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GENERALIZATION OF LEVINSON'S INEQUALITY

IMRAN ABBAS BALOCH, JOSIP PEČARIĆ AND MARJAN PRALJAK

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Abstract. Mercer [5] gave a generalization of Levinson's inequality that replaces the assumption of symmetry of the two sequences with a weaker assumptions of equality of variances. Witkowski [10] further loosened this assumption and extended the result to the class of 3-convex functions.

We generalize these results to a newly defined, larger class of functions. We also prove the converse in case the function is continuous. In particular, we show that if Levinson's inequality holds under Mercer's assumptions, then the function is 3-convex.

1. Introduction

A well-known inequality due to Levinson [4] is given in the following theorem.

THEOREM 1.1. *If* $f:(0,2c)\to\mathbb{R}$ *satisfies* $f'''\geqslant 0$ *and* p_i,x_i,y_i, $i=1,2,\dots,n$, *are such that* $p_i>0$, $\sum_{i=1}^{n}p_i=1$, $0\leqslant x_i\leqslant c$ *and*

$$x_1+y_1=x_2+y_2=\ldots=x_n+y_n=2c, \tag{1}$$

then the inequality

$$\sum_{i=1}^{n}p_if(x_i)-f(\overline{x})\leqslant\sum_{i=1}^{n}p_if(y_i)-f(\overline{y}) \tag{2}$$

holds, where $\overline{x}=\sum_{i=1}^{n}p_ix_i$ *and* $\overline{y}=\sum_{i=1}^{n}p_iy_i$ *denote the weighted arithmetic means.*

The assumptions on the differentiability of f can be weakened by working with the divided differences. A kth order divided difference of a function $f:I\to\mathbb{R}$, where I is an interval in $\mathbb{R}$, at distinct points $x_0,\dots,x_k\in I$ is defined recursively by

$$[x_i]f=f(x_i),\qquad \text{for } i=0,\dots,k$$

and

$$[x_0,\dots,x_k]f=\frac{[x_1,\dots,x_k]f-[x_0,\dots,x_{k-1}]f}{x_k-x_0}.$$

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A function $f : I \to \mathbb{R}$ is called k-convex if $[x_0,...,x_k]f \geqslant 0$ for all choices of $k+1$ distinct points $x_0,...,x_k \in I$. If the kth derivative $f^{(k)}$ of a k-convex function exists, then $f^{(k)} \geqslant 0$, but $f^{(k)}$ may not exist (for properties of divided differences and k-convex functions see [8]).

Popoviciu [9] showed that in Theorem 1.1 it is enough to assume that f is 3-convex. Bullen [1] gave another proof of Popoviciu's result, as well as a converse of Levinson's inequality (rescaled to a general interval $[a,b]$). Bullen's result is the following:

THEOREM 1.2. *(a) If $f : [a,b] \to \mathbb{R}$ is 3-convex and p_i, x_i, y_i, $i = 1,2,\ldots,n$, are such that $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $a \leqslant x_i, y_i \leqslant b$, (1) holds (for some $c \in [a,b]$) and*

$$\max(x_1,...,x_n) \leqslant \min(y_1,...,y_n), \tag{3}$$

then (2) holds.
(b) If for a continuous function f inequality (2) holds for all n, all $c \in [a,b]$, all $2n$ distinct points satisfying (1) and (3) and all weights $p_i > 0$ such that $\sum_{i=1}^{n} p_i = 1$, then f is 3-convex.

Pečarić [6] proved that one can weaken the assumption (3) and still guarantee that inequality (2) holds, i. e. the following result holds

THEOREM 1.3. *If $f : [a,b] \to \mathbb{R}$ is 3-convex and p_i, x_i, y_i, $i = 1,2,\ldots,n$, are such that $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $a \leqslant x_i, y_i \leqslant b$, (1) holds (for some $c \in [a,b]$) and*

$$\left.\begin{aligned} x_i + x_{n-i+1} &\leqslant 2c, \\ \frac{p_i x_i + p_{n-i+1} x_{n-i+1}}{p_i + p_{n-i+1}} &\leqslant c, \end{aligned}\right\} \quad \text{for } i = 1,2,\ldots,n, \tag{4}$$

then (2) holds.

The inequality from Theorem 1.3 for uniform weights $p_i = \frac{1}{n}$ was proven by Lawrence and Segalman [3]. A shorter proof of Lawrence and Segalman's result for a wider class of functions was obtained by Pečarić [7]. More recently, Hussain, Pečarić and Perić [2] gave a refinement of the inequality from Theorem 1.3.

All of the generalizations of Levinson's inequality mentioned so far assume that (1) holds, i. e. that the distribution of the points x_i is equal to the distribution of the points y_i reflected around the point $c \in [a,b]$. Recently, Mercer [5] made a significant improvement by replacing this condition of symmetric distribution with the weaker one that the variances of the two sequences are equal.

THEOREM 1.4. *If $f : [a,b] \to \mathbb{R}$ satisfies $f''' \geqslant 0$ and p_i, x_i, y_i, $i = 1,2,\ldots,n$, are such that $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $a \leqslant x_i, y_i \leqslant b$, (3) holds and*

$$\sum_{i=1}^{n} p_i (x_i - \bar{x})^2 = \sum_{i=1}^{n} p_i (y_i - \bar{y})^2, \tag{5}$$

then (2) holds.

Witkowski [10] showed that, similarly as before, the assumptions on differentiability of f can be weakened and for Theorem 1.4 to hold it is enough to assume that f is 3-convex. Furthermore, Witkowski weakened the assumption (5) as well and showed that equality of variances can be replaced by inequality in certain direction.

THEOREM 1.5. *If $f:(a,b)\to\mathbb{R}$ is 3-convex, $p_i>0$ for $i=1,2,\dots,n$, $\sum_{i=1}^n p_i = 1$, $a\leqslant x_i, y_i\leqslant b$ are such that (3) holds and*

$$(a)\quad f''_-(\max x_i)>0 \text{ and } \sum_{i=1}^n p_i(x_i-\bar{x})^2 \leqslant \sum_{i=1}^n p_i(y_i-\bar{y})^2,$$

or

$$(b)\quad f''_+(\min y_i)<0 \text{ and } \sum_{i=1}^n p_i(x_i-\bar{x})^2 \geqslant \sum_{i=1}^n p_i(y_i-\bar{y})^2,$$

or

$$(c)\quad f''_-(\max x_i)\leqslant 0\leqslant f''_+(\min y_i),$$

then (2) holds.

Witkowski [10] also gave the result for 3-concave functions.

THEOREM 1.6. *If $f:(a,b)\to\mathbb{R}$ is 3-concave, $p_i>0$ for $i=1,2,\dots,n$, $\sum_{i=1}^n p_i = 1$, $a\leqslant x_i, y_i\leqslant b$ are such that*

$$(a)\quad f''_-(\max x_i)<0 \text{ and } \sum_{i=1}^n p_i(x_i-\bar{x})^2 \leqslant \sum_{i=1}^n p_i(y_i-\bar{y})^2,$$

or

$$(b)\quad f''_+(\min y_i)>0 \text{ and } \sum_{i=1}^n p_i(x_i-\bar{x})^2 \geqslant \sum_{i=1}^n p_i(y_i-\bar{y})^2,$$

or

$$(c)\quad f''_-(\min y_i)\leqslant 0\leqslant f''_+(\max x_i),$$

then (2) holds with the reverse inequality.

In this paper we will build on and extend the methods of Witkowski [10]. We will introduce a new class of functions $\mathcal{K}_1^c(a,b)$ that extends 3-convex functions and can be interpreted as functions that are "3-convex at point c". We will prove some of the properties of this new class, in particular that a function is 3-convex on an interval if and only if it is 3-convex at every point of the interval. The main result of this paper is that $\mathcal{K}_1^c(a,b)$ is the largest class of functions for which Levinson's inequality holds under Mercer's assumptions, i. e. that $f\in\mathcal{K}_1^c(a,b)$ if and only if inequality (2) holds for arbitrary weights $p_i>0$, $\sum_{i=1}^n p_i=1$, and sequences x_i and y_i that satisfy $x_i\leqslant c\leqslant y_i$ for $i=1,2,\ldots,n$. Analogues results for the reverse of inequality (2) and the class $\mathcal{K}_2^c(a,b)$ of functions that are "3-concave at point c" hold.

2. Main results

We will generalize Theorem 1.4 by weakening the assumptions on the function f. Before stating our main results, we will introduce a new class of functions and show some of its properties.

DEFINITION 2.1. Let $f:(a,b)\to\mathbb{R}$ be a function and $c\in(a,b)$. We say that $f\in\mathscr{K}_1^c(a,b)$ $(f\in\mathscr{K}_2^c(a,b))$ if there exists a constant A such that the function $F(x)=f(x)-\frac{A}{2}x^2$ is concave (convex) on $(a,c]$ and convex (concave) on $[c,b)$.

REMARK 2.2. If $f\in\mathscr{K}_i^c(a,b)$, $i=1,2$, and $f''(c)$ exists, then $f''(c)=A$. We will show this for $f\in\mathscr{K}_1^c(a,b)$: due to the concavity and convexity of F for every distinct points $x_j\in(a,c]$ and $y_j\in[c,b)$, $j=1,2,3$, we have

$$[x_1,x_2,x_3]F=[x_1,x_2,x_3]f-A/2\leqslant 0\leqslant[y_1,y_2,y_3]f-A/2=[y_1,y_2,y_3]F.$$

Therefore, if $f''_-(c)$ and $f''_+(c)$ exist, letting $x_j\nearrow c$ and $y_j\searrow c$, we get

$$f''_-(c)\leqslant A\leqslant f''_+(c).$$

□

REMARK 2.3. If $f:(a,b)\to\mathbb{R}$ is 3-convex (3-concave), then $f\in\mathscr{K}_1^c(a,b)$ $(f\in\mathscr{K}_2^c(a,b))$ for every $c\in(a,b)$. Indeed, if f is 3-convex, then f', f''_- and f''_+ exist and f' is convex (see [8]). Hence, for every $\alpha_1,\alpha_2\in(a,c]$ and $\beta_1,\beta_2\in[c,b)$ it holds

$$\frac{f'(\alpha_2)-f'(\alpha_1)}{\alpha_2-\alpha_1}\leqslant f''_-(c)\leqslant f''_+(c)\leqslant\frac{f'(\beta_2)-f'(\beta_1)}{\beta_2-\beta_1}.$$

Therefore, for every $A\in[f''_-(c),f''_+(c)]$ the function $F(x)=f(x)-\frac{A}{2}x^2$ satisfies

$$\frac{F'(\alpha_2)-F'(\alpha_1)}{\alpha_2-\alpha_1}\leqslant 0\leqslant\frac{F'(\beta_2)-F'(\beta_1)}{\beta_2-\beta_1},$$

so F' is nonincreasing on $(a,c]$ and nondecreasing on $[c,b)$. The next theorem shows that this property characterizes 3-convex (3-concave) functions.

On the other hand, $f(x)=x^4$ is an example of a function that belongs to $\mathscr{K}_1^2(-1,3)$, but is not 3-convex on $(-1,3)$. Furthermore, $f(x)=|x|$ is an example of a function that belongs to $\mathscr{K}_1^0(-1,1)$, but f is not differentiable at zero, a point in the interval $(-1,1)$. □

THEOREM 2.4. *If $f\in\mathscr{K}_1^c(a,b)$ $(f\in\mathscr{K}_2^c(a,b))$ for every $c\in(a,b)$, then f is 3-convex (3-concave).*

Proof. We will give the proof for $f\in\mathscr{K}_1^c(a,b)$. It is enough to prove that f' exists and is convex. For this purpose we will use the following characterization of convexity (see [8]): g is convex if and only if the function

$$(x,y)\mapsto[x,y]g=\frac{g(x)-g(y)}{x-y}$$

is nondecreasing in both variables.

For every $c \in (a,b)$ there exists constant A_c such that the function $F_c(x) = f(x) - \frac{A_c}{2}x^2$ is concave on $(a,c]$ and convex on $[c,b)$. Therefore F'_{c-} and F'_{c+} exist and $F'_{c-}(x) \geqslant F'_{c+}(x)$ for $x \in (a,c)$ and $F'_{c-}(x) \leqslant F'_{c+}(x)$ for $x \in (c,b)$. Since the function $x \mapsto \frac{A_c}{2}x^2$ is differentiable, f'_- and f'_+ also exist. Let $x \in (a,b)$ be arbitrary and $c_1 < x < c_2$. We have $f'_-(x) \leqslant f'_+(x)$ due to convexity of F_{c_1} and $f'_-(x) \geqslant f'_+(x)$ due to concavity of F_{c_2}, so f' exists. Furthermore, due to concavity and convexity of F_c we also have, for every $x_1 \neq x_2 \leqslant c \leqslant y_1 \neq y_2$,

$$\frac{F'_c(x_2) - F'_c(x_1)}{x_2 - x_1} = \frac{f'_c(x_2) - f'_c(x_1)}{x_2 - x_1} - A_c \leqslant 0 \leqslant \frac{f'_c(y_2) - f'_c(y_1)}{y_2 - y_1} - A_c = \frac{F'_c(y_2) - F'_c(y_1)}{y_2 - y_1}.$$

In particular, for $z_1 < z_2 < z_3$

$$\frac{f'(z_2) - f'(z_1)}{z_2 - z_1} \leqslant A_{z_2} \leqslant \frac{f'(z_3) - f'(z_2)}{z_3 - z_2}. \tag{6}$$

Now, let $x_1, x_2, y \in (a,b)$ be arbitrary. If $y < x_1 < x_2$, applying (6) we get

$$\frac{f'(x_1) - f'(y)}{x_1 - y} \leqslant A_{x_1} \leqslant \frac{f'(x_2) - f'(x_1)}{x_2 - x_1} = \frac{f'(x_2) - f'(y)}{x_2 - x_1} - \frac{f'(x_1) - f'(y)}{x_2 - x_1}.$$

By multiplying the above inequality with $\frac{x_2 - x_1}{x_2 - y} > 0$ and rearranging we get

$$\frac{f'(x_1) - f'(y)}{x_1 - y} \leqslant \frac{f'(x_2) - f'(y)}{x_2 - y}.$$

We can treat the cases $x_1 < y < x_2$ and $x_1 < x_2 < y$ similarly and conclude that the function $(x,y) \mapsto [x,y]f'$ is nondecreasing in x. By symmetry, the same thing holds for y and the proof is finished. $\square$

REMARK 2.5. Taking into account Remark 2.3 and Theorem 2.4, we can describe the property from the definition of $\mathscr{K}_1^c(a,b)$ as "3-convexity at point c". Therefore, we have shown that a function f is 3-convex on (a,b) if and only if it is 3-convex at every $c \in (a,b)$.

The following theorem is our main result and it generalizes Theorem 1.4.

THEOREM 2.6. *Let* $a < x_i \leqslant c \leqslant y_i < b$, $p_i > 0$ *for* $i = 1,2,\ldots,n$, $\sum_{i=1}^n p_i = 1$ *and* (5) *holds. If* $f \in \mathscr{K}_1^c(a,b)$, *then inequality* (2) *holds and if* $f \in \mathscr{K}_2^c(a,b)$, *then* (2) *holds with reverse sign of inequality.*

Proof. For $0 \leqslant t \leqslant 1$, let $x_i(t) = \overline{x} + t(x_i - \overline{x})$ and $y_i(t) = \overline{y} + t(y_i - \overline{y})$. We define the function

$$U(t) = \sum_{i=1}^n p_i f(y_i(t)) - f(\overline{y}) - \sum_{i=1}^n p_i f(x_i(t)) + f(\overline{x}).$$

We will first show that for $f \in \mathscr{K}_1^c(a,b)$ the function U is convex. Let $t_1,t_2,t_3 \in [0,1]$, $t_i \neq t_j$ for $i \neq j$, and $x_i \neq \overline{x}$. Since $F(x) = f(x) - \frac{A}{2}x^2$ is concave on $(a,c]$

$$\begin{aligned}0 \geqslant [x_i(t_1),x_i(t_2),x_i(t_3)]F &= \frac{F(x_i(t_1))}{(x_i(t_1)-x_i(t_2))(x_i(t_1)-x_i(t_3))} \\ &+ \frac{F(x_i(t_2))}{(x_i(t_2)-x_i(t_3))(x_i(t_2)-x_i(t_1))} + \frac{F(x_i(t_3))}{(x_i(t_3)-x_i(t_1))(x_i(t_3)-x_i(t_2))} \\ &= \frac{f(x_i(t_1)) - \frac{A}{2}(x_i(t_1))^2}{(t_1-t_2)(t_1-t_3)(x_i-\overline{x})^2} + \frac{f(x_i(t_2)) - \frac{A}{2}(x_i(t_2))^2}{(t_2-t_3)(t_2-t_1)(x_i-\overline{x})^2} \\ &\quad + \frac{f(x_i(t_3)) - \frac{A}{2}(x_i(t_3))^2}{(t_3-t_1)(t_3-t_2)(x_i-\overline{x})^2}\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{f(x_i(t_1))}{(t_1-t_2)(t_1-t_3)} + \frac{f(x_i(t_2))}{(t_2-t_3)(t_2-t_1)} + \frac{f(x_i(t_3))}{(t_3-t_1)(t_3-t_2)} \\ &\qquad - \frac{A}{2}\left[\frac{(x_i(t_1))^2}{(t_1-t_2)(t_1-t_3)} + \frac{(x_i(t_2))^2}{(t_2-t_3)(t_2-t_1)} + \frac{(x_i(t_3))^2}{(t_3-t_1)(t_3-t_2)}\right] \leqslant 0 \quad (7)\end{aligned}$$

holds for $x_i \neq \overline{x}$. If $x_i = \overline{x}$ then (7) also holds with left-hand side equal to zero.

Similarly, since F is convex on $[c,b)$ the inequality

$$\begin{aligned}&\frac{f(y_i(t_1))}{(t_1-t_2)(t_1-t_3)} + \frac{f(y_i(t_2))}{(t_2-t_3)(t_2-t_1)} + \frac{f(y_i(t_3))}{(t_3-t_1)(t_3-t_2)} \\ &\qquad - \frac{A}{2}\left[\frac{(y_i(t_1))^2}{(t_1-t_2)(t_1-t_3)} + \frac{(y_i(t_2))^2}{(t_2-t_3)(t_2-t_1)} + \frac{(y_i(t_3))^2}{(t_3-t_1)(t_3-t_2)}\right] \geqslant 0 \quad (8)\end{aligned}$$

holds for every y_i and distinct points $t_1,t_2,t_3 \in [0,1]$.

Now, consider

$$\begin{aligned}[t_1,t_2,t_3]U &= \frac{U(t_1)}{(t_1-t_2)(t_1-t_3)} + \frac{U(t_2)}{(t_2-t_3)(t_2-t_1)} + \frac{U(t_3)}{(t_3-t_1)(t_3-t_2)} \\ &= \frac{1}{(t_1-t_2)(t_1-t_3)}\left(\sum_{i=1}^n p_i f(y_i(t_1)) - f(\overline{y}) - \sum_{i=1}^n p_i f(x_i(t_1)) + f(\overline{x})\right) \\ &+ \frac{1}{(t_2-t_3)(t_2-t_1)}\left(\sum_{i=1}^n p_i f(y_i(t_2)) - f(\overline{y}) - \sum_{i=1}^n p_i f(x_i(t_2)) + f(\overline{x})\right) \\ &+ \frac{1}{(t_3-t_1)(t_3-t_2)}\left(\sum_{i=1}^n p_i f(y_i(t_3)) - f(\overline{y}) - \sum_{i=1}^n p_i f(x_i(t_3)) + f(\overline{x})\right) \\ &= \sum_{i=1}^n p_i\left[\frac{f(y_i(t_1))}{(t_1-t_2)(t_1-t_3)} + \frac{f(y_i(t_2))}{(t_2-t_3)(t_2-t_1)} + \frac{f(y_i(t_3))}{(t_3-t_1)(t_3-t_2)}\right]\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{n} p_i\left[\frac{f(x_i(t_1))}{(t_1-t_2)(t_1-t_3)}+\frac{f(x_i(t_2))}{(t_2-t_3)(t_2-t_1)}+\frac{f(x_i(t_3))}{(t_3-t_1)(t_3-t_2)}\right]\\
&-(f(\overline{y})-f(\overline{x}))\left[\frac{1}{(t_1-t_2)(t_1-t_3)}+\frac{1}{(t_2-t_3)(t_2-t_1)}+\frac{1}{(t_3-t_1)(t_3-t_2)}\right]\\
&=\sum_{i=1}^{n} p_i\left[\frac{f(y_i(t_1))}{(t_1-t_2)(t_1-t_3)}+\frac{f(y_i(t_2))}{(t_2-t_3)(t_2-t_1)}+\frac{f(y_i(t_3))}{(t_3-t_1)(t_3-t_2)}\right.\\
&\left.-\frac{A}{2}\left(\frac{(y_i(t_1))^2}{(t_1-t_2)(t_1-t_3)}+\frac{(y_i(t_2))^2}{(t_2-t_3)(t_2-t_1)}+\frac{(y_i(t_3))^2}{(t_3-t_1)(t_3-t_2)}\right)\right]\\
&+\frac{A}{2}\sum_{i=1}^{n} p_i\left(\frac{(y_i(t_1))^2}{(t_1-t_2)(t_1-t_3)}+\frac{(y_i(t_2))^2}{(t_2-t_3)(t_2-t_1)}+\frac{(y_i(t_3))^2}{(t_3-t_1)(t_3-t_2)}\right)\\
&+\sum_{i=1}^{n} p_i\left[\frac{A}{2}\left(\frac{(x_i(t_1))^2}{(t_1-t_2)(t_1-t_3)}+\frac{(x_i(t_2))^2}{(t_2-t_3)(t_2-t_1)}+\frac{(x_i(t_3))^2}{(t_3-t_1)(t_3-t_2)}\right)\right.\\
&\left.-\frac{f(x_i(t_1))}{(t_1-t_2)(t_1-t_3)}+\frac{f(x_i(t_2))}{(t_2-t_3)(t_2-t_1)}+\frac{f(x_i(t_3))}{(t_3-t_1)(t_3-t_2)}\right]\\
&-\frac{A}{2}\sum_{i=1}^{n} p_i\left(\frac{(x_i(t_1))^2}{(t_1-t_2)(t_1-t_3)}+\frac{(x_i(t_2))^2}{(t_2-t_3)(t_2-t_1)}+\frac{(x_i(t_3))^2}{(t_3-t_1)(t_3-t_2)}\right)\\
&\geqslant \frac{A}{2}\sum_{i=1}^{n} p_i\left(\frac{(y_i(t_1))^2-(x_i(t_1))^2}{(t_1-t_2)(t_1-t_3)}+\frac{(y_i(t_2))^2-(x_i(t_2))^2}{(t_2-t_3)(t_2-t_1)}+\frac{(y_i(t_3))^2-(x_i(t_3))^2}{(t_3-t_1)(t_3-t_2)}\right) \quad (9)
\end{aligned}
$$

where the last inequality follows from (7) and (8). Notice that

$$
\begin{aligned}
\sum_{i=1}^{n} p_i x_i(t_j)^2 &= \sum_{i=1}^{n} p_i\left(\overline{x}^2+2t_j\overline{x}(x_i-\overline{x})+t_j^2(x_i-\overline{x})^2\right)\\
&=\overline{x}^2+t_j^2\sum_{i=1}^{n} p_i(x_i-\overline{x})^2 \quad (10)
\end{aligned}
$$

and, similarly,

$$
\sum_{i=1}^{n} p_i y_i(t_j)^2=\overline{y}^2+t_j^2\sum_{i=1}^{n} p_i(y_i-\overline{y})^2 \quad (11)
$$

Subtracting (10) from (11) and taking into account assumption (5) we have

$$
\sum_{i=1}^{n} p_i\left(y_i(t_j)^2-x_i(t_j)^2\right)=\overline{y}^2-\overline{x}^2,
$$

so the last line in (9) is equal to

$$
\frac{A}{2}\left(\overline{y}^2-\overline{x}^2\right)\left[\frac{1}{(t_1-t_2)(t_1-t_3)}+\frac{1}{(t_2-t_3)(t_2-t_1)}+\frac{1}{(t_3-t_1)(t_3-t_2)}\right]=0.
$$

Therefore $[t_1,t_2,t_3]U \geqslant 0$ for every choice of t_j, $j=1,2,3$, so U is convex.

Next we will show that the right hand derivative of U at zero is nonnegative. Firstly, since $F(x) = f(x) - \frac{A}{2}x^2$ is concave on $(a,c]$ and convex on $[c,b)$, both F'_- and F'_+ exist and are nonincreasing on (a,c) with $F'_- \geqslant F'_+$ and nondecreasing on (c,b) with $F'_- \leqslant F'_+$. Since $x \mapsto \frac{A}{2}x^2$ is differentiable, f'_- and f'_+ also exist and

$$F'_-(x) = f'_-(x) - Ax \quad \text{and} \quad F'_+(x) = f'_+(x) - Ax. \tag{12}$$

Notice that, as $t \searrow 0$, the expression $y_i(t) = \bar{y} + t(y_i - \bar{y})$ increases (decreases) to $\bar{y}$ for $y_i < \bar{y}$ ($y_i > \bar{y}$) and $y_i(t) \equiv \bar{y}$ when $y_i = \bar{y}$. The analogous claim holds for $x_i(t)$ and $\bar{x}$. Since $U(0) = 0$ we have

$$\begin{aligned}
U'_+(0) &= \lim_{t\searrow 0} \frac{U(t)}{t} = \lim_{t\searrow 0} \left[\sum_{i=1}^n p_i \frac{f(y_i(t)) - f(\bar{y})}{t} - \sum_{i=1}^n p_i \frac{f(x_i(t)) - f(\bar{x})}{t} \right] \\
&= \lim_{t\searrow 0} \left[\sum_{i=1}^n p_i \frac{f(\bar{y} + t(y_i - \bar{y})) - f(\bar{y})}{t(y_i - \bar{y})} (y_i - \bar{y}) \right. \\
&\qquad\qquad \left. - \sum_{i=1}^n p_i \frac{f(\bar{x} + t(x_i - \bar{x})) - f(\bar{x})}{t(x_i - \bar{x})} (x_i - \bar{x}) \right] \\
&= f'_-(\bar{y}) \sum_{y_i < \bar{y}} p_i (y_i - \bar{y}) + f'_+(\bar{y}) \sum_{y_i > \bar{y}} p_i (y_i - \bar{y}) \\
&\qquad\qquad - f'_-(\bar{x}) \sum_{x_i < \bar{x}} p_i (x_i - \bar{x}) - f'_+(\bar{x}) \sum_{x_i > \bar{x}} p_i (x_i - \bar{x}) \\
&= F'_-(\bar{y}) \sum_{y_i < \bar{y}} p_i (y_i - \bar{y}) + F'_+(\bar{y}) \sum_{y_i > \bar{y}} p_i (y_i - \bar{y}) + A\bar{y} \sum_{i=1}^n p_i (y_i - \bar{y}) \\
&\quad - F'_-(\bar{x}) \sum_{x_i < \bar{x}} p_i (x_i - \bar{x}) - F'_+(\bar{x}) \sum_{x_i > \bar{x}} p_i (x_i - \bar{x}) - A\bar{x} \sum_{i=1}^n p_i (x_i - \bar{x}) \\
&= F'_-(\bar{y}) \sum_{i=1}^n p_i (y_i - \bar{y}) + \big(F'_+(\bar{y}) - F'_-(\bar{y})\big) \sum_{y_i > \bar{y}} p_i (y_i - \bar{y}) \\
&\quad - F'_-(\bar{x}) \sum_{i=1}^n p_i (x_i - \bar{x}) - \big(F'_+(\bar{x}) - F'_-(\bar{x})\big) \sum_{x_i > \bar{x}} p_i (x_i - \bar{x}) \\
&= \big(F'_+(\bar{y}) - F'_-(\bar{y})\big) \sum_{y_i > \bar{y}} p_i (y_i - \bar{y}) - \big(F'_+(\bar{x}) - F'_-(\bar{x})\big) \sum_{x_i > \bar{x}} p_i (x_i - \bar{x}) \geqslant 0,
\end{aligned}$$

where the last inequality follows since $\bar{y} \in (c,b)$ and $\bar{x} \in (a,c)$ (if $\bar{y} = c$ or $\bar{x} = c$, then the sequences $y_i = \bar{y}$ and $x_i = \bar{x}$ are constant and inequality (2) is trivial).

Therefore, for $f \in \mathcal{K}_1^c(a,b)$ the function U is convex and $U'_+(0) \geqslant 0$, so $U(0) \leqslant U(1)$, which is inequality (2). The proof for $f \in \mathcal{K}_2^c(a,b)$ is analogous with U being concave and $U_+(0) \leqslant 0$. $\square$

The following theorem shows that for continuous functions inequality (2) characterizes the class $\mathcal{K}_i^c(a,b)$, $i = 1,2$.

THEOREM 2.7. *Let $f:(a,b)\to\mathbb{R}$ be continuous and $c\in(a,b)$. If inequality (2) (the reverse of (2)) holds for every $n\in\mathbb{N}$ and sequences p_i, x_i, y_i, $i=1,\dots,n$, such that $p_i>0$, $\sum_{i=1}^n p_i=1$, $a<x_i\leqslant c\leqslant y_i<b$ and (5) holds, then $f\in\mathscr{K}_1^c(a,b)$ ($f\in\mathscr{K}_2^c(a,b)$).*

Proof. We will give the proof for $f\in\mathscr{K}_1^c(a,b)$. Throughout the proof it is assumed that the x's are in $(a,c]$ and the y's in $[c,b)$.

Let $n=2$, $x_1\neq x_3$ and $0<p<1$. Then for $x_2=px_1+(1-p)x_3$ it holds $x_2\neq x_1$, $x_2\neq x_3$ and

$$pf(x_1)+(1-p)f(x_3)-f(px_1+(1-p)x_3)=p(1-p)(x_3-x_1)^2[x_1,x_2,x_3]f. \tag{13}$$

Furthermore

$$p(x_1-x_2)^2+(1-p)(x_3-x_2)^2=p(1-p)(x_3-x_1)^2,$$

so condition (5) applied to points x_1,x_3 and y_1,y_3, with $p_1=p$, $p_2=1-p$, is equivalent to

$$|x_3-x_1|=|y_3-y_1|. \tag{14}$$

If (14) holds, then (13) and (5) imply that

$$[x_1,x_2,x_3]f\leqslant[y_1,y_2,y_3]f, \tag{15}$$

where y_2 is the point such that

$$\frac{x_3-x_2}{x_3-x_1}=\frac{y_3-y_2}{y_3-y_1}\in(0,1). \tag{16}$$

Most of the proof will consist in showing that (15) holds for arbitrary x_i's and y_i's, $i=1,2,3$. For clarity of presentation we will break the proof into several steps.

Step 1: *If $x_3-x_1=k(y_3-y_1)$ for some $k\in\mathbb{Z}\setminus\{0\}$ and (16) holds, then*

$$\frac{[x_3,x_3-qd]f-[x_1+(1-q)d,x_1]f}{d}\leqslant\big(2|k|-1\big)[y_1,y_2,y_3]f,$$

where $q=(y_3-y_2)/(y_3-y_1)$ and $d=y_3-y_1$ if $k\in\mathbb{N}$ and $d=y_1-y_3$ if $-k\in\mathbb{N}$.

We will prove the claim for $k\in\mathbb{N}$ and the other case is analogous. Denote $z_{2j}=x_1+jd$ and $z_{2j+1}=x_1+(1-q)d+jd$. Notice that $|z_{j+1}-z_{j-1}|=|y_3-y_1|$ and $(z_{2j}-z_{2j-1})/(z_{2j}-z_{2j-2})=(z_{2j}-z_{2j-1})/(z_{2j+1}-z_{2j-1})=q$, so (14) and (16) are satisfied and we can apply (15) to get $[z_{j-1},z_j,z_{j+1}]f\leqslant[y_1,y_2,y_3]f$. Summing these inequalities for $j=1,\dots,2k-1$ we get

$$\sum_{j=1}^{2k-1}[z_{j-1},z_j,z_{j+1}]f=\frac{1}{d}\sum_{j=1}^{2k-1}\big([z_j,z_{j+1}]f-[z_{j-1},z_j]f\big)=$$

$$=\frac{[z_{2k},z_{2k-1}]f-[z_1,z_0]f}{d}\leqslant\big(2k-1\big)[y_1,y_2,y_3]f.$$

Since $z_{2k} = x_3$, $z_{2k-1} = x_3 - qd$, $z_1 = x_1 + (1-q)d$ and $z_0 = x_1$, the claim follows.

Step 2: If $x_3 - x_1 = k(y_3 - y_1)$ for some $k \in \mathbb{Z}\backslash\{0\}$ and (16) holds, then (15) holds.

Again we will prove the case $k \in \mathbb{N}$, with the other case proven analogously. Denote $d = y_3 - y_1$, $q = (y_3 - y_2)/(y_3 - y_1)$, $z_j = x_3 - jqd$, $\tilde{z}_j = x_1 + j(1-q)d$. Since $z_j - \tilde{z}_j = (k-j)d$, $z_j - qd = z_{j+1}$ and $\tilde{z}_j + (1-q)d = \tilde{z}_j$, applying the inequality from Step 1 for $j = 0, 1, \dots, k-1$ and summing up we get

$$\frac{1}{d}\left(\sum_{j=0}^{k-1}[z_j, z_{j+1}]f - \sum_{j=0}^{k-1}[\tilde{z}_j, \tilde{z}_{j+1}]f\right) \leqslant [y_1, y_2, y_3]f \sum_{j=0}^{k-1}\big(2(k-j)-1\big). \tag{17}$$

Denote $x_2 = qx_1 + (1-q)x_3$ and notice that $z_k = \tilde{z}_k = x_2$, $x_3 - x_2 = kqd$ and $x_2 - x_1 = k(1-q)d$. Since $\sum_{j=1}^{k}(2j-1) = k^2$,

$$\sum_{j=0}^{k-1}[z_j, z_{j+1}]f = \frac{1}{qd}\sum_{j=0}^{k-1}\big(f(z_j) - f(z_{j+1})\big) = \frac{f(x_3) - f(x_2)}{qd}$$

and, similarly, $\sum_{j=0}^{k-1}[\tilde{z}_j, \tilde{z}_{j+1}]f = \frac{f(x_2)-f(x_1)}{(1-q)d}$, dividing the inequality (17) by k^2 we get the claim.

Step 3: If $m(x_3 - x_1) = k(y_3 - y_1)$ for some $m \in \mathbb{N}$, $k \in \mathbb{Z}\backslash\{0\}$ and (16) holds, then

$$\big(2m-1\big)[x_1, x_2, x_3]f \leqslant \frac{[y_3, y_3 - pd]f - [y_1 + (1-p)d, y_1]f}{d},$$

where $p = (x_3 - x_2)/(x_3 - x_1)$ and $d = (x_3 - x_1)/k$.

Denote $z_{2j} = y_1 + jd$ and $z_{2j+1} = y_1 + (1-p)d + jd$. Notice that $x_3 - x_1 = k(z_{j+1} - z_{j-1})$ and $(z_{2j} - z_{2j-1})/(z_{2j} - z_{2j-2}) = (z_{2j} - z_{2j-1})/(z_{2j+1} - z_{2j-1}) = p$, so we can apply inequality from Step 2 to get $[x_1, x_2, x_3]f \leqslant [z_{j-1}, z_j, z_{j+1}]f$. Summing these inequalities for $j = 1, \dots, 2m-1$ we get

$$\begin{aligned}\big(2m-1\big)[x_1, x_2, x_3]f &\leqslant \sum_{j=1}^{2m-1}[z_{j-1}, z_j, z_{j+1}]f = \\ &= \frac{1}{d}\sum_{j=1}^{2m-1}\big([z_j, z_{j+1}]f - [z_{j-1}, z_j]f\big) = \frac{[z_{2m}, z_{2m-1}]f - [z_1, z_0]f}{d}.\end{aligned}$$

Since $z_{2k} = y_3$, $z_{2k-1} = y_3 - pd$, $z_1 = y_1 + (1-p)d$ and $z_0 = y_1$, the claim follows.

Step 4: If the ratio $(x_3 - x_1)/(y_3 - y_1)$ is rational and (16) holds, then (15) holds.

There exist $m \in \mathbb{N}$, $k \in \mathbb{Z}\backslash\{0\}$ such that $m(x_3 - x_1) = k(y_3 - y_1)$. Denote $d = (x_3 - x_1)/k$, $p = (x_3 - x_2)/(x_3 - x_1)$, $z_j = y_3 - jpd$, $\tilde{z}_j = y_1 + j(1-p)d$. Since $z_j -$

$\tilde{z}_j = (m-j)d$, $z_j - pd = z_{j+1}$ and $\tilde{z}_j + (1-p)d = \tilde{z}_j$, applying the inequality from Step 3 for $j = 0, 1, \ldots, m-1$ and summing up we get

$$[x_1, x_2, x_3]f \sum_{j=0}^{m-1} \big(2(m-j)-1\big) \leqslant \frac{1}{d}\left(\sum_{j=0}^{m-1}[z_j, z_{j+1}]f - \sum_{j=0}^{k-1}[\tilde{z}_j, \tilde{z}_{j+1}]f\right). \tag{18}$$

Denote $y_2 = py_1 + (1-p)y_3$ and notice that $z_k = \tilde{z}_k = y_2$, $y_3 - y_2 = mpd$ and $y_2 - y_1 = m(1-p)d$. Since $\sum_{j=1}^{m}(2j-1) = m^2$,

$$\sum_{j=0}^{m-1}[z_j, z_{j+1}]f = \frac{1}{pd}\sum_{j=0}^{m-1}\big(f(z_j) - f(z_{j+1})\big) = \frac{f(y_3) - f(y_2)}{pd}$$

and, similarly, $\sum_{j=0}^{m-1}[\tilde{z}_j, \tilde{z}_{j+1}]f = \frac{f(y_2)-f(y_1)}{(1-p)d}$, dividing the inequality (18) by m^2 we get the claim.

Step 5: *If the ratio* $(x_3 - x_1)/(y_3 - y_1)$ *is an arbitrary real number and (16) holds, then (15) holds.*

Since f is continuous, for fixed z_1 the mapping $(z_2, z_3) \mapsto [z_1, z_2, z_3]f$ is continuous. Therefore, for any $\varepsilon > 0$ there exists a small enough neighbourhood around the point $(y_2, y_3) \in \mathbb{R}^2$ such that for any point $(\tilde{y}_2, \tilde{y}_3)$ in the neighbourhood $[y_1, y_2, y_3]f - [y_1, \tilde{y}_2, \tilde{y}_3]f > -\varepsilon$. Moreover, we can choose the points $\tilde{y}_2$ and $\tilde{y}_3$ in such a way that the ratio $(x_3 - x_1)/(\tilde{y}_3 - y_1)$ is rational and $\frac{x_3 - x_2}{x_3 - x_1} = \frac{\tilde{y}_3 - \tilde{y}_2}{\tilde{y}_3 - y_1}$. Therefore, applying the inequality from Step 4 we obtain

$$0 \leqslant [y_1, \tilde{y}_2, \tilde{y}_3]f - [x_1, x_2, x_3]f < [y_1, y_2, y_3]f + \varepsilon - [x_1, x_2, x_3]f.$$

Letting $\varepsilon \searrow 0$ we get the claim.

Step 6: *(15) holds for arbitrary x's and y's.*

Let us, for $q \in (0,1)$, denote the set

$$\mathscr{D}_q = \Big\{p \in (0,1) : \text{for any } x\text{'s and } y\text{'s such that } p = \frac{x_3 - x_2}{x_3 - x_1} \text{ and } q = \frac{y_3 - y_2}{y_3 - y_1} \text{ inequality (15) holds}\Big\}.$$

Our goal is to prove that $\mathscr{D}_q = (0,1)$ for every $q \in (0,1)$. We will first show that $\mathscr{D}_q$ is dense in $(0,1)$. So far, in Step 5, we have shown that $q \in \mathscr{D}_q$. Next, let $p \in \mathscr{D}_q$ and $(x_3 - x_2)/(x_3 - x_1) = 1 - p$. Since $(x_1 - x_2)/(x_1 - x_3) = p$ and the divided differences are symmetric in x's (i. e. $[x_1, x_2, x_3]f = [x_3, x_2, x_1]f$) it follows that $1 - p \in \mathscr{D}_q$.

Next we will show that if $p_1, p_2 \in \mathscr{D}_q$, then $p = p_1 \cdot p_2 \in \mathscr{D}_q$. We will make use of the following identity

$$[x_1, x_2, x_3]f = (1-\alpha)[x_1, \tilde{x}, x_3]f + \alpha[\tilde{x}, x_2, x_3]f, \qquad \text{where } \alpha = \frac{x_2 - \tilde{x}}{x_2 - x_1}. \tag{19}$$

Notice that, when $\tilde{x}$ is between x_1 and x_2, then $0 < \alpha < 1$. Let x_1, x_2, x_3 and $\tilde{x}$ be such that $p = (x_3 - x_2)/(x_3 - x_1)$ and $p_1 = (x_3 - \tilde{x})/(x_3 - x_1)$, with $p < p_1$ (i.e. $\tilde{x}$ is between x_1 and x_2). Then $p_2 = (x_3 - x_2)/(x_3 - \tilde{x})$ and, since $p_1, p_2 \in \mathscr{D}_q$, applying (19) we get

$$[x_1, x_2, x_3]f = (1-\alpha)[x_1, \tilde{x}, x_3]f + \alpha[\tilde{x}, x_2, x_3]f \leqslant [y_1, y_2, y_3]f.$$

Therefore $p \in \mathscr{D}_q$. Since $q \in \mathscr{D}_q$, by properties proven so far, we have that for every $k, m \in \mathbf{N}$, numbers $q^k, 1 - q^k, (1-q^k)^m \in \mathscr{D}_q$. It is enough to prove that the numbers of the latter form are dense in $(0,1)$. Let $p \in (0,1)$ and $\varepsilon > 0$ be arbitrary. Then $\tilde{p} \in (p - \varepsilon, p + \varepsilon)$ if and only if $\log \tilde{p} \in (d, e)$, where $d = \log(p - \varepsilon)$, $e = \log(p + \varepsilon)$. One can choose large enough k such that $r = 1 - q^k$ satisfies $|\log r| < e - d$. For such r there exists m such that $m \log r \in (d, e)$, i. e. $(1 - q^k)^m \in (p - \varepsilon, p + \varepsilon)$.

Let x's and y's be arbitrary with $q = (y_3 - y_2)/(y_3 - y_1)$ and $p = (x_3 - x_2)/(x_3 - x_1)$ in $(0,1)$. Since $\mathscr{D}_q$ is dense, there exists $\tilde{p} \in \mathscr{D}_q$ arbitrarily close to p, i. e. $\tilde{x} = \tilde{p}x_1 + (1 - \tilde{p})x_3$ is arbitrarily close to x_2. Applying again identity (19) we get

$$\begin{aligned}[x_1, x_2, x_3]f = (1-\alpha)[x_1, \tilde{x}, x_3]f + \alpha[\tilde{x}, x_2, x_3]f \leqslant \\ \leqslant (1-\alpha)[y_1, y_2, y_3]f + \alpha[\tilde{x}, x_2, x_3]f. \quad (20)\end{aligned}$$

As $\tilde{x}$ is approaching x_2, α is approaching zero and, moreover, the second term on the right hand side of (20) is also approaching zero since

$$|\alpha[\tilde{x}, x_2, x_3]f| = |\alpha| \left| \frac{[x_2, x_3]f - [\tilde{x}, x_2]f}{x_3 - \tilde{x}} \right| \leqslant |\alpha| \left| \frac{[x_2, x_3]f}{x_3 - \tilde{x}} \right| + \frac{|f(x_2) - f(\tilde{x})|}{|(x_3 - \tilde{x})(x_2 - x_1)|}$$

and f is continuous. Therefore, from (20) we conclude that (15) holds.

Step 7: $f \in \mathscr{K}_1^c(a,b)$.

Since inequality (15) holds for arbitrary x_i's and y_i's, $i = 1,2,3$, the supremum of the expression on the left hand side is less then or equal to the infimum of the expression on the right hand side. Let A be an arbitrary real number such that

$$\sup_{x_1, x_2, x_3} [x_1, x_2, x_3]f \leqslant A \leqslant \inf_{y_1, y_2, y_3} [y_1, y_2, y_3]f$$

and let $F(x) = f(x) - \frac{A}{2}x^2$. The function F satisfies

$$[x_1, x_2, x_3]F = [x_1, x_2, x_3]f - A \leqslant 0 \leqslant [y_1, y_2, y_3]f - A = [y_1, y_2, y_3]F,$$

so $f \in \mathscr{K}_1^c(a,b)$. $\square$

REMARK 2.8. If we assume additional assumptions on the differentiability of f, then the proof of Theorem 2.7 becomes significantly shorter. For example, if we assume that f has a continuous first derivative, then we first prove, as in Theorem 2.7, that for x_1, x_2, y_1, y_2 and $p \in (0,1)$ such that $x_1 \neq x_2$ and $|x_2 - x_1| = |y_2 - y_1|$, with $\overline{x} = px_1 + (1-p)x_2$ and $\overline{y} = py_1 + (1-p)y_3$, the inequality

$$[x_1, \overline{x}, x_2]f \leqslant [y_1, \overline{y}, y_2]f \tag{21}$$

holds. Letting $p \searrow 0$ in (21) we obtain

$$\frac{f'(x_2) - \frac{f(x_2)-f(x_1)}{x_2-x_1}}{x_2 - x_1} \leqslant \frac{f'(y_2) - \frac{f(y_2)-f(y_1)}{y_2-y_1}}{y_2 - y_1} \tag{22}$$

and letting $p \nearrow 1$ in (21) we obtain

$$\frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - f'(x_1)}{x_2 - x_1} \leqslant \frac{\frac{f(y_2)-f(y_1)}{y_2-y_1} - f'(y_1)}{y_2 - y_1}. \tag{23}$$

Adding (22) and (23) we get that

$$\frac{f'(x_2) - f'(x_1)}{x_2 - x_1} \leqslant \frac{f'(y_2) - f'(y_1)}{y_2 - y_1} \tag{24}$$

if $|x_2 - x_1| = |y_2 - y_1|$. Suppose, next, that $x_2 - x_1 = k(y_2 - y_1)$ for some $k \in \mathbb{N}$. Denoting $d = y_2 - y_1$ and $z_i = x_1 + i \cdot d$ and applying inequality (24) for $z_i - z_{i-1} = y_2 - y_1$ for $i = 1, \ldots, k$ and summing up we get

$$\sum_{i=1}^{k} \frac{f'(z_i) - f'(z_{i-1})}{d} \leqslant k \frac{f'(y_2) - f'(y_1)}{y_2 - y_1}$$

and, dividing by k, we see that (24) holds in this case as well. The case $m(x_2 - x_1) = k(y_2 - y_1)$ for some $k, m \in \mathbf{N}$ is treated similarly and (24) for arbitrary x's and y's follows by continuity of f'. Now, let A be any number such that

$$\sup_{x_1 \neq x_2} \frac{f'(x_2) - f'(x_1)}{x_2 - x_1} \leqslant A \leqslant \inf_{y_1 \neq y_2} \frac{f'(y_2) - f'(y_1)}{y_2 - y_1}$$

and $F(x) = f(x) - \frac{A}{2}x^2$.Then F satisfies

$$\frac{F'(x_2) - F'(x_1)}{x_2 - x_1} \leqslant 0 \leqslant \frac{F'(y_2) - F'(y_1)}{y_2 - y_1},$$

so F' is nonincreasing on $(a,c]$ and nondecreasing on $[c,b)$ and $f \in \mathscr{K}_1^c(a,b)$.

If we assume the existence of the second derivative f'', then the proof is shortened more. For arbitrary $x \in (a,c]$ and $y \in [c,b)$ there exists small enough $\Delta > 0$ such that the points $x - 2\Delta$ and $y + 2\Delta$ are in (a,b). From (21) we conclude that

$$[x - 2\Delta, x - \Delta, x]f \leqslant [y, y + \Delta, y + 2\Delta]f$$

and letting $\Delta \searrow 0$ we get that

$$f''(x) \leqslant f''(y) \qquad \text{for } a < x \leqslant c \leqslant y < b.$$

Therefore, the function $F(x) = f(x) - \frac{1}{2} f''(c)$ satisfies

$$F''(x) \leqslant 0 \leqslant F''(y) \qquad \text{for } a < x \leqslant c \leqslant y < b,$$

so $f \in \mathscr{K}_1^c(a,b)$.

□

The following result is the converse of Theorem 1.4 and states that Levinson's inequality under Mercer's conditions holds if and only if f is 3-convex.

COROLLARY 2.9. *Let $f : (a,b) \to \mathbb{R}$ be continuous. If inequality (2) (the reverse of (2)) holds for every $n \in \mathbb{N}$ and sequences p_i, x_i, y_i, $i = 1,\ldots,n$, such that $p_i > 0$, $\sum_{i=1}^n p_i = 1$, $a < x_i, y_i < b$ and (3) and (5) hold, then f is 3-convex (3-concave).*

Proof. By Theorem 2.7, $f \in \mathscr{K}_1^c(a,b)$ ($f \in \mathscr{K}_2^c(a,b)$) for every $c \in (a,b)$. Therefore, by Theorem 2.4, f is 3-convex (3-concave). □

The next result weakens the assumption (5) and is a generalization of Theorem 1.5.

THEOREM 2.10. *Let $f : (a,b) \to \mathbb{R}$ and, for $i = 1,2,\ldots,n$, $p_i > 0$, $\sum_{i=1}^n p_i = 1$, $a \leqslant x_i, y_i \leqslant b$ are such that (3) holds and $f \in \mathscr{K}_1^c(a,b)$ for some $c \in [\max x_i, \min y_i]$. Then, if*

$$(a) \quad f''_-(\max x_i) \geqslant 0 \text{ and } \sum_{i=1}^n p_i(x_i - \bar{x})^2 \leqslant \sum_{i=1}^n p_i(y_i - \bar{y})^2,$$

or

$$(b) \quad f''_+(\min y_i) \leqslant 0 \text{ and } \sum_{i=1}^n p_i(x_i - \bar{x})^2 \geqslant \sum_{i=1}^n p_i(y_i - \bar{y})^2,$$

or

$$(c) \quad f''_-(\max x_i) < 0 < f''_+(\min y_i) \text{ and } f \text{ is 3-convex}$$

then (2) holds.

Proof. If we subtract (10) from (11) without assuming (5) and insert the obtained identity into (9) we get that (9) is equal to

$$\frac{A}{2}\Big(\sum_{i=1}^n p_i(y_i - \bar{y})^2 - \sum_{i=1}^n p_i(x_i - \bar{x})^2\Big)\left[\frac{t_1^2}{(t_1 - t_2)(t_1 - t_3)} + \frac{t_2^2}{(t_2 - t_3)(t_2 - t_1)} + \right.$$
$$\left. + \frac{t_3^2}{(t_3 - t_1)(t_3 - t_2)}\right] = \frac{A}{2}\Big(\sum_{i=1}^n p_i(y_i - \bar{y})^2 - \sum_{i=1}^n p_i(x_i - \bar{x})^2\Big). \quad (25)$$

Similarly as in Remark 2.2, we can show that for distinct points $\tilde{x}_j \in (a, \max x_i]$ and $\tilde{y}_j \in [\min y_i, b)$, $j = 1,2,3$, we have

$$[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3]f \leqslant A \leqslant [\tilde{y}_1, \tilde{y}_2, \tilde{y}_3]f.$$

Letting $\tilde{x}_j \nearrow \max x_i$ and $\tilde{y}_j \searrow \min y_i$ we get

$$f''_-(\max x_i) \leqslant A \leqslant f''_+(\min y_i).$$

Therefore, if the assumption (a) or (b) holds, from (25) we can still deduce convexity of the function U as in the proof of Theorem 2.6. If the assumption (c) holds, then f''_- is left-continuous, f''_+ is right-continuous, they are both nondecreasing and $f''_- \leqslant f''_+$. Therefore there exists $\tilde{c} \in [\max x_i, \min y_i]$ such that $f \in \mathscr{K}_1^{\tilde{c}}(a,b)$ with the associated constant $\tilde{A} = 0$ and we can again deduce convexity of U.

The proof that $U'_+(0) \geqslant 0$ is the same as in Theorem 2.6 and we conclude that (2) holds. $\square$

The generalization of Theorem 1.6 is proven in the same way and we only give its statement.

THEOREM 2.11. *Let $f : (a,b) \to \mathbb{R}$ and, for $i = 1,2,\ldots,n$, $p_i > 0$, $\sum_{i=1}^n p_i = 1$, $a \leqslant x_i, y_i \leqslant b$ are such that (3) holds and $f \in \mathscr{K}_2^c(a,b)$ for some $c \in [\max x_i, \min y_i]$. Then, if*

$$(a) \quad f''_-(\max x_i) \leqslant 0 \text{ and } \sum_{i=1}^n p_i(x_i - \overline{x})^2 \leqslant \sum_{i=1}^n p_i(y_i - \overline{y})^2,$$

or

$$(b) \quad f''_+(\min y_i) \geqslant 0 \text{ and } \sum_{i=1}^n p_i(x_i - \overline{x})^2 \geqslant \sum_{i=1}^n p_i(y_i - \overline{y})^2,$$

or

$$(c) \quad f''_-(\max x_i) < 0 < f''_+(\min y_i) \text{ and } f \text{ is 3-concave}$$

then the reverse of (2) holds.

Following the idea of Witkowski [10] we can apply the Hermite-Hadamard inequality to the convex function U and obtain the following refinement of the Levinson inequality.

COROLLARY 2.12. *Let $a < x_i \leqslant c \leqslant y_i < b$, $p_i > 0$ for $i = 1,2,\ldots,n$, $\sum_{i=1}^n p_i = 1$ and (5) holds. If $f \in \mathscr{K}_1^c(a,b)$, then the following inequalities hold*

$$\begin{aligned} 0 &\leqslant \sum_{i=1}^n p_i f\Big(\frac{y_i + \overline{y}}{2}\Big) - f(\overline{y}) - \sum_{i=1}^n p_i f\Big(\frac{x_i + \overline{x}}{2}\Big) + f(\overline{x}) \\ &\leqslant \sum_{i=1}^n p_i \frac{\int_{\overline{y}}^{y_i} f(t)dt}{y_i - \overline{y}} - f(\overline{y}) - \sum_{i=1}^n p_i \frac{\int_{\overline{x}}^{x_i} f(t)dt}{x_i - \overline{x}} + f(\overline{x}) \\ &\leqslant \frac{1}{2}\Big[\sum_{i=1}^n p_i f(y_i) - f(\overline{y}) - \sum_{i=1}^n p_i f(x_i) + f(\overline{x})\Big] \end{aligned}$$

If $f \in \mathscr{K}_2^c(a,b)$, then the reversed inequalities hold.

Note that the rightmost inequality can be rewritten in a nice symmetric form

$$\sum_{i=1}^{n} p_i \left(\frac{f(x_i)+f(\overline{x})}{2} - \frac{\int_{\overline{x}}^{x_i} f(t)dt}{x_i - \overline{x}} \right) \leqslant \sum_{i=1}^{n} p_i \left(\frac{f(y_i)+f(\overline{y})}{2} - \frac{\int_{\overline{y}}^{y_i} f(t)dt}{y_i - \overline{y}} \right),$$

while the leftmost inequality is

$$\sum_{i=1}^{n} p_i \left(\frac{\int_{\overline{x}}^{x_i} f(t)dt}{x_i - \overline{x}} - f\left(\frac{x_i + \overline{x}}{2} \right) \right) \leqslant \sum_{i=1}^{n} p_i \left(\frac{\int_{\overline{y}}^{y_i} f(t)dt}{y_i - \overline{y}} - f\left(\frac{y_i + \overline{y}}{2} \right) \right).$$

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Imran Abbas Baloch
Abdus Salam School of Mathematical Sciences, GC University
Lahore, Pakistan
e-mail: iabbasbaloch@gmail.com

Josip Pečarić
University Of Zagreb, Faculty Of Textile Technology
Zagreb, Croatia
and
Abdus Salam School of Mathematical Sciences, GC University
Lahore, Pakistan
e-mail: pecaric@mahazu.hazu.hr

Marjan Praljak University Of Zagreb, Faculty of Food Technology and Biotechnology
Zagreb, Croatia
and
Abdus Salam School of Mathematical Sciences, GC University
Lahore, Pakistan
e-mail: mpraljak@pbf.hr