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SOME ESTIMATES FOR HAUSDORFF OPERATORS

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Abstract. In this paper, we give some sufficient conditions for the boundedness of three types of Hausdorff operators on the Lebesgue spaces with power weights. In some cases, these conditions are also necessary and the corresponding operator norms are worked out. We extend and improve some known results in [6, 11].

1. Introduction

The one-dimensional Hausdorff operator is defined by

$$h_\Phi f(x) = \int_{\mathrm{R}} \frac{\Phi(x/t)}{|t|} f(t)dt,$$

where $\Phi \in L^1(\mathrm{R})$. Liflyand and Móricz [22] proved that h_Φ is a bounded linear operator on the real Hardy space $H^1(\mathrm{R})$ by the theory of Fourier transform and Hilbert transform. Furthermore, Hausdorff operators were considered in various spaces, for example, see [2, 17, 23, 25]. If we choose $\Phi(t) = \alpha(1-t)^{\alpha-1}\chi(0,1)(t)$ for $\alpha = 1,2,...$, then $H_\Phi = C_\alpha$ is called the Cesàro operator of order α. A brief history of the study of the Cesàro operator can be found in [17].

On the other hand, the operator h_Φ contains the classical Hardy operator and its adjoint operator if we choose suitable functions Φ. For $x > 0$, when one chooses $\Phi(t)$ as $t^{-1}\chi_{(1,\infty)}(t)$ and $\chi_{(0,1]}(t)$, we obtain the classical Hardy operator h and the adjoint Hardy operator h^* respectively, where

$$hf(x) := \frac{1}{x}\int_0^x f(t)dt \quad \text{and} \quad h^*f(x) := \int_x^\infty \frac{f(t)}{t}dt.$$

It is well known that Hardy operators are important operators in Harmonic analysis, for instance, see [8, 15, 16].

Hausdorff operators (Hausdorff summability methods) have a deep root in the study of the one-dimensional Fourier analysis, particularly the summability of the classical Fourier series. A broad and comprehensive overview of the study for Hausdorff

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operators can be found in [21]. One can see [1–7, 10–13, 17–27] to find details of some recent developments for Hausdorff operators.

For multidimensional Hausdorff operators, there are many kinds of definitions [1, 3–5, 18–21, 24, 25]. One of the interesting definitions of the Hausdorff operators is

$$H_\Phi f(x) = \int_{\mathbb{R}^n} \frac{\Phi(x/|y|)}{|y|^n} f(y)dy.$$

Similar to h_Φ, H_Φ contains the high dimensional Hardy operator H and its adjoint operator H^* (see [4, 9]). Recently, the authors obtained the following theorem in [11].

THEOREM A. ([11]) *Let* $1 \leqslant p,\ q \leqslant \infty$ *and* $\alpha,\ \gamma \in \mathbb{R}$ *satisfy* $\frac{\gamma+n}{q} = \frac{\alpha+n}{p}$. *For any general function* $\Phi(x)$, *if*

$$K_{\Phi,s,n,p,\alpha} = \omega_{n-1}^{\frac{1}{p'}} \left(\int_{S^{n-1}} \left(\int_0^\infty |\Phi(\rho\varphi)|^s \rho^{-1+\frac{(\alpha+n)s}{p}} d\rho \right)^{\frac{q}{s}} d\varphi \right)^{\frac{1}{q}} < \infty,$$

where s satisfies $\frac{1}{q} = \frac{1}{p} + \frac{1}{s} - 1$, *then the operator* H_Φ *from* $L^p(\mathbb{R}^n, |x|^\alpha)$ *into* $L^q(\mathbb{R}^n, |x|^\gamma)$ *is bounded, i.e.,*

$$\|H_\Phi f\|_{L^q(\mathbb{R}^n,|x|^\gamma)} \leqslant K_{\Phi,s,n,p,\alpha} \|f\|_{L^p(\mathbb{R}^n,|x|^\alpha)}$$

for all $f \in L^p(\mathbb{R}^n, |x|^\alpha)$.

Here we point out that some partial cases of Theorem A were given in [4] and [26]. In this paper, we firstly prove that in some case, $K_{\Phi,s,n,p,\alpha} < \infty$ is necessary for the boundedness of H_Φ on the Lebesgue spaces with power weights. See Section 2 for the details. In Section 3, we consider another multidimensional Hausdorff operator $\widetilde{H_\Phi}$ (see the below definition) and obtain its boundedness on some Lebesgue spaces with power weights. At the same time, we prove some best estimate of $\widetilde{H_\Phi}$ on the Lebesgue spaces with power weights. In last section, we consider the following multilinear Hausdorff operator.

For a locally integrable function $F(u_1, u_2, ..., u_m)$, we define

$$T_\Phi(F)(x) = \int_{\mathbb{R}^{nm}} \frac{\Phi(x/|u|)}{|u|^{nm}} F(u_1, u_2, ..., u_m)du,$$

where $x \in \mathbb{R}^n$, $u = (u_1, u_2, ..., u_m)$ with $u_i \in \mathbb{R}^n$ and $|u| = \sqrt{|u_1|^2 + |u_2|^2 + \cdots + |u_m|^2}$. When Φ is a radial function, Chen, Fan and Zhang in [6] proved

THEOREM B. ([6]) *Suppose* $\beta = n(m-1)$ *and* $p \geqslant 1$. *If*

$$K_1 = \omega_{nm-1}^{\frac{1}{p'}} \omega_{n-1}^{\frac{1}{p}} \int_0^\infty \frac{\Phi(r)}{r} r^{\frac{nm}{p}} dr < \infty,$$

then we have a constant $K_1 > 0$ *such that*

$$\|T_\Phi(F)\|_{L^p(\mathbb{R}^n,|x|^\beta dx)} \leqslant K_1 \|F\|_{L^p(\mathbb{R}^{nm})},$$

where S^{nm-1} is the unit sphere in R^{nm} and S^{n-1} is the unit sphere in R^n with Lebesgue measures ω_{nm-1} and ω_{n-1}, respectively.

In Section 4, we will remove the radial condition for Φ in the above theorem and obtain the same boundedness. See Theorem 4.1.

Throughout this paper, ω_{nm-1} denotes the area of the unit sphere S^{nm-1} in R^{nm} with Lebesgue measures for $m,n \in \mathbb{Z}^+$.

2. The best estimate of H_Φ on $L^p(\mathrm{R}^n, |x|^\alpha)$

THEOREM 2.1. *Let $1 \leqslant p \leqslant \infty$, $\alpha \in \mathrm{R}$ and $\Phi \geqslant 0$. Then H_Φ is a bounded operator on $L^p(\mathrm{R}^n, |x|^\alpha)$ if and only if*

$$K_{\Phi,n,p,\alpha} = \omega_{n-1}^{\frac{1}{p'}} \left(\int_{S^{n-1}} \left(\int_0^\infty \Phi(\rho\varphi)\rho^{-1+\frac{\alpha+n}{p}} d\rho \right)^p d\varphi \right)^{\frac{1}{p}} < \infty. \tag{2.1}$$

Moreover, when (2.1) holds, the operator norm of H_Φ on $L^p(\mathrm{R}^n, |x|^\alpha)$ is given by

$$\|H_\Phi\|_{L^p(\mathrm{R}^n,|x|^\alpha)\to L^p(\mathrm{R}^n,|x|^\alpha)} = K_{\Phi,n,p,\alpha}. \tag{2.2}$$

Proof. Sufficiency. If we choose $\alpha = \gamma$ in Theorem A, then $p = q$. So using Theorem A, we obtain H_Φ is a bounded operator on $L^p(\mathrm{R}^n, |x|^\alpha)$ if the inequality (2.1) holds. See [11] for the detailed proof.

Necessity. If H_Φ is a bounded operator on $L^p(\mathrm{R}^n, |x|^\alpha)$, then there exists a constant $C > 0$ such that

$$\|H_\Phi f\|_{L^p(\mathrm{R}^n,|x|^\alpha)} \leqslant C\|f\|_{L^p(\mathrm{R}^n,|x|^\alpha)}$$

for all $f \in L^p(\mathrm{R}^n, |x|^\alpha)$. Next we take

$$f_\varepsilon(x) = |x|^{-\frac{\alpha+n+\varepsilon}{p}} \chi_{\{|x|>1\}}(x)$$

for any $\varepsilon > 0$, then $f_\varepsilon \in L^p(\mathrm{R}^n, |x|^\alpha)$ and $\|f_\varepsilon\|_{L^p(\mathrm{R}^n,|x|^\alpha)} = \omega_{n-1}^{\frac{1}{p}} \varepsilon^{-\frac{1}{p}}$. Therefore,

$$\|H_\Phi f_\varepsilon\|_{L^p(\mathrm{R}^n,|x|^\alpha)} \leqslant C\|f_\varepsilon\|_{L^p(\mathrm{R}^n,|x|^\alpha)}. \tag{2.3}$$

On the other hand, we express H_Φ in polar coordinates by writing $x = |x|x'$. Then

$$\begin{aligned} H_\Phi f_\varepsilon(x) &= \int_{\mathrm{R}^n} \frac{\Phi(|x|x'/|y|)}{|y|^n} |y|^{-\frac{\alpha+n+\varepsilon}{p}} \chi_{\{|y|>1\}}(y)dy \\ &= \int_1^\infty \int_{S^{n-1}} \frac{\Phi(|x|x'/t)}{t} t^{-\frac{\alpha+n+\varepsilon}{p}} d\theta dt \\ &= \omega_{n-1}|x|^{-\frac{\alpha+n+\varepsilon}{p}} \int_0^{|x|} \frac{\Phi(\rho x')}{\rho} \rho^{\frac{\alpha+n+\varepsilon}{p}} d\rho. \end{aligned}$$

Then we obtain

$$\begin{aligned}\|H_\Phi f_\varepsilon\|_{L^p(\mathbb{R}^n,|x|^\alpha)} &= \omega_{n-1}\left(\int_{\mathbb{R}^n}\left(|x|^{-\frac{n+\varepsilon}{p}}\int_0^{|x|}\frac{\Phi(\rho x')}{\rho}\rho^{\frac{\alpha+n+\varepsilon}{p}}d\rho\right)^p dx\right)^{\frac{1}{p}}\\
&\geqslant \omega_{n-1}\left(\int_{|x|\geqslant\frac{1}{\varepsilon}}\left(|x|^{-\frac{n+\varepsilon}{p}}\int_0^{\frac{1}{\varepsilon}}\frac{\Phi(\rho x')}{\rho}\rho^{\frac{\alpha+n+\varepsilon}{p}}d\rho\right)^p dx\right)^{\frac{1}{p}}\\
&= \omega_{n-1}\left(\int_{\frac{1}{\varepsilon}}^{\infty}\int_{S^{n-1}}\left(\int_0^{\frac{1}{\varepsilon}}\frac{\Phi(\rho x')}{\rho}\rho^{\frac{\alpha+n+\varepsilon}{p}}d\rho\right)^p t^{-\varepsilon-1}dt\,dx'\right)^{\frac{1}{p}}\\
&= \omega_{n-1}\varepsilon^{\varepsilon}\varepsilon^{-\frac{1}{p}}\left(\int_{S^{n-1}}\left(\int_0^{\frac{1}{\varepsilon}}\frac{\Phi(\rho x')}{\rho}\rho^{\frac{\alpha+n+\varepsilon}{p}}d\rho\right)^p dx'\right)^{\frac{1}{p}}.\end{aligned}$$

Note that $\|f_\varepsilon\|_{L^p(\mathbb{R}^n,|x|^\alpha)} = \omega_{n-1}^{\frac{1}{p}}\varepsilon^{-\frac{1}{p}}$, so we have

$$\|H_\Phi f_\varepsilon\|_{L^p(\mathbb{R}^n,|x|^\alpha)} \geqslant \omega_{n-1}^{\frac{1}{p'}}\|f_\varepsilon\|_{L^p(\mathbb{R}^n,|x|^\alpha)}\varepsilon^{\varepsilon}\left(\int_{S^{n-1}}\left(\int_0^{\frac{1}{\varepsilon}}\frac{\Phi(\rho\varphi)}{\rho}\rho^{\frac{\alpha+n+\varepsilon}{p}}d\rho\right)^p d\varphi\right)^{\frac{1}{p}}.$$

Applying the inequality (2.3) and the above inequality, we get

$$\omega_{n-1}^{\frac{1}{p'}}\left(\int_{S^{n-1}}\left(\int_0^{\frac{1}{\varepsilon}}\frac{\Phi(\rho\varphi)}{\rho}\rho^{\frac{\alpha+n+\varepsilon}{p}}d\rho\right)^p d\varphi\right)^{\frac{1}{p}} \leqslant \frac{C}{\varepsilon^{\varepsilon}}.$$

Letting $\varepsilon \to 0^+$ in the above inequality, we obtain the inequality (1), i.e.

$$K_{\Phi,n,p,\alpha} = \omega_{n-1}^{\frac{1}{p'}}\left(\int_{S^{n-1}}\left(\int_0^{\infty}\Phi(\rho\varphi)\rho^{-1+\frac{\alpha+n}{p}}d\rho\right)^p d\varphi\right)^{\frac{1}{p}} < \infty.$$

When the inequality (2.1) holds, the operator H_Φ is bounded and

$$\|H_\Phi f\|_{L^p(\mathbb{R}^n,|x|^\alpha)} \leqslant K_{\Phi,n,p,\alpha}\|f\|_{L^p(\mathbb{R}^n,|x|^\alpha)}.$$

Therefore, we have

$$\|H_\Phi\|_{L^p(\mathbb{R}^n,|x|^\alpha)\to L^p(\mathbb{R}^n,|x|^\alpha)} \leqslant K_{\Phi,n,p,\alpha}.$$

On the other hand, using the above f_ε, we have

$$\|H_\Phi\|_{L^p(\mathbb{R}^n,|x|^\alpha)\to L^p(\mathbb{R}^n,|x|^\alpha)} \geqslant K_{\Phi,n,p,\alpha}.$$

So we obtain the inequality (2.2). □

3. Some estimates of $\widetilde{H_\Phi}$

In this section, we consider the following multidimensional Hausdorff operator,

$$\widetilde{H_\Phi} f(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f\left(\frac{x}{|y|}\right) dy.$$

Using Minkowski's inequality, we obtain if

$$\widetilde{K}_{\Phi,p,n} = \int_{\mathbb{R}^n} |\Phi(y)||y|^{-n+\frac{n}{p}} dy < \infty,$$

then

$$\|\widetilde{H_\Phi} f\|_{L^p(\mathbb{R}^n)} \leqslant \|f\|_{L^p(\mathbb{R}^n)}.$$

In general, we will prove the following results.

THEOREM 3.1. *Let* $1 \leqslant p \leqslant q \leqslant \infty$ *and* $\alpha,\ \gamma \in \mathbb{R}$ *satisfy* $\frac{\gamma+n}{q} = \frac{\alpha+n}{p}$. *For any general function* $\Phi(x)$, *if*

$$\widetilde{K}_{\Phi,s,p,n,\alpha} = \omega_{n-1}^{\frac{1}{s'}} \left(\int_{\mathbb{R}^n} |\Phi(y)|^s |y|^{-n+\frac{(n+\alpha)s}{p}} dy \right)^{\frac{1}{s}} < \infty,$$

where s *satisfies* $\frac{1}{q} = \frac{1}{p} + \frac{1}{s} - 1$, *then we have*

$$\|\widetilde{H_\Phi} f\|_{L^q(\mathbb{R}^n,|x|^\gamma)} \leqslant \widetilde{K}_{\Phi,s,p,n,\alpha} \left(\int_{S^{n-1}} \left(\int_0^\infty |f(\rho\varphi)|^p \rho^{\alpha+n-1} d\rho \right)^{\frac{q}{p}} d\varphi \right)^{\frac{1}{q}}.$$

In particular, we obtain

$$\|\widetilde{H_\Phi} f\|_{L^q(\mathbb{R}^n,|x|^\gamma)} \leqslant \widetilde{K}_{\Phi,s,p,n,\alpha} \omega_{n-1}^{-\frac{1}{s'}} \|f\|_{L^p_{rad}(\mathbb{R}^n,|x|^\alpha)},$$

where $L^p_{rad}(\mathbb{R}^n,|x|^\alpha) = \{f \in L^p(\mathbb{R}^n,|x|^\alpha) : f \text{ is a radial function}\}$.

Proof. By polar coordinates, we have

$$\widetilde{H_\Phi} f(x) = \int_0^\infty \int_{S^{n-1}} \frac{\Phi(t\theta)}{t} f\left(\frac{x}{t}\right) d\theta dt$$

and

$$\|\widetilde{H_\Phi} f\|^q_{L^q(\mathbb{R}^n,|x|^\gamma)} = \int_0^\infty \int_{S^{n-1}} \left| \int_0^\infty \int_{S^{n-1}} \Phi(t\theta) f\left(\frac{\rho\varphi}{t}\right) d\theta \frac{dt}{t} \right|^q \rho^{\gamma+n} d\varphi \frac{d\rho}{\rho}.$$

We apply $\frac{\gamma+n}{q} = \frac{\alpha+n}{p}$ and Fubini's theorem for interchange of integrals in ρ and φ. Then

$$\begin{aligned}
&\|\widetilde{H_\Phi} f\|^q_{L^q(\mathbb{R}^n,|x|^\gamma)} \\
&= \int_0^\infty \int_{S^{n-1}} \left| \int_0^\infty \int_{S^{n-1}} \Phi(t\theta) t^{\frac{n+\alpha}{p}} f(\rho\varphi t^{-1}) (\rho t^{-1})^{\frac{n+\alpha}{p}} d\theta \frac{dt}{t} \right|^q d\varphi \frac{d\rho}{\rho} \\
&\leqslant \int_{S^{n-1}} \int_0^\infty \left(\int_{S^{n-1}} \int_0^\infty |\Phi(t\theta)| t^{\frac{n+\alpha}{p}} |f(\rho\varphi t^{-1})| (\rho t^{-1})^{\frac{n+\alpha}{p}} \frac{dt}{t} d\theta \right)^q \frac{d\rho}{\rho} d\varphi.
\end{aligned}$$

Using Minkowski's inequality, we have

$$\begin{aligned}&\left(\int_0^\infty\left(\int_{S^{n-1}}\int_0^\infty|\Phi(t\theta)|t^{\frac{n+\alpha}{p}}|f(\rho\varphi t^{-1})|(\rho t^{-1})^{\frac{n+\alpha}{p}}\frac{dt}{t}d\theta\right)^q\frac{d\rho}{\rho}\right)^{\frac{1}{q}}\\&\leqslant\int_{S^{n-1}}\left(\int_0^\infty\left|\int_0^\infty|\Phi(t\theta)|t^{\frac{n+\alpha}{p}}|f(\rho\varphi t^{-1})|(\rho t^{-1})^{\frac{n+\alpha}{p}}\frac{dt}{t}\right|^q\frac{d\rho}{\rho}\right)^{\frac{1}{q}}d\theta.\end{aligned}$$

For

$$\int_0^\infty|\Phi(t\theta)|t^{\frac{n+\alpha}{p}}|f(\rho\varphi t^{-1})|(\rho t^{-1})^{\frac{n+\alpha}{p}}\frac{dt}{t},$$

we can regard it as a convolution inequality on the multiplicative group R^+ with Haar measure $\frac{dt}{t}$. Applying Young's inequality (see [14]) for $\frac{1}{q}=\frac{1}{s}+\frac{1}{p}-1$, we have

$$\begin{aligned}&\left(\int_0^\infty\left(\int_0^\infty|\Phi(t\theta)|t^{\frac{n+\alpha}{p}}|f(\rho\varphi t^{-1})|(\rho t^{-1})^{\frac{n+\alpha}{p}}\frac{dt}{t}\right)^q\frac{d\rho}{\rho}\right)^{\frac{1}{q}}\\&\leqslant\left(\int_0^\infty|\Phi(\rho\theta)|^s\rho^{\frac{(n+\alpha)s}{p}}\frac{d\rho}{\rho}\right)^{\frac{1}{s}}\left(\int_0^\infty|f(\rho\varphi)|^p\rho^{n+\alpha}\frac{d\rho}{\rho}\right)^{\frac{1}{p}}\\&=\left(\int_0^\infty|\Phi(\rho\theta)|^s\rho^{-1+\frac{(n+\alpha)s}{p}}d\rho\right)^{\frac{1}{s}}\left(\int_0^\infty|f(\rho\varphi)|^p\rho^{\alpha+n-1}d\rho\right)^{\frac{1}{p}}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}&\|\widetilde{H_\Phi}f\|^q_{L^q(\mathrm{R}^n,|x|^\gamma)}\\&\leqslant\int_{S^{n-1}}\left(\int_{S^{n-1}}\left(\int_0^\infty|\Phi(\rho\theta)|^s\rho^{-1+\frac{(n+\alpha)s}{p}}d\rho\right)^{\frac{1}{s}}d\theta\right)^q\left(\int_0^\infty|f(\rho\varphi)|^p\rho^{\alpha+n-1}d\rho\right)^{\frac{q}{p}}d\varphi\\&=\left(\int_{S^{n-1}}\left(\int_0^\infty|\Phi(\rho\theta)|^s\rho^{-1+\frac{(n+\alpha)s}{p}}d\rho\right)^{\frac{1}{s}}d\theta\right)^q\left(\int_{S^{n-1}}\left(\int_0^\infty|f(\rho\varphi)|^p\rho^{\alpha+n-1}d\rho\right)^{\frac{q}{p}}d\varphi.\end{aligned}$$

Applying Hölder's inequality $(\frac{1}{s}+\frac{1}{s'}=1)$, we obtain

$$\begin{aligned}&\int_{S^{n-1}}\left(\int_0^\infty|\Phi(\rho\theta)|^s\rho^{-1+\frac{(n+\alpha)s}{p}}d\rho\right)^{\frac{1}{s}}d\theta\\&\leqslant\omega_{n-1}^{\frac{1}{s'}}\left(\int_{S^{n-1}}\int_0^\infty|\Phi(\rho\theta)|^s\rho^{-1+\frac{(n+\alpha)s}{p}}d\rho d\theta\right)^{\frac{1}{s}}\\&=\omega_{n-1}^{\frac{1}{s'}}\left(\int_{\mathrm{R}^n}|\Phi(y)|^s|y|^{-n+\frac{(n+\alpha)s}{p}}dy\right)^{\frac{1}{s}}\\&:=\widetilde{K}_{\Phi,s,p,n,\alpha}.\end{aligned}$$

Hence, we have

$$\|\widetilde{H_\Phi}f\|_{L^q(\mathrm{R}^n,|x|^\gamma)}\leqslant\widetilde{K}_{\Phi,s,p,n,\alpha}\left(\int_{S^{n-1}}\left(\int_0^\infty|f(\rho\varphi)|^p\rho^{\alpha+n-1}d\rho\right)^{\frac{q}{p}}d\varphi\right)^{\frac{1}{q}}.$$

In particular, when $f(x)$ is a radial function, noting that $\frac{1}{q}=\frac{1}{p}+\frac{1}{s}-1$, we easily obtain that

$$\begin{aligned}
&\left(\int_{S^{n-1}}\left(\int_0^\infty |f(\rho\varphi)|^p\rho^{\alpha+n-1}d\rho\right)^{\frac{q}{p}}d\varphi\right)^{\frac{1}{q}}\\
&=\omega_{n-1}^{\frac{1}{q}}\left(\int_0^\infty |f(\rho)|^p\rho^{\alpha+n-1}d\rho\right)^{\frac{1}{p}}\\
&=\omega_{n-1}^{\frac{1}{q}-\frac{1}{p}}\left(\int_0^\infty\int_{S^{n-1}} |f(\rho)|^p\rho^{\alpha+n-1}d\rho\right)^{\frac{1}{p}}\\
&=\omega_{n-1}^{-\frac{1}{s'}}\|f\|_{L^p_{rad}(\mathrm{R}^n,|x|^\alpha)}.
\end{aligned}$$

Therefore, we obtain

$$\|\widetilde{H_\Phi}f\|_{L^q(\mathrm{R}^n,|x|^\gamma)}\leqslant \widetilde{K}_{\Phi,s,p,n,\alpha}\,\omega_{n-1}^{-\frac{1}{s'}}\|f\|_{L^p_{rad}(\mathrm{R}^n,|x|^\alpha)}. \qquad \square$$

COROLLARY 3.1. (See [4]) *Let* $1\leqslant p\leqslant\infty$ *and* $\alpha\in\mathrm{R}$. *For any general function* $\Phi(x)$, *if*

$$\widetilde{K}_{\Phi,p,n,\alpha}=\int_{\mathrm{R}^n}|\Phi(y)||y|^{-n+\frac{n+\alpha}{p}}dy<\infty,$$

then

$$\|\widetilde{H_\Phi}f\|_{L^p(\mathrm{R}^n,|x|^\alpha)}\leqslant \widetilde{K}_{\Phi,p,n,\alpha}\|f\|_{L^p(\mathrm{R}^n,|x|^\alpha)}.$$

Proof. In Theorem 3.1, if we choose $\alpha=\gamma$, then $p=q$ and $s=1$. Therefore we obtain the desired result by Theorem 3.1. $\square$

THEOREM 3.2. *Let* $1\leqslant p\leqslant\infty$, $\alpha\in\mathrm{R}$ *and* $\Phi\geqslant 0$. *Then* $\widetilde{H_\Phi}$ *is a bounded operator on* $L^p(\mathrm{R}^n,|x|^\alpha)$ *if and only if*

$$\widetilde{K}_{\Phi,p,n,\alpha}=\int_{\mathrm{R}^n}\Phi(y)|y|^{-n+\frac{n+\alpha}{p}}dy<\infty. \tag{3.1}$$

Moreover, when (3.1) holds, the operator norm of H_Φ *on* $L^p(\mathrm{R}^n,|x|^\alpha)$ *is given by*

$$\|\widetilde{H_\Phi}\|_{L^p(\mathrm{R}^n,|x|^\alpha)\to L^p(\mathrm{R}^n,|x|^\alpha)}=\widetilde{K}_{\Phi,p,n,\alpha}.$$

Proof. Sufficiency. The proof is obvious by Corollary 3.1.

Necessity. The proof is similar to that of Theorem 2.1. Here note that we choose the same radial function f_ε in Theorem 2.1, i.e.,

$$f_\varepsilon(x)=|x|^{-\frac{\alpha+n+\varepsilon}{p}}\chi_{\{|x|>1\}}(x).$$

Then we have

$$\widetilde{H_\Phi}f_\varepsilon(x)=|x|^{-\frac{\alpha+n+\varepsilon}{p}}\int_{|y|<|x|}\frac{\Phi(y)}{|y|^n}|y|^{\frac{\alpha+n+\varepsilon}{p}}dy.$$

Therefore, we obtain

$$
\begin{aligned}
\|\widetilde{H_\Phi} f_\varepsilon\|_{L^p(\mathbb{R}^n,|x|^\alpha)} &= \left(\int_{\mathbb{R}^n}\left(|x|^{-\frac{n+\varepsilon}{p}}\int_{|y|<|x|}\frac{\Phi(y)}{|y|^n}|y|^{\frac{\alpha+n+\varepsilon}{p}}dy\right)^p dx\right)^{\frac{1}{p}}\\
&\geqslant \left(\int_{|x|\geqslant\frac{1}{\varepsilon}}\left(|x|^{-\frac{n+\varepsilon}{p}}\int_{|y|\leqslant\frac{1}{\varepsilon}}\frac{\Phi(y)}{|y|^n}|y|^{\frac{\alpha+n+\varepsilon}{p}}dy\right)^p dx\right)^{\frac{1}{p}}\\
&= \omega_{n-1}^{\frac{1}{p}}\varepsilon^{-\frac{1}{p}}\varepsilon^{\varepsilon}\int_{|y|\leqslant\frac{1}{\varepsilon}}\frac{\Phi(y)}{|y|^n}|y|^{\frac{\alpha+n+\varepsilon}{p}}dy\\
&= \varepsilon^{\varepsilon}\|f_\varepsilon\|_{L^p(\mathbb{R}^n,|x|^\alpha)}\int_{|y|\leqslant\frac{1}{\varepsilon}}\frac{\Phi(y)}{|y|^n}|y|^{\frac{\alpha+n+\varepsilon}{p}}dy.
\end{aligned}
$$

The remaining proof is the same as that of Theorem 2.1. So we omit it. □

4. Multilinear Hausdorff operator

We firstly recall the definition of multilinear Hausdorff operator $T_\Phi(F)$. Let $x\in\mathbb{R}^n$, $u=(u_1,u_2,\dots,u_m)$ with $u_i\in\mathbb{R}^n$ and $|u|=\sqrt{|u_1|^2+|u_2|^2+\cdots+|u_m|^2}$. The operator $T_\Phi(F)$ is defined by

$$T_\Phi(F)(x)=\int_{\mathbb{R}^{nm}}\frac{\Phi(x/|u|)}{|u|^{nm}}F(u_1,u_2,\dots,u_m)du.$$

THEOREM 4.1. *Suppose $\beta=n(m-1)$ and $p\geqslant 1$. If Φ satisfies*

$$K_{\Phi,p,n,m}=\omega_{nm-1}^{\frac{1}{p'}}\left(\int_{S^{n-1}}\left(\int_0^\infty|\Phi(\rho\varphi)|\rho^{\frac{mn}{p}-1}d\rho\right)^p d\varphi\right)^{\frac{1}{p}}<\infty,$$

then we have

$$\|T_\Phi(F)\|_{L^p(\mathbb{R}^n,|x|^\beta dx)}\leqslant K_{\Phi,p,n,m}\|F\|_{L^p(\mathbb{R}^{nm})}.$$

Proof. By polar coordinates, we have

$$T_\Phi(F)(x)=\int_0^\infty\int_{S^{nm-1}}\frac{F(t\theta)}{t}\Phi\left(\frac{x}{t}\right)d\theta dt.$$

and

$$\|T_\Phi(F)\|^p_{L^p(\mathbb{R}^n,|x|^\beta dx)}=\int_0^\infty\int_{S^{n-1}}\left|T_\Phi(F)(\rho\varphi)\right|^p\rho^{mn}d\varphi\frac{d\rho}{\rho},$$

where $\beta=n(m-1)$. Therefore,

$$
\begin{aligned}
&\|T_\Phi(F)\|^p_{L^p(\mathbb{R}^n,|x|^\beta dx)}\\
&\leqslant \int_0^\infty\int_{S^{n-1}}\left(\int_0^\infty\int_{S^{nm-1}}\left|\Phi(\rho\varphi t^{-1})\right|\left|F(t\theta)\right|d\theta\frac{dt}{t}\right)^p\rho^{mn}d\varphi\frac{d\rho}{\rho}\\
&= \int_0^\infty\int_{S^{n-1}}\left(\int_0^\infty\int_{S^{nm-1}}\left|\Phi(\rho\varphi t^{-1})(\rho t^{-1})^{\frac{mn}{p}}\right|\left|F(t\theta)t^{\frac{mn}{p}}\right|d\theta\frac{dt}{t}\right)^p d\varphi\frac{d\rho}{\rho}\\
&= \int_{S^{n-1}}\int_0^\infty\left(\int_{S^{nm-1}}\int_0^\infty\left|\Phi(\rho\varphi t^{-1})(\rho t^{-1})^{\frac{mn}{p}}\right|\left|F(t\theta)t^{\frac{mn}{p}}\right|\frac{dt}{t}d\theta\right)^p\frac{d\rho}{\rho}d\varphi.
\end{aligned}
$$

By Hölder's inequality, we have

$$\left(\int_{S^{nm-1}}\int_0^\infty \left|\Phi(\rho\varphi t^{-1})(\rho t^{-1})^{\frac{mn}{p}}\right|\left|F(t\theta)t^{\frac{mn}{p}}\right|\frac{dt}{t}d\theta\right)^p$$
$$\leqslant \omega_{nm-1}^{\frac{p}{p'}}\left(\int_{S^{nm-1}}\int_0^\infty \left|\Phi(\rho\varphi t^{-1})(\rho t^{-1})^{\frac{mn}{p}}\right|\left|F(t\theta)t^{\frac{mn}{p}}\right|\frac{dt}{t}d\theta\right)^p.$$

Hence,

$$\|T_\Phi(F)\|^p_{L^p(\mathrm{R}^n,|x|^\beta dx)}$$
$$\leqslant \omega_{nm-1}^{\frac{p}{p'}}\int_{S^{n-1}}\int_{S^{nm-1}}\int_0^\infty\left(\int_0^\infty \left|\Phi(\rho\varphi t^{-1})(\rho t^{-1})^{\frac{mn}{p}}\right|\left|F(t\theta)t^{\frac{mn}{p}}\right|\frac{dt}{t}\right)^p\frac{d\rho}{\rho}d\theta d\varphi.$$

As the proof of Theorem 3.1, using Young's inequality, we have

$$\left(\int_0^\infty\left(\int_0^\infty \left|\Phi(\rho\varphi t^{-1})(\rho t^{-1})^{\frac{mn}{p}}\right|\left|F(t\theta)t^{\frac{mn}{p}}\right|\frac{dt}{t}\right)^p\frac{d\rho}{\rho}\right)^{\frac{1}{p}}$$
$$\leqslant \left(\int_0^\infty |F(\rho\theta)|^p\rho^{mn-1}d\rho\right)^{\frac{1}{p}}\left(\int_0^\infty|\Phi(\rho\varphi)|\rho^{\frac{mn}{p}-1}d\rho\right).$$

So applying Minkowski's integral inequality, we obtain

$$\|T_\Phi(F)\|^p_{L^p(\mathrm{R}^n,|x|^\beta dx)} \leqslant \omega_{nm-1}^{\frac{p}{p'}}\int_{S^{n-1}}\left(\int_{S^{nm-1}}\int_0^\infty |F(\rho\theta)|^p\rho^{mn-1}d\rho d\theta\right)$$
$$\times\left(\int_0^\infty|\Phi(\rho\varphi)|\rho^{\frac{mn}{p}-1}d\rho\right)^p d\varphi$$
$$= \omega_{nm-1}^{\frac{p}{p'}}\int_{S^{n-1}}\left(\int_0^\infty|\Phi(\rho\varphi)|\rho^{\frac{mn}{p}-1}d\rho\right)^p d\varphi\cdot\|F\|^p_{L^p(\mathrm{R}^{nm})}.$$

Therefore, we have

$$\|T_\Phi(F)\|_{L^p(\mathrm{R}^n,|x|^\beta dx)} \leqslant K_{\Phi,p,n,m}\|F\|_{L^p(\mathrm{R}^{nm})},$$

where

$$K_{\Phi,p,n,m} = \omega_{nm-1}^{\frac{1}{p'}}\left(\int_{S^{n-1}}\left(\int_0^\infty|\Phi(\rho\varphi)|\rho^{\frac{mn}{p}-1}d\rho\right)^p d\varphi\right)^{\frac{1}{p}}. \quad \square$$

REMARK 4.1. As described in [6], if we take

$$F(u_1,u_2,\dots,u_m) = f_1(u_1)f_2(u_2)\dots f_m(u_m),$$

then $T_\Phi(F)$ becomes an m-linear operator

$$T_\Phi(f_1,f_2,\dots,f_m)(x) = \int_{\mathrm{R}^{nm}}\frac{\Phi(x/|u|)}{|u|^{nm}}\prod_{j=1}^m f_j(u_j)du.$$

So by Theorem 4.1 and Hölder's inequality, we obtain

$$\|T_\Phi(f_1,f_2,\ldots,f_m)\|_{L^p(\mathbb{R}^n,|x|^\beta dx)} \leqslant K_{\Phi,p,n,m}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(\mathbb{R}^n)},$$

where $\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}=\frac{1}{p}$, $p_j,p\geqslant 1$ and $j=1,\ldots,m$.

REMARK 4.2. Obviously, when Φ is a radial function in Theorem 4.1, we obtain Theorem B in the introduction.

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