Journal of
Mathematical
Inequalities
Volume 9, Number 2 (2015), 425–435 doi:10.7153/jmi-09-36

GENERALIZED COMPOSITION OPERATORS FROM ZYGMUND TYPE SPACES TO Q_K SPACES

HAIYING LI, TIANSHUI MA AND ZHITAO GUO

(*Communicated by J. Pečarić*)

Abstract. Let φ be an analytic self-map of $\mathbb{D}$ and $g \in H(\mathbb{D})$. The boundedness and compactness of generalized composition operators

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi,\ z \in \mathbb{D},\ f \in H(\mathbb{D})$$

from Zygmund type spaces to Q_K spaces are investigated.

1. Introduction

Let φ be an analytic self-map of the open unit disc $\mathbb{D}$ of the complex plane. Let $H(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ and $g \in H(\mathbb{D})$.

The composition operator is defined by $C_\varphi f(z) = f(\varphi(z))$, $f \in H(\mathbb{D})$. This operator is well studied for many years. We refer to the books [1, 2], which are excellent sources for the development of the theory of composition operators in function spaces.

The following, so called, generalized composition operator C_φ^g

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi,\ z \in \mathbb{D},\ f \in H(\mathbb{D}),$$

was introduced by Li and Stević in [3], and studied later, for example, in [4, 5, 6]. An n-dimensional extension of the operator was introduced in [7] and studied in [8, 9, 10, 11].

A fundamental problem in the study of generalized composition operators C_φ^g is to investigate the relations between function theoretic properties of φ and g and operator theoretic properties of the restriction of C_φ^g to various Banach spaces of analytic functions. Lots of attentions have been attracted to the study of the problem on many Banach spaces of analytic functions in recent years.

For $0 < \alpha < \infty$, a function $f \in H(\mathbb{D})$ belongs to the Zygmund type space $\mathscr{Z}_\alpha$ if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f''(z)| < \infty$$

Mathematics subject classification (2010): 47B38.
Keywords and phrases: Zygmund type space, Q_K space, generalized composition operator, bounded, compact.

equipped with the norm

$$\|f\|_{\mathscr{Z}_\alpha} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f''(z)| < \infty.$$

The little Zygmund type space $\mathscr{Z}_\alpha^0$ consists of all $f \in \mathscr{Z}_\alpha$ such that

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha |f''(z)| = 0.$$

When $\alpha = 1$, $\mathscr{Z}_\alpha$ is the classical Zygmund space $\mathscr{Z}$ and $\mathscr{Z}^0$. A systematic study of operators on Zygmund and Zygmund type spaces was started by Li and Stević in [3, 12, 13, 14]. For other papers in the area see, also, [7], [15]–[17] and [18]–[20].

Let $K : [0,\infty) \to [0,\infty)$ be a nonnegative nondecreasing continuous function. An analytic function f on D is said to belong to Q_K if

$$\|f\|_{Q_K} = \left\{ \sup_{a\in D} \int_D |f'(z)|^2 K(g(z,a))dA(z) \right\}^{\frac{1}{2}} < \infty$$

and an analytic function $f \in Q_{K,0}$ if

$$\lim_{|a|\to 1} \int_D |f'(z)|^2 K(g(z,a))dA(z) = 0,$$

where dA denotes the normalized Lebesgue area measure on D, $g(z,a) = \log \frac{1}{|\phi_a(z)|}$ is a Green function, $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$. If $K(t) = t^p, Q_K = Q_p$. Q_K is a Banach space under the norm $|f(0)| + \|f\|_{Q_K}$ and Q_K is Möbius invariant. If Q_K consists of just constant functions, we say that it is trivial. We assume throughout this paper that

$$\int_0^{\frac{1}{e}} K(-\log r) r dr < \infty.$$

Moreover, under this assumption, Q_K is nontrivial. For more details on Q_K space, see in [21]–[24]. Zhang and Liu studied generalized composition operators from Bloch type spaces to Q_K type spaces in [6]. This paper is devoted to investigating the boundedness and compactness of generalized composition operators C_φ^g from Zygmund type spaces to Q_K spaces. Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other.

2. Preliminaries

To derive our results, we need the following lemmas. The following Lemma 2.1 follows easily, from, for example, the arguments in [12], Lemma 2.2 in [25] and Lemma 2.4 in [26] (see also [16]).

LEMMA 2.1. *For every $f \in \mathscr{Z}_\alpha$ and $0 < \alpha < \infty$. Then*
(i) For $0 < \alpha < 1$. $|f'(z)| \leqslant \frac{2}{1-\alpha} \|f\|_{\mathscr{Z}_\alpha}$ and $|f(z)| \leqslant \frac{2}{1-\alpha} \|f\|_{\mathscr{Z}_\alpha}$;

(ii) For $\alpha = 1$. $|f'(z)| \leqslant 2\|f\|_{\mathscr{Z}_\alpha} \ln \frac{2}{1-|z|^2}$ *and* $|f(z)| \leqslant \|f\|_{\mathscr{Z}_\alpha}$*;*

(iii) For $\alpha > 1$. $|f'(z)| \leqslant \frac{2}{\alpha-1} \frac{\|f\|_{\mathscr{Z}_\alpha}}{(1-|z|^2)^{\alpha-1}}$*;*

(iv) For $1 < \alpha < 2$. $|f(z)| \leqslant \frac{2}{(\alpha-1)(\alpha-2)} \|f\|_{\mathscr{Z}_\alpha}$*;*

(v) For $\alpha = 2$. $|f(z)| \leqslant 2\|f\|_{\mathscr{Z}_2} \ln \frac{2}{1-|z|^2}$*;*

(vi) For $\alpha > 2$. $|f(z)| \leqslant \frac{2}{(\alpha-1)(\alpha-2)} \frac{\|f\|_{\mathscr{Z}_\alpha}}{(1-|z|^2)^{\alpha-2}}$.

LEMMA 2.2. *Let K be a nonnegative nondecreasing continuous function on* $[0,\infty)$. *Assume that φ is an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g : \mathscr{Z}_\alpha \to Q_K$ is compact if and only if for every bounded sequence $\{f_k\}$ in $\mathscr{Z}_\alpha$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$,* $\lim_{k\to\infty} \|C_\varphi^g f_k\|_{Q_K} = 0$.

LEMMA 2.3. *Let K be a nonnegative nondecreasing continuous function on* $[0,\infty)$. *Assume that φ is an analytic self-map of $\mathbb{D}$ and $0 < \alpha < \infty$. If $C_\varphi^g : \mathscr{Z}_\alpha \to Q_K$ is compact, then for any $\varepsilon > 0$ there exists a δ, $0 < \delta < 1$ such that for all f in $\mathscr{Z}_\alpha$,*

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f'(\varphi(z))|^2 |g(z)|^2 K(g(z,a))\,dA(z) < \varepsilon$$

holds whenever $\delta < r < 1$.

Proof. The lemma can be obtained by the similar methods in [19, 23, 24]. Here we omit the details. □

LEMMA 2.4. ([27]) *Let f be a holomorphic function in $\mathbb{D}$ with the gap series expansion*

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}, \quad z \in \mathbb{D},$$

where for a constant $q > 1$ the natural numbers n_k, $k \geqslant 1$, satisfy $n_{k+1}/n_k \geqslant q$, $k \geqslant 1$. Then

$$f \in \mathscr{B}_\alpha \text{ if and only if } \limsup_{k\to\infty} |a_k| n_k^{1-\alpha} < \infty.$$

The following lemma can be found in [28].

LEMMA 2.5. *Assume that $\{n_k\}$ is an increasing sequence of positive integers satisfying $\frac{n_{k+1}}{n_k} \geqslant \lambda > 1$ for all $k \in \mathbb{N}$. Let $0 < p < \infty$. Then there are two positive constants C_1 and C_2, depending only on p and λ such that*

$$C_1 \left(\sum_{k=0}^{\infty} |a_k|^2 \right)^{\frac{1}{2}} \leqslant \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^{\infty} a_k e^{i n_k \theta} \right|^p d\theta \right)^{\frac{1}{p}} \leqslant C_2 \left(\sum_{k=0}^{\infty} |a_k|^2 \right)^{\frac{1}{2}}.$$

3. The boundedness and compactness of $C_\varphi^g : \mathscr{Z}_\alpha \to Q_K$

THEOREM 3.1. *Let K be a nonnegative nondecreasing continuous function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$ and $0<\alpha<1$. Then $C_\varphi^g : \mathscr{Z}_\alpha \to Q_K$ is bounded if and only if*

$$M := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g(z)|^2 K(g(z,a))\,dA(z) < \infty, \tag{1}$$

moreover,

$$\|C_\varphi^g\|^2_{\mathscr{Z}_\alpha\to Q_K} \asymp M.$$

Proof. Let $f\in\mathscr{Z}_\alpha$, by Lemma 2.1 (i), we have

$$|f'(z)| \leqslant \frac{2}{1-\alpha}\|f\|_{\mathscr{Z}_\alpha}.$$

Hence

$$\begin{aligned}\|C_\varphi^g f\|^2_{Q_K} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(C_\varphi^g f)'(z)|^2 K(g(z,a))\,dA(z)\\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(\varphi(z))|^2 |g(z)|^2 K(g(z,a))\,dA(z)\\ &\leqslant C\|f\|^2_{\mathscr{Z}_\alpha} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g(z)|^2 K(g(z,a))\,dA(z).\end{aligned}$$

Hence $C_\varphi^g : \mathscr{Z}_\alpha \to Q_K$ is bounded by (1).

Conversely, suppose that $C_\varphi^g : \mathscr{Z}_\alpha \to Q_K$ is bounded. Let $h(z)=z\in\mathscr{Z}_\alpha$, then

$$\begin{aligned}\infty &> \|C_\varphi^g h\|^2_{Q_K}\\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(C_\varphi^g h)'(z)|^2 K(g(z,a))\,dA(z)\\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g(z)|^2 K(g(z,a))\,dA(z).\end{aligned}$$

Hence (1) holds.

Finally, the estimate of the norm $\|C_\varphi^g\|^2_{\mathscr{Z}_\alpha\to Q_K}$ is easily obtained. □

THEOREM 3.2. *Let K be a nonnegative nondecreasing continuous function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$ and $\alpha=1$. Then*

(a) If

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g(z)|^2 \left(\ln\frac{2}{1-|\varphi(z)|^2}\right)^2 K(g(z,a))\,dA(z) < \infty, \tag{2}$$

then $C_\varphi^g : \mathscr{Z} \to Q_K$ is bounded.

(b) If $C_\varphi^g : \mathscr{Z} \to Q_K$ is bounded, then

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g(z)|^2 \ln\frac{1}{1-|\varphi(z)|^2} K(g(z,a))\,dA(z) < \infty. \tag{3}$$

Proof. (a) Let $f \in \mathscr{Z}$, then by Lemma 2.1 (ii),

$$|f'(z)| \leqslant 2\|f\|_{\mathscr{Z}} \ln \frac{2}{1-|z|^2}.$$

Then we have

$$\begin{aligned}
\|C_\varphi^g f\|_{Q_K}^2 &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(C_\varphi^g f)'(z)|^2 K(g(z,a))dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(\varphi(z))|^2 |g(z)|^2 K(g(z,a))dA(z) \\
&\leqslant C\|f\|_{\mathscr{Z}}^2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g(z)|^2 \left(\ln \frac{2}{1-|\varphi(z)|^2}\right)^2 K(g(z,a))dA(z).
\end{aligned}$$

Hence $C_\varphi^g : \mathscr{Z} \to Q_K(p,q)$ is bounded by (2).

(b) Assume that $C_\varphi^g : \mathscr{Z} \to Q_K$ is bounded. Let $h(z) = z \in \mathscr{Z}$. By the boundedness of $C_\varphi^g : \mathscr{Z} \to Q_K$, then

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g(z)|^2 K(g(z,a))dA(z) < \infty.$$

Hence

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|\leqslant \frac{1}{\sqrt{2}}} |g(z)|^2 \ln \frac{1}{1-|\varphi(z)|^2} K(g(z,a))dA(z) \\
&\leqslant \ln 2 \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|\leqslant \frac{1}{\sqrt{2}}} |g(z)|^2 K(g(z,a))dA(z) \\
&\leqslant \ln 2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g(z)|^2 K(g(z,a))dA(z) < \infty. \qquad (4)
\end{aligned}$$

For $z \in \mathbb{D}$, such that $|z| = r \leqslant \frac{1}{\sqrt{2}}$. Let

$$t(z) = \sum_{k=0}^{\infty} \frac{1}{2^k+1} z^{2^k+1}.$$

By Lemma 2.4, $t'(z) = \sum_{k=0}^{\infty} z^{2^k} \in \mathscr{B}$. By the relationship of Bloch space $\mathscr{B}$ and Zygmund space $\mathscr{Z}$, then $t \in \mathscr{Z}$. Let

$$t_\theta(z) = t(e^{i\theta}z) = \sum_{k=0}^{\infty} \frac{1}{2^k+1} (e^{i\theta}z)^{2^k+1}.$$

Then we have $t_\theta \in \mathscr{Z}$ and $\|t_\theta\|_{\mathscr{Z}} = \|t\|_{\mathscr{Z}}$ by the definition of Zygmund space and basic calculation. Thus

$$\begin{aligned}
\infty &> \|C_\varphi^g\|^2 \|t_\theta\|_{\mathscr{Z}}^2 \\
&\geqslant \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(C_\varphi^g t_\theta)'(z)|^2 K(g(z,a))dA(z) \\
&\geqslant \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|> \frac{1}{\sqrt{2}}} \left| \sum_{k=0}^{\infty} e^{i(2^k+1)\theta} \varphi^{2^k}(z) \right|^2 |g(z)|^2 K(g(z,a))dA(z). \qquad (5)
\end{aligned}$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi}\|C_\varphi^g\|^2\|t_\theta\|_{\mathscr{Z}}^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi}\|C_\varphi^g\|^2\|t\|_{\mathscr{Z}}^2 d\theta = \|C_\varphi^g\|^2\|t_\theta\|_{\mathscr{Z}}^2,$$

by use of Lemma 2.5, Fubini's theorem and (5), then we have

$$\begin{aligned}
\infty &> \frac{1}{2\pi}\int_0^{2\pi}\|C_\varphi^g\|^2\|t_\theta\|_{\mathscr{Z}}^2 d\theta \\
&\geqslant \frac{1}{2\pi}\int_0^{2\pi}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>\frac{1}{\sqrt{2}}}\left|\sum_{k=0}^{\infty} e^{i(2^k+1)\theta}\varphi^{2^k}(z)\right|^2 |g(z)|^2 K(g(z,a))dA(z)d\theta \\
&\geqslant \sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>\frac{1}{\sqrt{2}}}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=0}^{\infty} e^{i(2^k+1)\theta}\varphi^{2^k}(z)\right|^2 d\theta\right\}|g(z)|^2 K(g(z,a))dA(z) \\
&\geqslant C\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>\frac{1}{\sqrt{2}}}\left(\sum_{k=0}^{\infty}|\varphi(z)|^{2^{k+1}}\right)|g(z)|^2 K(g(z,a))dA(z).
\end{aligned}$$

For any $0<r<1$,

$$\ln\frac{1}{1-r^2} = \sum_{k=1}^{\infty}\frac{r^{2k}}{k} = \sum_{k=0}^{\infty}\sum_{j=2^k}^{2^{k+1}-1}\frac{r^{2j}}{j} \leqslant \sum_{k=0}^{\infty}\left(\frac{1}{2^k}+\cdots+\frac{1}{2^k}\right)r^{2\cdot 2^k} = \sum_{k=0}^{\infty} r^{2^{k+1}}.$$

Thus

$$\begin{aligned}
\infty &> \frac{1}{2\pi}\int_0^{2\pi}\|C_\varphi^g\|^2\|t_\theta\|_{\mathscr{Z}}^2 d\theta \\
&\geqslant \sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>\frac{1}{\sqrt{2}}}|g(z)|^2\ln\frac{1}{1-|\varphi(z)|^2}K(g(z,a))dA(z). \qquad (6)
\end{aligned}$$

Then (3) holds by (4) and (6). $\square$

THEOREM 3.3. *Let K be a nonnegative nondecreasing continuous function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$ and $\alpha>1$. If*

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|g(z)|^2}{(1-|\varphi(z)|^2)^{2(\alpha-1)}}K(g(z,a))dA(z)<\infty, \qquad (7)$$

then $C_\varphi^g:\mathscr{Z}_\alpha\to Q_K$ is bounded.

Proof. Let $f\in\mathscr{Z}_\alpha$, then by Lemma 2.1 (iii),

$$|f'(z)| \leqslant \frac{2}{\alpha-1}\frac{\|f\|_{\mathscr{Z}_\alpha}}{(1-|z|^2)^{\alpha-1}}.$$

Then we have

$$\begin{aligned}\|C_\varphi^g f\|_{Q_K}^2 &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |(C_\varphi^g f)'(z)|^2 K(g(z,a))dA(z)\\ &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f'(\varphi(z))|^2|g(z)|^2 K(g(z,a))dA(z)\\ &\leqslant C\|f\|_{\mathscr{Z}_\alpha}^2 \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{|g(z)|^2K(g(z,a))}{(1-|\varphi(z)|^2)^{2(\alpha-1)}}dA(z).\end{aligned} \tag{8}$$

Hence $C_\varphi^g:\mathscr{Z}_\alpha\to Q_K$ is bounded by (7) and (8). □

THEOREM 3.4. *Let K be a nonnegative nondecreasing continuous function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$ and $0<\alpha<1$. Then $C_\varphi^g:\mathscr{Z}_\alpha\to Q_K$ is compact if and only if (1) and*

$$\lim_{r\to 1}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r} |g(z)|^2K(g(z,a))dA(z)=0. \tag{9}$$

Proof. Let $\{f_k\}$ be a bounded sequence in $\mathscr{Z}_\alpha$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$. By use of Lemma 2.2, we only need to prove that $\|C_\varphi^g f_k\|_{Q_K}\to 0$, $k\to\infty$. By (9), we have that, for any $\varepsilon>0$, there exists an r, $0<r<1$ such that

$$\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r} |g(z)|^2K(g(z,a))dA(z)<\varepsilon. \tag{10}$$

By Lemma 2.1 (i), we have

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |(C_\varphi^g f_k)'(z)|^2K(g(z,a))dA(z)\\ \leqslant{}& \sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r} |f_k'(\varphi(z))|^2|g(z)|^2K(g(z,a))dA(z)\\ &+\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|\leqslant r} |f_k'(\varphi(z))|^2|g(z)|^2K(g(z,a))dA(z)\\ \leqslant{}& C\|f_k\|_{\mathscr{Z}_\alpha}^2\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r} |g(z)|^2K(g(z,a))dA(z)\\ &+\sup_{|w|\leqslant r}|f_k'(w)|^2\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |g(z)|^2K(g(z,a))dA(z).\end{aligned}$$

By Cauchy's estimate, $\{f_k'\}$ also converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then $\sup_{|w|\leqslant r}|f_k'(w)|^2\to 0$, $k\to\infty$. Hence by use of (1) and (10), $\|C_\varphi^g f_k\|_{Q_K}\to 0$, $k\to\infty$. By Lemma 2.2, $C_\varphi^g:\mathscr{Z}_\alpha\to Q_K$ is compact.

Conversely, assume that $C_\varphi^g:\mathscr{Z}_\alpha\to Q_K$ is compact. Let $h(z)=z\in\mathscr{Z}_\alpha$, we have that (1) holds. By Lemma 2.3, then for any $\varepsilon>0$, there exists a δ, $0<\delta<1$ such that for any $f\in\mathscr{Z}_\alpha$, whenever $\delta<r<1$,

$$\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r} |f'(\varphi(z))|^2|g(z)|^2K(g(z,a))dA(z)<\varepsilon. \tag{11}$$

Let $f(z)=z\in\mathscr{Z}_\alpha$ in (11), then

$$\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}|g(z)|^2K(g(z,a))dA(z)<\varepsilon. \quad \square$$

THEOREM 3.5. *Let K be a nonnegative nondecreasing continuous function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$ and $\alpha=1$. Then*

(a) If (1) holds and

$$\lim_{r\to 1}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}|g(z)|^2\left(\ln\frac{2}{1-|\varphi(z)|^2}\right)^2K(g(z,a))dA(z)=0, \tag{12}$$

then $C_\varphi^g:\mathscr{Z}\to Q_K$ is compact.

(b) If $C_\varphi^g:\mathscr{Z}\to Q_K$ is compact, then (1) and

$$\lim_{r\to 1}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}|g(z)|^2\ln\frac{1}{1-|\varphi(z)|^2}K(g(z,a))dA(z)=0. \tag{13}$$

Proof. (a) Let $\{f_k\}$ be a bounded sequence in $\mathscr{Z}$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$. By use of Lemma 2.2, we only need to prove that $\|C_\varphi^g f_k\|_{Q_K}\to 0$, $k\to\infty$. By (12), we have that for any $\varepsilon>0$, there exists an r, $0<r<1$ such that

$$\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}|g(z)|^2\left(\ln\frac{2}{1-|\varphi(z)|^2}\right)^2K(g(z,a))dA(z)<\varepsilon. \tag{14}$$

By Lemma 2.1 (ii), then

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|(C_\varphi^g f_k)'(z)|^2K(g(z,a))dA(z)\\
\leqslant&\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}|f_k'(\varphi(z))|^2|g(z)|^2K(g(z,a))dA(z)\\
&+\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|\leqslant r}|f_k'(\varphi(z))|^2|g(z)|^2K(g(z,a))dA(z)\\
\leqslant&\sup_{|w|\leqslant r}|f_k'(w)|^2\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|g(z)|^2K(g(z,a))dA(z)\\
&+C\|f_k\|_{\mathscr{Z}_\alpha}^2\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}|g(z)|^2\left(\ln\frac{2}{1-|\varphi(z)|^2}\right)^2K(g(z,a))dA(z).
\end{aligned}$$

By Cauchy's estimate, $\{f_k'\}$ also converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then

$$\sup_{|w|\leqslant r}|f_k'(w)|^2\to 0,\quad k\to\infty. \tag{15}$$

Hence by (1), (14) and (15), $\|C_\varphi^g f_k\|_{Q_K}\to 0$, $k\to\infty$. By Lemma 2.2, $C_\varphi^g:\mathscr{Z}\to Q_K$ is compact.

(b) Suppose that $C_\varphi^g : \mathscr{Z} \to Q_K$ is compact. Let $h(z) = z \in \mathscr{Z}$, then (1) holds. Now we choose the function $t(z)$ given in the proof of Theorem 3.2, then $t \in \mathscr{Z}$. Choose a sequence $\{\lambda_j\}$ in $\mathbb{D}$ which converges to 1 as $j \to \infty$ and let $t_j(z) = t(\lambda_j z)$ for $j \in \mathbb{N}$. Then $t_j \in \mathscr{Z}$ and $\|t_j\|_{\mathscr{Z}} \leqslant C$. Let $t_{j,\theta}(z) = t_j(e^{i\theta}z)$, then $t_{j,\theta} \in \mathscr{Z}^0$. Replace f in Lemma 2.3 by $t_{j,\theta}$ and integrate both sides with respect to θ, then we have

$$\begin{aligned}\varepsilon &> \sup_{a\in\mathbb{D}} \frac{1}{2\pi}\int_{|\varphi(z)|>r}\left(\int_0^{2\pi}|t_j'(e^{i\theta}\varphi(z))|^2 d\theta\right)|g(z)|^2K(g(z,a))dA(z)\\ &= \sup_{a\in\mathbb{D}} \frac{1}{2\pi}\int_{|\varphi(z)|>r}\int_0^{2\pi}\left|\sum_{k=1}^{\infty}(\lambda_j\varphi(z)e^{i\theta})^{2^k}\right|^2 d\theta|\lambda_j|^2|g(z)|^2K(g(z,a))dA(z)\\ &= \sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\left(\sum_{k=1}^{\infty}|\lambda_j\varphi(z)|^{2^{k+1}}\right)|\lambda_j|^2|g(z)|^2K(g(z,a))dA(z).\end{aligned}$$

For $\frac{1}{\sqrt{2}} < r < 1$ and sufficiently large j, then we have

$$\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}|\lambda_j|^2\left(\ln\frac{1}{1-|\lambda_j\varphi(z)|^2}\right)|g(z)|^2K(g(z,a))dA(z) < \varepsilon.$$

Using Fatou's lemma, we obtain (13). □

THEOREM 3.6. *Let K be a nonnegative nondecreasing continuous function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$ and $\alpha > 1$. If (1) holds and*

$$\lim_{r\to 1}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\frac{|g(z)|^2}{(1-|\varphi(z)|^2)^{2(\alpha-1)}}K(g(z,a))dA(z) = 0, \tag{16}$$

then $C_\varphi^g : \mathscr{Z}_\alpha \to Q_K$ is compact.

Proof. Let $\{f_k\}$ be a bounded sequence in $\mathscr{Z}_\alpha$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$. By use of Lemma 2.2, we only need to prove that $\|C_\varphi^g f_k\|_{Q_K} \to 0$, $k \to \infty$. By (16), then for any $\varepsilon > 0$, there exists an r, $0 < r < 1$ such that

$$\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\frac{|g(z)|^2K(g(z,a))}{(1-|\varphi(z)|^2)^{2(\alpha-1)}}dA(z) < \varepsilon. \tag{17}$$

By Lemma 2.1 (iii), then

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|(C_\varphi^g f_k)'(z)|^2K(g(z,a))dA(z)\\ \leqslant&\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}|f_k'(\varphi(z))|^2|g(z)|^2K(g(z,a))dA(z)\\ &+\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|\leqslant r}|f_k'(\varphi(z))|^2|g(z)|^2K(g(z,a))dA(z)\\ \leqslant&\sup_{|w|\leqslant r}|f_k'(w)|^2\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|g(z)|^2K(g(z,a))dA(z)\\ &+C\|f_k\|_{\mathscr{Z}_\alpha}^2\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\frac{|g(z)|^2K(g(z,a))}{(1-|\varphi(z)|^2)^{2(\alpha-1)}}dA(z).\end{aligned}$$

By Cauchy's estimate, $\{f_k'\}$ also converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then $\sup_{|w|\leqslant r}|f_k'(w)|^2 \to 0$, $k\to\infty$. Hence by (1) and (17), $\|C_\varphi^g f_k\|_{Q_K}\to 0$, $k\to\infty$. By Lemma 2.2, $C_\varphi^g : \mathscr{Z}_\alpha \to Q_K$ is compact. □

Acknowledgement. This work was supported in part by the National Natural Science Foundation of China (Nos. 11201127; 11271112), the Young Core Teachers Program of Henan Province (No. 2011GGJS-062) and IRTSTHN (14IRTSTHN023).

REFERENCES

[1] C. C. COWEN AND B. D. MACCLUER, *Composition operators on spaces of analytic functions*, CRC Press, Boca Roton, 1995.

[2] J. H. SHAPIRO, *Composition Operators and Classical Function Theory*, Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.

[3] S. LI AND S. STEVIĆ, *Generalized composition operators on Zygmund spaces and Bloch type spaces*, J. Math. Anal. Appl. **338** (2008), 1282–1295.

[4] S. STEVIĆ, *Generalized composition operators from logarithmic Bloch spaces to mixed-norm spaces*, Util. Math. **77** (2008), 167–172.

[5] S. STEVIĆ AND AJAY K. SHARMA, *Generalized composition operators on weighted Hardy spaces*, Appl. Math. Comput. **218**, 17 (2012), 8347–8352.

[6] F. ZHANG AND Y. LIU, *Generalized composition operators from Bloch type spaces to Q_K type spaces*, J. Funct. Spaces Appl. **8**, 1 (2010), 55–66.

[7] S. STEVIĆ, *On an integral operator from Zygmund spaces to the Bloch-type space on the unit ball*, Glasg. Math. J. **51** (2009), 275–287.

[8] S. STEVIĆ, *Integral-type operators from a mixed norm space to a Bloch-type space on the unit ball*, Siberian Math. J. **50**, 6 (2009), 1098–1105.

[9] S. STEVIĆ, *On an integral operator between Bloch-type spaces on the unit ball*, Bull. Sci. Math. **134**, 4 (2010), 329–339.

[10] S. STEVIĆ AND SEI-ICHIRO UEKI, *On an integral-type operator between weighted-type spaces and Bloch-type spaces on the unit ball*, Appl. Math. Comput. **217**, 7 (2010), 3127–3136.

[11] S. STEVIĆ AND SEI-ICHIRO UEKI, *Integral-type operators acting between weighted-type spaces on the unit ball*, Appl. Math. Comput. **215**, 7 (2009), 2464–2471.

[12] S. LI AND S. STEVIĆ, *Volterra-type operators on Zygmund spaces*, J. Inequal. Appl. **2007**, Art. ID 32124, 10 pp.

[13] S. LI AND S. STEVIĆ, *Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to Zygmund spaces*, J. Math. Anal. Appl. **345**, 1 (2008), 40–52.

[14] S. LI AND S. STEVIĆ, *Weighted composition operators from Zygmund spaces into Bloch spaces*, Applied Mathematics and Computation **206**, 2 (2008), 825–831.

[15] B. CHOE, H. KOO AND W. SMITH, *Composition operators on small spaces*, Integr. Equ. Oper. Theory **56** (2006), 357–380.

[16] K. ESMAEILI AND M. LINDSTRÖM, *Weighted composition operators between Zygmund type spaces and their essential norms*, Integr. Equ. Oper. Theory **75** (2013), 473–490.

[17] S. YE AND Q. HU, *Weighted composition operators on the Zygmund space*, Abstract and Applied Analysis **2012** (2012), Article ID 462482.

[18] P. L. DUREN, *Theory of H^p Spaces*, San Diego: Pure and Applied Mathematics **38**, Academic Press, 1970.

[19] X. FU AND S. LI, *Composition operators from Zygmund spaces into Q_K spaces*, J. Inequal. Appl. **2013**, 175 (2013).

[20] Y. REN, *An integral-type operator from $Q_K(p,q)$ spaces to Zygmund-type spaces*, Appl. Math. Comput. **236** (2014), 27–32.

[21] R. ZHAO, *On a general family of function space*, Ann. Acad. Sci. Fenn. Math. Diss., 1996.

[22] M. ESSÉN, H. WULAN AND J. XIAO, *Several function theoretic characterization of Q_K spaces*, J. Funct. Anal. **230** (2006), 78–115.

[23] H. WULAN, *Compactness of the composition operators from the Bloch space to Q_K spaces*, Acta Math. Sin. **21** (2005), 1415–1424.

[24] S. LI, *On an integral-type operator from the Bloch space into $Q_K(p,q)$ spaces*, Filomat **26** (2012), 125–133.

[25] S. STEVIĆ, *On an integral operator on the unit ball in $\mathbb{C}^n$*, J. Inequal. Appl. **1** (2005), 81–88.

[26] S. STEVIĆ, *On an integral-type operator from Zygmund-type spaces to mixed-norm spaces on the unit ball*, Abstr. Appl. Anal. **2010**, Art. ID 198608, 7 pp.

[27] S. YAMASHITA, *Gap series and α-Bloch functions*, Yokohama Math. J. **28** (1980), 31–36.

[28] A. ZYGMUND, *Trigonometric series*, Cambridge University Press, London, 1959.

(Received May 13, 2014)

Haiying Li
College of Mathematics and Information Science
Henan Normal University
Xinxiang 453007, China
e-mail: haiyingli2012@yahoo.com

Tianshui Ma
College of Mathematics and Information Science
Henan Normal University
Xinxiang 453007, China

Zhitao Guo
College of Mathematics and Information Science
Henan Normal University
Xinxiang 453007, China