Journal of Mathematical Inequalities
Volume 9, Number 2 (2015), 541–551 doi:10.7153/jmi-09-47

ASYMPTOTIC FORMULAS FOR THE GAMMA FUNCTION BY GOSPER

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(*Communicated by N. Elezović*)

Abstract. The main aim of this paper is to give two general asymptotic expansions for the gamma function, which include the Gosper formula as their special cases. Furthermore, we present an inequality for the gamma function.

1. Introduction

Stirling's formula

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad n \in \mathbb{N} := \{1,2,\ldots\} \tag{1.1}$$

has many applications in statistical physics, probability theory and number theory. Actually it was discovered by A. De Moivre (1667-1754) in the form

$$n! \approx C \cdot \sqrt{n}\left(\frac{n}{e}\right)^n,$$

and Stirling (1692–1770) identified the constant C precisely $\sqrt{2\pi}$.

The following asymptotic formulas are well-known for the gamma function (see, for example, [1, p. 257]):

$$\begin{aligned}\Gamma(x+1) &\sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{m=1}^{\infty}\frac{B_{2m}}{2m(2m-1)x^{2m-1}}\right)\\ &= \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}-\frac{1}{1680x^7}+\cdots\right)\\ &\qquad (x\to\infty) \quad (\text{Stirling series})\end{aligned} \tag{1.2}$$

and

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}-\frac{571}{2488320x^4}+\cdots\right) \quad (x\to\infty) \quad (\text{Laplace formula}), \tag{1.3}$$

Mathematics subject classification (2010): Primary 33B15; Secondary 41A60.
Keywords and phrases: Gamma function, asymptotic formula, Bernoulli numbers, fundamental theorem of algebra, Newton's formulas.

where B_n $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ are the n-th Bernoulli numbers defined by the following generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \qquad |z| < 2\pi.$$

The Laplace formula (1.3) is sometimes incorrectly called Stirling series (see [14, pp. 2–3]). Stirling's formula is in fact the first approximation to the asymptotic formula (1.3).

Stirling's formula has attracted much interest of many mathematicians and have motivated a large number of research papers concerning various generalizations and improvements (see, for example, [3, 4, 5, 7, 8, 9, 11, 12, 16, 18, 20, 21, 22]). See also an overview at [19].

Gosper [16] replaced $\sqrt{2\pi n}$ by $\sqrt{2\pi(n+1/6)}$ in Stirling's formula to substantially improve it, to

$$n! \approx \sqrt{2\pi\left(n+\frac{1}{6}\right)}\left(\frac{n}{e}\right)^n. \tag{1.4}$$

Buric and Elezovic [9] obtained the following asymptotic expansion which includes the Gosper formula (1.4) as its special case:

$$n! \sim \sqrt{2\pi}\left(\frac{n}{e}\right)^n \sqrt{n+\frac{1}{6}}\left(\sum_{m=0}^{\infty} P_m n^{-m}\right) \tag{1.5}$$

with the coefficients P_m given by

$$P_0 = 1, \quad P_m = \frac{1}{m}\sum_{k=2}^{m}\left[\frac{B_{k+1}}{k+1} + \frac{(-1)^k}{2\cdot 6^k}\right] P_{m-k}, \tag{1.6}$$

where B_k are the Bernoulli numbers. Namely,

$$n! \sim \sqrt{2\pi}\left(\frac{n}{e}\right)^n \sqrt{n+\frac{1}{6}}\left(1 + \frac{1}{144n^2} - \frac{23}{6480n^3} + \frac{5}{41472n^4} + \frac{4939}{6531840n^5} + \frac{11839}{1343692800n^6} - \frac{1110829}{1881169920n^7} - \frac{14470283}{5417769369600n^8} + \cdots\right). \tag{1.7}$$

REMARK 1.1. A slight modification of the Theorem 4.1 from [9] gives the general algorithm in the following form:

The following formula is valid for all real $r \neq 0$,

$$n! \sim \sqrt{2\pi}\left(\frac{n}{e}\right)^n \sqrt{n+\frac{1}{6}}\left(\sum_{m=0}^{\infty} P_m n^{-m}\right)^{1/r}, \tag{1.8}$$

where

$$P_0 = 1, \quad P_m = \frac{r}{m}\sum_{k=2}^{m}\left[\frac{B_{k+1}}{k+1} + \frac{(-1)^k}{2\cdot 6^k}\right] P_{m-k} \tag{1.9}$$

and B_k are the Bernoulli numbers.

In this paper, we deal with the same problem. Our proving methods are different from ones in [9]. Furthermore, we present an inequality for the gamma function.

2. Main results

THEOREM 2.1. *Let r be a given nonzero real number. Then the gamma function has the following asymptotic expansion:*

$$\Gamma(x+1) \sim \sqrt{2\pi\left(x+\frac{1}{6}\right)}\left(\frac{x}{e}\right)^x\left(1+\sum_{j=1}^{\infty}\frac{a_j}{x^j}\right)^{1/r}, \qquad x\to\infty, \tag{2.1}$$

where the coefficients $a_j \equiv a_j(r)$ $(j=1,2,\ldots)$ are given by

$$a_j = (-1)^j \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{S_1}{1}\right)^{k_1}\left(\frac{S_2}{2}\right)^{k_2}\cdots\left(\frac{S_j}{j}\right)^{k_j}, \tag{2.2}$$

the summation being taken over all combinations of nonnegative integers k_j satisfying the equation

$$k_1+2k_2+\cdots+jk_j = j,$$

and

$$S_j = S_j(r) = r\left(\frac{(-1)^{j-1}B_{j+1}}{j+1}-\frac{1}{2\cdot 6^j}\right), \qquad j\in\mathbb{N}.$$

Proof. To determine a_j $(j=1,2,\ldots)$, we first express (2.1) as follows:

$$r\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x\left(1+\frac{1}{6x}\right)}\left(\frac{x}{e}\right)^x}\right) = \ln\left(1+\sum_{j=1}^{m}\frac{a_j}{x^j}\right)+O(x^{-m-1}) \qquad (x\to\infty).$$

By using the fundamental theorem of algebra, we see that there exist unique complex numbers $x_1,\ldots,x_m$ such that

$$1+\frac{a_1}{x}+\cdots+\frac{a_m}{x^m} = \left(1+\frac{x_1}{x}\right)\cdots\left(1+\frac{x_m}{x}\right). \tag{2.3}$$

By applying the following series expansion:

$$\ln\left(1+\frac{z}{x}\right) = \sum_{j=1}^{m}\frac{(-1)^{j-1}z^j}{jx^j}+O(x^{-m-1}) \qquad (|z|<|x|;\ x\to\infty), \tag{2.4}$$

we obtain

$$\ln\left(1+\frac{a_1}{x}+\cdots+\frac{a_m}{x^m}\right) = \sum_{j=1}^{m}\frac{(-1)^{j-1}S_j}{jx^j}+O(x^{-m-1}), \qquad x\to\infty, \tag{2.5}$$

where

$$S_j = x_1^j + \cdots + x_m^j, \qquad j = 1, \ldots, m.$$

It follows from (1.2) that

$$\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}\,(x/e)^x}\right) \sim \sum_{j=1}^{\infty} \frac{B_{j+1}}{j(j+1)x^j} \qquad (x \to \infty).$$

By using the Maclaurin expansion of $\ln(1+x)$:

$$\ln(1+x) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} x^j, \qquad |x| < 1,$$

we obtain

$$\frac{1}{2}\ln\left(1+\frac{1}{6x}\right) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2j \cdot 6^j x^j}, \qquad x \to \infty.$$

Moreover, we have, as $x \to \infty$,

$$\begin{aligned} r\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x\left(1+\frac{1}{6x}\right)}\left(\frac{x}{e}\right)^x}\right) &= r\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}\,(x/e)^x}\right) - \frac{r}{2}\ln\left(1+\frac{1}{6x}\right) \\ &\sim \sum_{j=1}^{\infty} \frac{r}{j}\left(\frac{B_{j+1}}{j+1} - \frac{(-1)^{j-1}}{2\cdot 6^j}\right)\frac{1}{x^j} \end{aligned} \tag{2.6}$$

or

$$r\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x\left(1+\frac{1}{6x}\right)}\left(\frac{x}{e}\right)^x}\right) \sim \sum_{j=1}^{m} \frac{r}{j}\left(\frac{B_{j+1}}{j+1} - \frac{(-1)^{j-1}}{2\cdot 6^j}\right)\frac{1}{x^j} + O(x^{-m-1}). \tag{2.7}$$

From (2.5) and (2.7), we obtain

$$S_j = S_j(r) = r\left(\frac{(-1)^{j-1}B_{j+1}}{j+1} - \frac{1}{2\cdot 6^j}\right), \qquad j = 1, 2, \ldots, m. \tag{2.8}$$

That is,

$$\begin{cases} x_1 + \cdots + x_m = S_1, \\ x_1^2 + \cdots + x_m^2 = S_2, \\ \qquad \cdots \\ x_1^m + \cdots + x_m^m = S_m. \end{cases} \tag{2.9}$$

Let

$$P_m(x) = x^m + b_1 x^{m-1} + \cdots + b_{m-1}x + b_m$$

be a polynomial with zeros: $x_1, \ldots, x_m$ which satisfy the system of equations (2.9). So we have

$$P_m(x) = (x - x_1)\cdots(x - x_m).$$

The Newton formulas (see, for example, [17] and references therein) give the connection between the coefficients b_j and the power sums S_j:

$$S_j + S_{j-1}b_1 + S_{j-2}b_2 + \cdots + S_1 b_{j-1} + jb_j = 0, \qquad j = 1,\ldots,m.$$

It is known (see also [17]) that b_j can be expressed in terms of S_j:

$$b_j = \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!} \left(\frac{S_1}{1}\right)^{k_1} \left(\frac{S_2}{2}\right)^{k_2} \cdots \left(\frac{S_j}{j}\right)^{k_j}.$$

Clearly,

$$\frac{(-1)^m}{x^m} P_m(-x) = \left(1+\frac{x_1}{x}\right)\cdots\left(1+\frac{x_m}{x}\right).$$

We thus have

$$1 + \frac{(-1)b_1}{x} + \frac{(-1)^2 b_2}{x^2} + \cdots + \frac{(-1)^m b_m}{x^m} = \left(1+\frac{x_1}{x}\right)\cdots\left(1+\frac{x_m}{x}\right). \tag{2.10}$$

By (2.3) and (2.10), the coefficients a_j are given by

$$\begin{aligned} a_j &= (-1)^j b_j \\ &= (-1)^j \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{(-1)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!} \left(\frac{S_1}{1}\right)^{k_1} \left(\frac{S_2}{2}\right)^{k_2} \cdots \left(\frac{S_j}{j}\right)^{k_j}, \end{aligned}$$

where S_j are given in (2.8). The proof of Theorem 2.1 is complete. □

Using mainly the Bell polynomials given below, Theorem 2.2 provides a recursive formula for determining the coefficients a_j $(j \in \mathbb{N})$ in (2.1). The representation using recursive algorithm is better for numerical evaluations.

The Bell polynomials, named in honor of Eric Temple Bell, are a triangular array of polynomials given by (see [13, pp. 133–134] and [15])

$$\begin{aligned} &B_{n,k}(x_1,x_2,\ldots,x_{n-k+1}) \\ &= \sum \frac{n!}{j_1!j_2!\cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}, \end{aligned} \tag{2.11}$$

where the sum is taken over all sequences $j_1, j_2, j_3, \ldots, j_{n-k+1}$ of non-negative integers such that

$$j_1 + j_2 + \cdots + j_{n-k+1} = k \quad \text{and} \quad j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n.$$

The sum

$$B_n(x_1,x_2,\ldots,x_n) = \sum_{k=1}^{n} B_{n,k}(x_1,x_2,\ldots,x_{n-k+1}) \tag{2.12}$$

is sometimes called the nth complete Bell polynomial.

In order to contrast them with complete Bell polynomials, the polynomials $B_{n,k}$ defined above are sometimes called partial Bell polynomials. The complete Bell polynomials appear in the exponential of a formal power series:

$$\exp\left(\sum_{n=1}^{\infty}\frac{x_n}{n!}u^n\right)=\sum_{n=0}^{\infty}\frac{B_n(x_1,\ldots,x_n)}{n!}u^n. \tag{2.13}$$

The Bell polynomials are quite general polynomials and they have been found in many applications in combinatorics. Comtet [13] devoted much to a thorough presentation of the Bell polynomials in the chapter on identities and expansions. For more results, the reader is referred to [10, Chapter 11] and [23, Chapter 5].

THEOREM 2.2. *Let r be a given nonzero real number. Then the gamma function has the following asymptotic expansion:*

$$\Gamma(x+1)\sim\sqrt{2\pi\left(x+\frac{1}{6}\right)}\left(\frac{x}{e}\right)^x\left(\sum_{k=0}^{\infty}\frac{a_k}{x^k}\right)^{1/r}, \quad x\to\infty, \tag{2.14}$$

where the coefficients $a_k\equiv a_k(r)$ $(k\in\mathbb{N}_0)$ are given by the recursion formula

$$a_0=1,\quad a_k=\frac{r}{k}\sum_{\ell=0}^{k-1}\left(\frac{B_{k-\ell+1}}{k-\ell+1}+\frac{(-1)^{k-\ell}}{2\cdot 6^{k-\ell}}\right)a_\ell,\quad k\in\mathbb{N}. \tag{2.15}$$

Proof. We can let

$$\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x\left(1+\frac{1}{6x}\right)}\left(\frac{x}{e}\right)^x}\right)^r=\sum_{k=0}^{\infty}\frac{a_k}{x^k},\qquad x\to\infty, \tag{2.16}$$

where $a_k\equiv a_k(r)$ (for $k\in\mathbb{N}_0$) are real numbers to be determined.

On the other hand, from (2.6) we obtain the following asymptotic expansion:

$$\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x\left(1+\frac{1}{6x}\right)}\left(\frac{x}{e}\right)^x}\right)^r\sim\exp\left(\sum_{j=1}^{\infty}\frac{c_j}{x^j}\right),\qquad x\to\infty, \tag{2.17}$$

where

$$c_j\equiv c_j(r)=\frac{r}{j}\left(\frac{B_{j+1}}{j+1}-\frac{(-1)^{j-1}}{2\cdot 6^j}\right),\qquad j\in\mathbb{N}. \tag{2.18}$$

Using (2.17) and (2.13), we have

$$\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x\left(1+\frac{1}{6x}\right)}\left(\frac{x}{e}\right)^x}\right)^r\sim\exp\left(\sum_{k=1}^{\infty}\frac{k!c_k}{k!}\frac{1}{x^k}\right)$$
$$\sim\sum_{k=0}^{\infty}\frac{B_k\Big(1!c_1,2!c_2\ldots,k!c_k\Big)}{k!}\frac{1}{x^k}.$$

Therefore it is seen that the coefficients a_k in (2.16) can be expressed in terms of the Bell polynomials:

$$a_k = \frac{B_k\Big(1!c_1, 2!c_2 \ldots, k!c_k\Big)}{k!}. \tag{2.19}$$

Bulò et al. [6, Theorem 1] proved that the complete Bell polynomials can be expressed using the following recursive formula:

$$B_k(x_1, x_2, \ldots, x_k) = \begin{cases} \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} x_{k-\ell} B_\ell(x_1, x_2, \ldots, x_\ell) & \text{if} \quad k \in \mathbb{N}, \\ 1 & \text{if} \quad k = 0. \end{cases}$$

Thus, the formula (2.19) can be rewritten as

$$\begin{aligned} a_0 &= 1 \quad \text{and} \\ a_k &= \frac{1}{k!} \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} (k-\ell)! c_{k-\ell} B_\ell\Big(1!c_1, 2!c_2 \ldots, \ell! c_\ell\Big) \\ &= \frac{1}{k!} \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} (k-\ell)! c_{k-\ell} \ell! a_\ell \\ &= \frac{r}{k} \sum_{\ell=0}^{k-1} \left(\frac{B_{k-\ell+1}}{k-\ell+1} + \frac{(-1)^{k-\ell}}{2 \cdot 6^{k-\ell}} \right) a_\ell \quad \text{for} \quad k \in \mathbb{N}. \end{aligned}$$

The proof of Theorem 2.2 is complete. □

By using another proving method, we prove Theorem 2.3, which includes Theorems 2.1 and 2.2 as its special case.

THEOREM 2.3. *Let r be a given nonzero real number and $\ell \geqslant 0$ be a given integer. Then the following asymptotic expression holds:*

$$\Gamma(x+1) \sim \sqrt{2\pi\left(x + \frac{1}{6}\right)} \left(\frac{x}{e}\right)^x \left(1 + \sum_{j=1}^{\infty} \frac{d_j}{x^j}\right)^{x^\ell / r} \tag{2.20}$$

with the coefficients $d_j \equiv d_j(\ell, r)\, (j \in \mathbb{N})$ given by

$$d_j = \sum_{(1+\ell)k_1 + (2+\ell)k_2 + \cdots + (j+\ell)k_j = j} \frac{c_1^{k_1} c_2^{k_2} \cdots c_j^{k_j}}{k_1! k_2! \cdots k_j!}, \tag{2.21}$$

where c_j are given in (2.18), summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1 + (2+\ell)k_2 + \cdots + (j+\ell)k_j = j.$$

Proof. To determine $d_j\,(j\in\mathbb{N})$, we first express (2.20) as follows:

$$\left(\frac{\Gamma(x+1)}{\sqrt{2\pi\left(x+\frac{1}{6}\right)}\left(\frac{x}{e}\right)^x}\right)^{r/x^\ell}=1+\sum_{j=1}^{m}\frac{d_j}{x^j}+O(x^{-m-1}). \tag{2.22}$$

Write (2.17) as

$$\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x\left(1+\frac{1}{6x}\right)}\left(\frac{x}{e}\right)^x}\right)^{r}=\exp\left(\sum_{k=1}^{m}\frac{c_k}{x^k}+\mathscr{R}_m(x)\right),\qquad x\to\infty,$$

where $\mathscr{R}_m(x)=O(x^{-m-1})$. Further, we have

$$\begin{aligned}\left(\frac{\Gamma(x+1)}{\sqrt{2\pi\left(x+\frac{1}{6}\right)}\left(\frac{x}{e}\right)^x}\right)^{r/x^\ell}&=e^{\mathscr{R}(x)/x^\ell}\exp\left(\sum_{k=1}^{m}\frac{c_k}{x^{k+\ell}}\right)\\
&=e^{\mathscr{R}(x)/x^\ell}\prod_{k=1}^{m}\left[1+\left(\frac{c_k}{x^{k+\ell}}\right)+\frac{1}{2!}\left(\frac{c_k}{x^{k+\ell}}\right)^2+\cdots\right]\\
&=e^{\mathscr{R}(x)/x^\ell}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{1}{k_1!k_2!\cdots k_m!}\left(\frac{c_1}{x^{1+\ell}}\right)^{k_1}\left(\frac{c_2}{x^{2+\ell}}\right)^{k_2}\cdots\left(\frac{c_m}{x^{m+\ell}}\right)^{k_m}\\
&=e^{\mathscr{R}(x)/x^\ell}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{c_1^{k_1}c_2^{k_2}\cdots c_m^{k_m}}{k_1!k_2!\cdots k_m!}\frac{1}{x^{(1+\ell)k_1+(2+\ell)k_2+\cdots+(m+\ell)k_m}}.\end{aligned} \tag{2.23}$$

Equating the coefficients by the equal powers of x in (2.22) and (2.23), we see that

$$d_j=\sum_{(1+\ell)k_1+(2+\ell)k_2+\cdots+(j+\ell)k_j=j}\frac{c_1^{k_1}c_2^{k_2}\cdots c_j^{k_j}}{k_1!k_2!\cdots k_j!}.$$

This completes the proof of Theorem 2.3. □

Setting $(\ell,r)=(0,1)$ in (2.20), yields (1.7). Here, from (2.20), we give several explicit expressions: as $x\to\infty$,

$$\Gamma(x+1)\sim\sqrt{2\pi\left(x+\frac{1}{6}\right)}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{24x^2}-\frac{23}{1080x^3}+\cdots\right)^{1/6}, \tag{2.24}$$

$$\Gamma(x+1)\sim\sqrt{2\pi\left(x+\frac{1}{6}\right)}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{12x^2}-\frac{23}{540x^3}+\cdots\right)^{1/12}, \tag{2.25}$$

$$\Gamma(x+1)\sim\sqrt{2\pi\left(x+\frac{1}{6}\right)}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{6x^2}-\frac{23}{270x^3}+\cdots\right)^{1/24}, \tag{2.26}$$

$$\Gamma(x+1) \sim \sqrt{2\pi\left(x+\frac{1}{6}\right)}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{144x^3}-\frac{23}{6480x^4}+\cdots\right)^x. \tag{2.27}$$

From (2.6) we obtain the following asymptotic expansion:

$$\Gamma(x+1) \sim \sqrt{2\pi\left(x+\frac{1}{6}\right)}\left(\frac{x}{e}\right)^x \exp\left(\frac{1}{144x^2}-\frac{23}{6480x^3}+\frac{1}{10368x^4}+\cdots\right). \tag{2.28}$$

The formula (2.28) motivate us to observe the following theorem.

THEOREM 2.4. *For* $x \geqslant 1$,

$$\sqrt{2\pi\left(x+\frac{1}{6}\right)}\left(\frac{x}{e}\right)^x < \Gamma(x+1) < \sqrt{2\pi\left(x+\frac{1}{6}\right)}\left(\frac{x}{e}\right)^x \exp\left(\frac{1}{144x^2}\right). \tag{2.29}$$

Proof. The first inequality in (2.29) has been proved in [3, 21]. Here we only prove the second inequality in (2.29). It follows from a known result (see [2, Theorem 8]) that, for $x > 0$,

$$\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}\,(x/e)^x}\right) < \frac{1}{12x}. \tag{2.30}$$

It is well-known that

$$\sum_{j=1}^{2n}\frac{(-1)^{j-1}}{j}x^j < \ln(1+x) < \sum_{j=1}^{2n+1}\frac{(-1)^{j-1}}{j}x^j, \qquad |x|<1,$$

which yields

$$\sum_{j=1}^{2n}\frac{(-1)^{j-1}}{j}\frac{1}{6^j x^j} < \ln\left(1+\frac{1}{6x}\right) < \sum_{j=1}^{2n+1}\frac{(-1)^{j-1}}{j}\frac{1}{6^j x^j}, \qquad |x|>\frac{1}{6}.$$

In particular, we have for $x > \frac{1}{6}$,

$$\frac{1}{6x}-\frac{1}{72x^2} < \ln\left(1+\frac{1}{6x}\right). \tag{2.31}$$

By using (2.30) and (2.31), we have, for $x \geqslant 1$,

$$\begin{aligned}
&\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x\left(1+\frac{1}{6x}\right)}\left(\frac{x}{e}\right)^x}\right)-\frac{1}{144x^2}\\
&= \ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}\,(x/e)^x}\right)-\frac{1}{2}\ln\left(1+\frac{1}{6x}\right)-\frac{1}{144x^2}\\
&< \frac{1}{12x}-\frac{1}{2}\left(\frac{1}{6x}-\frac{1}{72x^2}\right)-\frac{1}{144x^2}=0.
\end{aligned}$$

The proof of Theorem 2.4 is complete. □

REMARK 2.1. Batir [3, Theorem 1.6] proved that for $x \geqslant 1$,

$$\sqrt{2\pi(x+a)}\left(\frac{x}{e}\right)^x < \Gamma(x+1) \leqslant \sqrt{2\pi(x+b)}\left(\frac{x}{e}\right)^x \tag{2.32}$$

with the best possible constants

$$a = \frac{1}{6} = 0.166666\ldots \quad \text{and} \quad b = \frac{e^2}{2\pi} - 1 = 0.176005.$$

For $x > 1.6425$, the upper in (2.29) is better than one in (2.32).

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (Editors), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series **55**, Ninth printing, National Bureau of Standards, Washington, D. C., 1972.

[2] H. ALZER, *On some inequalities for the gamma and psi functions*, Math. Comput. **66** (1997), 373–389.

[3] N. BATIR, *Inequalities for the gamma function*, Arch. Math. (Basel) **91** (2008), 554–563.

[4] N. BATIR, *Very accurate approximations for the factorial function*, J. Math. Inequal. **4** (2010), 335–344.

[5] J. BUKAC, T. BURIĆ AND N. ELEZOVIĆ, *Stirling's formula revisited via some classical and new inequalities*, Math. Inequal. Appl. **14** (2011), 235–245.

[6] S. R. BULÒ, E. R. HANCOCKB, F. AZIZB AND M. PELILLOA, *Efficient computation of Ihara coefficients using the Bell polynomial recursion*, Linear Algebra Appl. **436** (2012), 1436–1441.

[7] T. BURIĆ AND N. ELEZOVIĆ, *Bernoulli polynomials and asymptotic expansions of the quotient of gamma functions*, J. Comput. Appl. Math. **235** (2011), 3315–3331.

[8] T. BURIĆ AND N. ELEZOVIĆ, *New asymptotic expansions of the quotient of gamma functions*, Integral Transforms Spec. Funct. **23** (2012), 355–368.

[9] T. BURIĆ AND N. ELEZOVIĆ, *New asymptotic expansions of the gamma function and improvements of Stirling's type formulas*, J. Comput. Anal. Appl. **13** (2011), 785–795.

[10] C. A. CHARALAMBIDES, *Enumerative Combinatorics, in: CRC Press Series on DiscreteMathematics and its Applications*, Chapman & Hall/CRC, Boca Raton, FL, 2002.

[11] C.-P. CHEN, *Unified treatment of several asymptotic formulas for the gamma function*, Numer. Algor. **64** (2013), 311–319.

[12] C.-P. CHEN AND L. LIN, *Remarks on asymptotic expansions for the gamma function*, Appl. Math. Lett. **25** (2012), 2322–2326.

[13] L. COMTET, *Advanced Combinatorics*, D. Reidel Publishing Co., Dordrecht, 1974.

[14] E. T. COPSON, *Asymptotic Expansions*, Cambridge University Press, 1965.

[15] D. CVIJOVIĆ, *New identities for the partial Bell polynomials*, Appl. Math. Lett. **24** (2011), 1544–1547.

[16] R. W. GOSPER, *Decision procedure for indefinite hypergeometric summation*, Proc. Natl. Acad. Sci. USA **75** (1978) 40–42.

[17] H. W. GOULD, *The Girard-Waring power sum formulas for symmetric functions and Fibonacci sequences*, Fibonacci Quart. **37** (1999), 135–140.

[18] E. A. KARATSUBA, *On the asymptotic representation of the Euler gamma function by Ramanujan*, J. Comput. Appl. Math. **135** (2001), 225–240.

[19] P. LUSCHNY, *Approximation formulas for the factorial function n!*, `http://www.luschny.de/math/factorial/approx/SimpleCases.html`.

[20] C. MORTICI, *Sharp inequalities related to Gosper's formula*, C. R. Math. Acad. Sci. Paris **348** (2010), 137–140.

[21] C. MORTICI, *On Gospers formula for the Gamma function*, J. Math. Inequal. **5** (2011), 611–614.

[22] G. NEMES, *More accurate approximations for the gamma function*, Thai J. Math. **9** (2011), 21–28.
[23] J. RIORDAN, *Combinatorial Identities*, Reprint of the 1968 original, Robert E. Krieger Publishing Co., Huntington, NY, 1979.

(Received May 25, 2012)

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