tag{2.18}$$

Substituting (2.18) into (2.17), we get (2.16). □

Proof of Theorem 1. By Lemma 1 and Lemma 3, we have

$$\begin{aligned} V(\Omega_B) &= \frac{V(\Omega)}{\prod\limits_{i=0}^{n}(F-2F_i)} \cdot \left\| \begin{matrix} -F_0 & F_1 & \cdots & F_n \\ F_0 & -F_1 & \cdots & F_n \\ \cdots & \cdots & \cdots & \cdots \\ F_0 & F_1 & \cdots & -F_n \end{matrix} \right\| \\ &= \frac{V(\Omega)\cdot \prod\limits_{i=0}^{n} F_i}{\prod\limits_{i=0}^{n}(F-2F_i)} \cdot \left\| \begin{matrix} -1 & 1 & \cdots & 1 \\ 1 & -1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & -1 \end{matrix} \right\| \\ &= \frac{2^n(n-1)\prod\limits_{i=0}^{n} F_i}{\prod\limits_{i=0}^{n}(F-2F_i)} \cdot V(\Omega) \end{aligned} \tag{2.19}$$

Using (2.19) and (2.6), we get

$$V(\Omega_B) \leqslant 2^n(n-1)\left[\frac{(n!)^{2(n+1)}\prod\limits_{i=0}^{n}F_i^2}{n^{(n+1)(n+2)}}\right]^{\frac{1}{n-1}} \cdot \frac{V(\Omega)}{(n^2-1)^{n+1}r^{2(n+1)}}. \tag{2.20}$$

Using the known inequality in [4, 9], we have

$$\sum_{i=0}^{n} F_i^2 \leqslant \frac{1}{(n!)^2 n^{n-4}(n+1)^{n-2}} \cdot \Big(\sum_{0\leqslant i<j\leqslant n} a_{ij}^2\Big)^{n-1}, \tag{2.21}$$

with equality if Ω is regular simplex.

Using the known result in [5], we have

$$\sum_{0\leqslant i<j\leqslant n} a_{ij}^2 = (n+1)^2(R^2-\overline{OG}^2) \leqslant (n+1)^2R^2. \tag{2.22}$$

Equality holds if Ω is regular simplex.

By the arithmetic-geometric mean inequality,inequalities (2.21) and (2.22), we have

$$\Big(\prod_{i=0}^{n}F_i^2\Big)^{\frac{1}{n-1}} \leqslant \Big(\frac{1}{n+1}\sum_{i=0}^{n}F_i^2\Big)^{\frac{n+1}{n-1}} \leqslant \Big[\frac{(n+1)^{n-1}}{(n!)^2n^{n-4}}\Big]^{\frac{n+1}{n-1}} \cdot R^{2(n+1)}. \tag{2.23}$$

From (2.20) and (2.23), we obtain inequality (1.4). It is easy to know that equality holds in (1.4) if Ω is regular simplex. □

Proof of Theorem 2. For $i=0,1,\cdots,n$, let $V(\Omega_i)$ and $V(\Omega_i')$ denote the volumes of the n-dimensional simplexes $\Omega_i = \operatorname{conv}\{A_0,\cdots,A_{i-1},G,A_{i+1},\cdots,A_n\}$ and $\Omega_i' =$

$\operatorname{conv}\{A_0', \cdots, A_{i-1}', G, A_{i+1}', \cdots, A_n'\}$ respectively. It is easy to see that point G is an interior point of simplexes Ω and Ω' (see [4]). From this we have

$$V(\Omega) = \sum_{i=0}^{n} V(\Omega_i), \qquad V(\Omega') = \sum_{i=0}^{n} V(\Omega_i').$$

By equality (20) in chapter XVIII in [4], we have

$$R^2 - \overline{OG}^2 = |GA_i||GA_i'| \quad (i = 0, 1, \cdots, n). \tag{2.24}$$

By the formula for volume of a simplex we have

$$\frac{V(\Omega_i')}{V(\Omega_i)} = \frac{\prod_{j=0, j\neq i}^{n} |GA_j'|}{\prod_{j=0, j\neq i}^{n} |GA_j|}. \tag{2.25}$$

Using $(*)$, (2.25), (2.24), we get

$$\begin{aligned}
\frac{V(\Omega')}{V(\Omega)} &= \frac{1}{n+1}\sum_{i=0}^{n}\frac{V(\Omega_i')}{V(\Omega_i)} = \frac{1}{n+1}\sum_{i=0}^{n}\Big(\prod_{j=0,j\neq i}^{n}\frac{|GA_j'|}{|GA_j|}\Big)\\
&= \frac{1}{n+1}\sum_{i=0}^{n}\Big(\prod_{j=0,j\neq i}^{n}\frac{R^2-\overline{OG}^2}{|GA_j|^2}\Big) = \frac{1}{n+1}\sum_{i=0}^{n}\Big[(R^2-\overline{OG}^2)\cdot|GA_i|^2\cdot\prod_{j=0}^{n}\frac{1}{|GA_j|^2}\Big].\\
&= \frac{(R^2-\overline{OG}^2)^n}{n+1}\cdot\frac{\sum_{i=0}^{n}|GA_i|^2}{\prod_{j=0}^{n}|GA_j|^2} = \frac{(n+1)^{2n-1}}{n^{2n}}\cdot(R^2-\overline{OG}^2)^n\cdot\frac{\sum_{i=0}^{n}m_i^2}{\prod_{j=0}^{n}m_i^2}.
\end{aligned} \tag{2.26}$$

From (2.26) and (2.16), we get

$$\frac{V(\Omega')}{V(\Omega)} \leqslant \frac{(n+1)^{n+1}}{(n!)^2 n^n\cdot V(\Omega)^2}\cdot(R^2-\overline{OG}^2)^n. \tag{2.27}$$

It is easy to know that equality holds if Ω is regular.

From (2.4) we get

$$V(\Omega) \geqslant \frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{n!}\cdot r^n. \tag{2.28}$$

Substituting (2.28) into the right of (2.27), we obtain (1.5). It is easy to see that equality holds in (1.5) if Ω is regular.

At last, we have a conjecture as fllows

CONJECTURE. $V(\Omega_B) \leqslant (\frac{2}{n-1})^n\cdot V(\Omega)$ *or* $V(\Omega_B) \geqslant (\frac{2}{n-1})^n\cdot V(\Omega)$.

REFERENCES

[1] Y. ZHANG, *A conjecture of pedal simplex*, Journal of Systems Sci. and Math. Sci. **12** (1992), 371–375.
[2] T. Y. MA, L. Z. ZHAO, J. YUAN, *Inequalities for the incenter simplex*, Math. Inequal. Appl. **10** (2007), 703–709.
[3] M. S. KLAMKIN, *Problem 78-20*, SIAM Rev. **21** (1979), 569–570.
[4] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND V. VOLENEC, *Recent advances in Geometric Inequalities*, Kluwer Academic Publ., Dordrecht, 1989.
[5] S. G. YANG, J. WANG, *Improvements of n-dimensional Euler inequality*, J. Geom. **51** (1994), 190–195.
[6] H. M. SU, *An inequality for a simplex*, Bull.of Math. **5** (1985), 43–46.
[7] S. G. YANG, *Some inequalities on areas of bisection planes of dihedral angles of a simplex*, Geom. Ded. **62** (1996), 161–165.
[8] W. X. SHENG, *Introduction to simplexes*, Hunan Normal Univ. Publ., Changsha, China, 2004.
[9] L. YANG, J. Z. ZHANG, *A class of geometric inequalities on finite set of points*, Acta Math. Sinica **23** (1980), 740–749.

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