Journal of
Mathematical
Inequalities
Volume 9, Number 2 (2015), 397–407 doi:10.7153/jmi-09-33

ADDITIVE ρ–FUNCTIONAL INEQUALITIES IN NON–ARCHIMEDEAN NORMED SPACES

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(*Communicated by A. Gilányi*)

Abstract. In this paper, we solve the additive ρ-functional inequalities

$$\|f(x+y)-f(x)-f(y)\| \leqslant \left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\| \tag{0.1}$$

and

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\| \leqslant \|\rho\left(f(x+y)-f(x)-f(y)\right)\|, \tag{0.2}$$

where ρ is a fixed non-Archimedean number with $|\rho|<1$.

Furthermore, we prove the Hyers-Ulam stability of the additive ρ-functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces and prove the Hyers-Ulam stability of additive ρ-functional equations associated with the additive ρ-functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces.

1. Introduction and preliminaries

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s| \leqslant |r|+|s|, \qquad \forall r,s \in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \leqslant \max\{|r|,|s|\}, \qquad \forall r,s \in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \leqslant 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Mathematics subject classification (2010): Primary 46S10, 39B62, 39B52, 47S10, 12J25.

Keywords and phrases: Hyers-Ulam stability, additive ρ-functional equation, additive ρ-functional inequality, non-Archimedean normed space.

This work was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A2004299).

DEFINITION 1.1. ([7]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\|: X \to [0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\| \qquad (r \in K, x \in X)$;

(iii) the strong triangle inequality

$$\|x+y\| \leqslant \max\{\|x\|, \|y\|\}, \qquad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

DEFINITION 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \leqslant \varepsilon$$

for all $n, m \geqslant N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \leqslant \varepsilon$$

for all $n \geqslant N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

The stability problem of functional equations originated from a question of Ulam [12] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [10] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [3] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$$

is called the *Jensen equation*.

In [4], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leqslant \|f(xy)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(xy)+f(xy^{-1}).$$

See also [11]. Gilányi [5] and Fechner [2] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [9] proved the Hyers-Ulam stability of additive functional inequalities.

In [8], Park defined additive ρ-functional inequalities and additive ρ-functional equations and proved the Hyers-Ulam stability of the additive ρ-functional inequalities and the additive ρ-functional equations in (Archimedean) Banach spaces.

In Section 2, we solve the additive functional inequality (0.1) and prove the Hyers-Ulam stability of the additive functional inequality (0.1) in non-Archimedean Banach spaces. We moreover prove the Hyers-Ulam stability of an additive functional equation associated with the functional inequality (0.1) in non-Archimedean Banach spaces.

In Section 3, we solve the additive functional inequality (0.2) and prove the Hyers-Ulam stability of the additive functional inequality (0.2) in non-Archimedean Banach spaces. We moreover prove the Hyers-Ulam stability of an additive functional equation associated with the functional inequality (0.2) in non-Archimedean Banach spaces

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2|\neq 1$ and let ρ be a non-Archimedean number with $|\rho|<1$.

2. Additive ρ-functional inequality (0.1)

We solve the additive ρ-functional inequality (0.1) in non-Archimedean normed spaces.

LEMMA 2.1. *Let G be an Abelian semigroup with division by 2. A mapping $f:G\to Y$ satisfies*

$$\|f(x+y)-f(x)-f(y)\|\leqslant\left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\| \tag{2.1}$$

for all $x,y\in G$ if and only if $f:G\to Y$ is additive.

Proof. Assume that $f:G\to Y$ satisfies (2.1).

Letting $x=y=0$ in (2.1), we get

$$\|f(0)\|\leqslant 0.$$

So $f(0)=0$.

Letting $y=x$ in (2.1), we get

$$\|f(2x)-2f(x)\|\leqslant 0$$

and so $f(2x)=2f(x)$ for all $x\in G$. Thus

$$f\left(\frac{x}{2}\right)=\frac{1}{2}f(x) \tag{2.2}$$

for all $x \in G$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y)-f(x)-f(y)\| &\leqslant \left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\| \\ &= |\rho|\|f(x+y)-f(x)-f(y)\| \end{aligned}$$

and so

$$f(x+y)=f(x)+f(y)$$

for all $x,y \in G$.

The converse is obviously true. □

COROLLARY 2.2. *Let G be an Abelian semigroup with division by 2. A mapping $f: G \to Y$ satisfies*

$$f(x+y)-f(x)-f(y)=\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right) \tag{2.3}$$

for all $x,y \in G$ if and only if $f: G \to Y$ is additive.

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (2.1) in Banach spaces.

THEOREM 2.3. *Let $r<1$ and θ be nonnegative real numbers, and let $f: X \to Y$ be a mapping such that*

$$\|f(x+y)-f(x)-f(y)\| \leqslant \left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\| + \theta(\|x\|^r+\|y\|^r) \tag{2.4}$$

for all $x,y \in X$. Then there exists a unique additive mapping $A: X \to Y$ such that

$$\|f(x)-A(x)\| \leqslant \frac{2\theta}{|2|^r}\|x\|^r \tag{2.5}$$

for all $x \in X$.

Proof. Letting $y=x$ in (2.4), we get

$$\|f(2x)-2f(x)\| \leqslant 2\theta\|x\|^r \tag{2.6}$$

for all $x \in X$. So

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| \leqslant \frac{2}{|2|^r}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \qquad (2.7)\\
&\leqslant \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&= \max\left\{|2|^l \left\|f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |2|^{m-1} \left\|f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\leqslant \max\left\{\frac{|2|^l}{|2|^{rl+1}}, \cdots, \frac{|2|^{m-1}}{|2|^{r(m-1)+1}}\right\} 2\theta \|x\|^r\\
&= \frac{2\theta}{|2|^{(r-1)l+1}} \|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a non-Archimedean Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.7), we get (2.5).

Now, let $T : X \to Y$ be another additive mapping satisfying (2.5). Then we have

$$\begin{aligned}
\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\|\\
&\leqslant \max\left\{\left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\|, \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\|\right\}\\
&\leqslant \frac{2\theta}{|2|^{(r-1)q+1}} \|x\|^r,
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A.

It follows from (2.4) that

$$\begin{aligned}
\|A(x+y) - A(x) - A(y)\| &= \lim_{n\to\infty} \left\|2^n \left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\|\\
&\leqslant \lim_{n\to\infty} \left\|2^n \rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\|\\
&\quad + \lim_{n\to\infty} \frac{|2|^n \theta}{|2|^{nr}} (\|x\|^r + \|y\|^r)\\
&= \left\|\rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\|A(x+y) - A(x) - A(y)\| \leqslant \left\|\rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive. □

THEOREM 2.4. *Let $r>1$ and θ be positive real numbers, and let $f:X\to Y$ be a mapping satisfying* (2.4). *Then there exists a unique additive mapping $A:X\to Y$ such that*

$$\|f(x)-A(x)\|\leqslant\frac{2\theta}{|2|}\|x\|^r \tag{2.8}$$

for all $x\in X$.

Proof. It follows from (2.6) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\|\leqslant\frac{2\theta}{|2|}\|x\|^r$$

for all $x\in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{2^l}f(2^lx)-\frac{1}{2^m}f(2^mx)\right\| \qquad (2.9)\\
&\leqslant\max\left\{\left\|\frac{1}{2^l}f\left(2^lx\right)-\frac{1}{2^{l+1}}f\left(2^{l+1}x\right)\right\|,\cdots,\left\|\frac{1}{2^{m-1}}f\left(2^{m-1}x\right)-\frac{1}{2^m}f\left(2^mx\right)\right\|\right\}\\
&=\max\left\{\frac{1}{|2|^l}\left\|f\left(2^lx\right)-\frac{1}{2}f\left(2^{l+1}x\right)\right\|,\cdots,\frac{1}{|2|^{m-1}}\left\|f\left(2^{m-1}x\right)-\frac{1}{2}f\left(2^mx\right)\right\|\right\}\\
&\leqslant\max\left\{\frac{|2|^{rl}}{|2|^{l+1}},\cdots,\frac{|2|^{r(m-1)}}{|2|^{(m-1)+1}}\right\}2\theta\|x\|^r\\
&=\frac{2\theta}{|2|^{(1-r)l+1}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m>l$ and all $x\in X$. It follows from (2.9) that the sequence $\{\frac{1}{2^n}f(2^nx)\}$ is a Cauchy sequence for all $x\in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^nx)\}$ converges. So one can define the mapping $A:X\to Y$ by

$$A(x):=\lim_{n\to\infty}\frac{1}{2^n}f(2^nx)$$

for all $x\in X$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (2.9), we get (2.8).

The rest of the proof is similar to the proof of Theorem 2.3. □

Let $A(x,y):=f(x+y)-f(x)-f(y)$ and $B(x,y):=\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)$ for all $x,y\in X$.

For $x,y\in X$ with $\|A(x,y)\|\leqslant\|B(x,y)\|$,

$$\|A(x,y)\|-\|B(x,y)\|\leqslant\|A(x,y)-B(x,y)\|.$$

For $x,y\in X$ with $\|A(x,y)\|>\|B(x,y)\|$,

$$\begin{aligned}
\|A(x,y)\|&=\|A(x,y)-B(x,y)+B(x,y)\|\\
&\leqslant\max\{\|A(x,y)-B(x,y)\|,\|B(x,y)\|\}\\
&=\|A(x,y)-B(x,y)\|\\
&\leqslant\|A(x,y)-B(x,y)\|+\|B(x,y)\|,
\end{aligned}$$

since $\|A(x,y)\| > \|B(x,y)\|$. So we have

$$\begin{aligned}\|f(x+y)-f(x)-f(y)\| &- \left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\| \\ &\leqslant \left\|f(x+y)-f(x)-f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\|.\end{aligned}$$

As corollaries of Theorems 2.3 and 2.4, we obtain the Hyers-Ulam stability results for the additivec ρ-functional equation (2.3) in non-Archimedean Banach spaces.

COROLLARY 2.5. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\left\|f(x+y)-f(x)-f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\| \leqslant \theta(\|x\|^r+\|y\|^r) \quad (2.10)$$

for all $x,y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (2.5).

COROLLARY 2.6. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.10). *Then there exists a unique additive mapping $A : X \to Y$ satisfying* (2.8).

3. Additive ρ-functional inequality (0.2)

We solve the additive ρ-functional inequality (0.2) in non-Archimedean normed spaces.

LEMMA 3.1. *Let G be an Abelian semigroup with division by 2. A mapping $f : G \to Y$ satisfis $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\| \leqslant \|\rho\left(f(x+y)-f(x)-f(y)\right)\| \quad (3.1)$$

for all $x,y \in G$ if and if $f : G \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\|2f\left(\frac{x}{2}\right)-f(x)\right\| \leqslant 0 \quad (3.2)$$

and so $f\left(\frac{x}{2}\right)-\frac{1}{2}f(x)$ for all $x \in G$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}\|f(x+y)-f(x)-f(y)\| &= \left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\| \\ &\leqslant |\rho|\|f(x+y)-f(x)-f(y)\|\end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in G$.

The converse is obviously true. $\square$

COROLLARY 3.2. *Let G be an Abelian semigroup with division by 2. A mapping $f : G \to Y$ satisfies $f(0) = 0$ and*

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \rho(f(x+y) - f(x) - f(y)) \tag{3.3}$$

for all $x, y \in G$ if and only if $f : G \to Y$ is additive.

Now, we prove the Hyers-Ulam stability of the additive ρ-functional inequality (3.1) in non-Archimedean Banach spaces.

THEOREM 3.3. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leqslant \|\rho(f(x+y) - f(x) - f(y))\| + \theta(\|x\|^r + \|y\|^r) \tag{3.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leqslant \theta \|x\|^r \tag{3.5}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.4), we get

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| = \left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leqslant \theta \|x\|^r \tag{3.6}$$

for all $x \in X$. So

$$\begin{aligned} &\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \\ &\quad \leqslant \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\quad = \max\left\{|2|^l \left\|f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |2|^{m-1} \left\|f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\quad \leqslant \max\left\{\frac{|2|^l}{|2|^{rl}}, \cdots, \frac{|2|^{m-1}}{|2|^{r(m-1)}}\right\} \theta \|x\|^r \\ &\quad = \frac{\theta}{|2|^{(r-1)l}} \|x\|^r \end{aligned} \tag{3.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.7) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a non-Archimedean

Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A: X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.7), we get (3.5).

The rest of the proof is similar to the proof of Theorem 2.3. □

THEOREM 3.4. *Let $r > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and* (3.4)*. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leqslant \frac{|2|^r}{|2|}\theta\|x\|^r \tag{3.8}$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leqslant \frac{|2|^r}{|2|}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| \\
&\leqslant \max\left\{\left\|\frac{1}{2^l}f\left(2^l x\right) - \frac{1}{2^{l+1}}f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{2^{m-1}}f\left(2^{m-1}x\right) - \frac{1}{2^m}f\left(2^m x\right)\right\|\right\} \\
&= \max\left\{\frac{1}{|2|^l}\left\|f\left(2^l x\right) - \frac{1}{2}f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|2|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{2}f\left(2^m x\right)\right\|\right\} \\
&\leqslant \max\left\{\frac{|2|^{rl}}{|2|^{l+1}}, \cdots, \frac{|2|^{r(m-1)}}{|2|^{(m-1)+1}}\right\}|2|^r\theta\|x\|^r \\
&= \frac{|2|^r\theta}{|2|^{(1-r)l+1}}\|x\|^r
\end{aligned} \tag{3.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A: X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.9), we get (3.8).

The rest of the proof is similar to the proof of Theorem 2.3. □

Let $A(x,y) := 2f\left(\frac{x+y}{2}\right) - f(x) - f(y)$ and $B(x,y) := \rho\left(f(x+y) - f(x) - f(y)\right)$ for all $x,y \in X$.

For $x,y \in X$ with $\|A(x,y)\| \leqslant \|B(x,y)\|$,

$$\|A(x,y)\| - \|B(x,y)\| \leqslant \|A(x,y) - B(x,y)\|.$$

For $x,y \in X$ with $\|A(x,y)\| > \|B(x,y)\|$,

$$\begin{aligned}
\|A(x,y)\| &= \|A(x,y) - B(x,y) + B(x,y)\| \\
&\leqslant \max\{\|A(x,y) - B(x,y)\|, \|B(x,y)\|\} \\
&= \|A(x,y) - B(x,y)\| \\
&\leqslant \|A(x,y) - B(x,y)\| + \|B(x,y)\|,
\end{aligned}$$

since $\|A(x,y)\| > \|B(x,y)\|$. So we have

$$\begin{aligned}
&\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| - \left\|\rho\left(f(x+y) - f(x) - f(y)\right)\right\| \\
&\leqslant \left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) - f(x) - f(y)\right)\right\|.
\end{aligned}$$

As corollaries of Theorems 3.3 and 3.4, we obtain the Hyers-Ulam stability results for the additivec ρ-functional equation (3.3) in non-Archimedean Banach spaces.

COROLLARY 3.5. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) - f(x) - f(y)\right)\right\| \leqslant \theta(\|x\|^r + \|y\|^r) \tag{3.10}$$

for all $x,y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (3.5).

COROLLARY 3.6. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and* (3.10). *Then there exists a unique additive mapping $A : X \to Y$ satisfying* (3.8).

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(Received April 13, 2014)

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