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A RESIDUAL–BASED POSTERIORI ERROR ESTIMATES FOR hp FINITE ELEMENT SOLUTIONS OF GENERAL BILINEAR OPTIMAL CONTROL PROBLEMS

ZULIANG LU

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Abstract. In this paper, we investigate a residual-based posteriori error estimates for the hp finite element approximation of general optimal control problems governed by bilinear elliptic equations. By using the hp discontinuous Galerkin finite element approximation for the control and the hp finite element approximation for both the state and the co-state, we derive a posteriori upper error bounds for the optimal control problems governed by bilinear elliptic equations in $L^2 - H^1$ norms. We also give a posteriori lower error bounds for the error estimate of the optimal control problems. These estimates can be readily used for constructing a reliable adaptive finite element approximation for such optimal control problems.

1. Introduction

Optimal control problems are now widely used in physical, biological, engineering design, fluid mechanics, and social-economic systems etc. The finite element method is undoubtedly the most widely used numerical method in computing optimal control problems. Finite element approximation of a class of elliptic optimal control problems has been studied by Falk as a pioneer in [13]. For some classes of linear and nonlinear optimal control problems, many researchers have obtained a priori error estimates for the standard finite element methods in [12, 14, 21, 27, 7] and for the mixed finite element methods in [5, 6, 7, 9, 11, 22, 23, 24], but there are very less published results on this topic for hp-finite element methods for bilinear optimal control problems.

Adaptive finite element approximation is among the most important means to boost accuracy and efficiency of the finite element discretization. There are three main versions in adaptive finite element approximation, i.e., p-version, h-version, and hp-version. The p-version of finite element methods uses a fixed mesh and improves the approximation of the solution by increasing degrees of piecewise polynomials. The h-version is based on mesh refinement and piecewise polynomials of low and fixed degrees. In hp-version adaptation, one have the option to split an element (h-refinement)

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or to increase its approximation order (p-refinement). Generally, a local p-refinement is the more efficient method on regions where the solution is smooth, while a local h-refinement is the strategy suitable on elements where the solution is not smooth. There have been many theoretical studies about hp finite element method in [1, 3, 4, 15].

Actually, there are many h-version of adaptive finite element methods for optimal control problems in [8, 16, 21, 19, 20, 17]. But, for high order element such as hp-version of finite element method for optimal control problems is very few. More recently, in [7], for the constrained optimal control problem governed by linear elliptic equations, the authors have derived a posteriori error estimates for the hp finite element solutions. Inspired by the work of [7], we consider a posteriori error estimates in L^2-H^1 norms for hp finite element solutions of general optimal control problems governed by bilinear elliptic equations. To our best knowledge for optimal control problems, these posteriori error estimates in L^2-H^1 norms for the general bilinear convex optimal control problems are new.

For $1\leqslant p<\infty$ and m any nonnegative integer let $W^{m,p}(\Omega)=\{v\in L^p(\Omega);\ D^\alpha v\in L^p(\Omega)$ if $|\alpha|\leqslant m\}$ denote the Sobolev spaces endowed with the norm $\|v\|_{m,p}^p=\sum\limits_{|\alpha|\leqslant m}\|D^\alpha v\|_{L^p(\Omega)}^p$, and the semi-norm $|\ v\ |_{m,p}^p=\sum\limits_{|\alpha|=m}\|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega)=\{v\in W^{m,p}(\Omega): v\mid_{\partial\Omega}=0\}$. For p=2, we denote $H^m(\Omega)=W^{m,2}(\Omega)$, $H_0^m(\Omega)=W_0^{m,2}(\Omega)$, and $\|\cdot\|_m=\|\cdot\|_{m,2}$, $\|\cdot\|=\|\cdot\|_{0,2}$. In this paper, we focus our attention on the following general bilinear convex optimal control problems:

$$\min_{u\in K}\{g(y)+j(u)\}, \tag{1.1}$$

$$-\mathrm{div}(A\nabla y)+uy=f \quad \text{in } \Omega, \quad y|_{\partial\Omega}=0, \tag{1.2}$$

where Ω and Ω_U are bounded open sets in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$ and $\partial\Omega_U$, g is a convex functional which is continuously differentiable function on the observation space $L^2(\Omega)$, $j(u)=\int_{\Omega_U}h(u)dx$, where $h(\cdot)$ is a strictly convex continuously differentiable function and $h'(\cdot)$ is locally lipschitz continuous in a neighborhood of u. Let $f\in L^2(\Omega)$, and $A(\cdot)=(a_{i,j}(\cdot))_{2\times 2}\in(W^{1,\infty}(\Omega))^{2\times 2}$, satisfying that there is a constant $c>0$ such that for any vector $X\in\mathbb{R}^2$, $X^tAX\geqslant c\|X\|_{\mathbb{R}^2}^2$. Let K be a closed convex set defined by $K=\{u\in L^2(\Omega_U):\int_{\Omega_U}udx\geqslant 0\}$.

The paper is organized as follows. In Section 2, we shall contruct the hp finite element approximation for the distributed convex optimal control problem governed by bilinear elliptic equations. In Section 3, we derive both hp a posteriori upper error bounds and hp a posteriori lower error bounds for the error estimates of the control, the state and the co-state. In Section 4, we give conclusions and some possible future work.

2. hp finite element of bilinear optimal control

In this section, we discuss the hp finite element approximation of general bilinear optimal control problems (1.1)–(1.2). Now, we shall take the state space $V=H_0^1(\Omega)$, the control space $U=L^2(\Omega_U)$, and $H=L^2(\Omega)$ to fix the idea. Let the observation space $Y = L^2(\Omega)$. To consider the hp finite element approximation of the general optimal control problems (1.1)–(1.2), we first give a weak formula for the state equation. Let

$$a(v,w) = \int_\Omega (A\nabla v)\cdot \nabla w dx, \quad \forall v,w \in V,$$
$$(f_1,f_2) = \int_\Omega f_1 f_2 dx, \quad \forall (f_1,f_2) \in H\times H,$$
$$(v,w)_U = \int_{\Omega_U} vw dx, \quad \forall (v,w) \in U\times U.$$

Then it is easy to see that

$$(j'(u),v)_U = \int_{\Omega_U} h'(u) v dx.$$

It follows from the assumptions on A that there are constants $c, C > 0$ such that

$$a(v,v) \geqslant c\|v\|_V^2, \quad |a(v,w)| \leqslant C|v|_V|w|_V, \quad \forall v,w\in V. \tag{2.1}$$

Then, the general optimal control problems (1.1)–(1.2) can be restated as follows:

$$\min_{u\in K\subset U}\{g(y)+j(u)\}, \tag{2.2}$$
$$a(y(u),w)+(uy,w) = (f,w), \quad \forall w\in V = H_0^1(\Omega). \tag{2.3}$$

It is well known (see, i.e., [18]) that the optimal control problems (2.2)–(2.3) has a solution (y,u), and that if a pair (y,u) is the solution of (2.2)–(2.3), then there is a co-state $p\in V$ such that the triplet (y,u,p) satisfies the following optimality conditions:

$$a(y,w)+(uy,w) = (f,w), \quad \forall w\in V = H_0^1(\Omega), \tag{2.4}$$
$$a(q,p)+(up,q) = (g'(y),q), \quad \forall q\in V = H_0^1(\Omega), \tag{2.5}$$
$$(h'(u)-yp, v-u)_U \geqslant 0, \quad \forall v\in K\subset U = L^2(\Omega_U), \tag{2.6}$$

where g' and h' are the derivatives of g and h. Here g' and h' can be viewed as functions in $Y = L^2(\Omega)$ and $U = L^2(\Omega_U)$, respectively.

Assume that Ω and Ω_U are polygonal. We consider the triangulation $\mathscr{T}$ of the set $\Omega\subset\mathbb{R}^2$ which is a collection of elements $\tau\in\mathscr{T}$, τ is a parallelogram or a triangle; associated with each element τ is an affine element map $F_\tau : \widehat{\tau}\to\tau$, where the reference element $\widehat{\tau}$ is the reference square $S = (0,1)^2$ if τ is a parallelogram and $\widehat{\tau}$ is the reference triangle $T = \{(x,y)\in\mathbb{R}^2 : 0<x<1, 0<y<\min(x,1-x)\}$ if τ is a triangle. We consider the triangulation $\mathscr{T}$ which satisfies the standard conditions defined in [25]. We write $h_\tau = \operatorname{diam}\tau$. Additionally we assume that triangulation $\mathscr{T}$ is γ-shape regular, i.e.,

$$h_\tau^{-1}\|F_\tau'\|_{L^\infty(\widehat{\tau})} + h_\tau\|(F_\tau')^{-1}\|_{L^\infty(\widehat{\tau})} \leqslant \gamma. \tag{2.7}$$

This implies that there exists a constant $C>0$ that depends solely on γ such that

$$C^{-1}h_\tau \leqslant h_{\tau'} \leqslant Ch_\tau, \quad \tau,\tau'\in\mathscr{T} \text{ with } \overline{\tau}\cap\overline{\tau}'\neq\emptyset, \tag{2.8}$$

and there exists a constant $M \in \mathbb{N}$ that depends solely on γ such that no more than M elements share a common vertex.

We also assume that the triangulation $\mathscr{T}_U$ of Ω_U which is a collection of elements $\tau_U \in \mathscr{T}_U$, is γ-shape regular which satisfies the standard conditions as $\mathscr{T}$. Associated with each element τ_U is an affine element map $F_{\tau_U} : \widehat{\tau} \to \tau_U$. We further assume the triangulation $\mathscr{T}$ satisfies the relation between the patch and the reference patch in [25]. For each element $\tau \in \mathscr{T}$, we denote $\mathscr{E}(\tau)$ the set of edges of τ and by $\mathscr{N}(\tau)$ the set of vertices of τ, and choose a polynomial degree $p_\tau \in \mathbb{N}$ and collect these numbers in the polynomial degree vector $p_1 = (p_\tau)_{\tau \in \mathscr{T}}$. Similarly, for each element $\tau_U \in \mathscr{T}_U$, we choose a polynomial degree vector $p_2 = (p_{\tau_U})_{\tau_U \in \mathscr{T}_U}$ ($p_{\tau_U} \in \mathbb{N}$). And $\mathscr{N}(\mathscr{T})$ denotes the set of all vertices of $\mathscr{T}$, $\mathscr{E}(\mathscr{T})$ denotes the set of all edges. Next, for $V \in \mathscr{N}(\mathscr{T})$, $e \in \mathscr{E}(\mathscr{T})$, we introduce the following notations:

$$\begin{aligned}
&\mathscr{N}(e) = \{V \in \mathscr{N}(\mathscr{T}) : V \in \overline{e}\},\\
&w_V = \{x \in \Omega : x \in \overline{\tau} \text{ and } \overline{\tau} \cap \{V\} \neq \emptyset\}^0,\\
&w_e^1 = \bigcup_{V \in \mathscr{N}(e)} w_V, \qquad w_\tau^1 = \bigcup_{V \in \mathscr{N}(\tau)} w_V,\\
&p_e = \max\{p_\tau : e \in \mathscr{E}(\tau)\}, \qquad h_{\tau_U} = \operatorname{diam}\tau_U,
\end{aligned} \tag{2.9}$$

where χ^0 denotes the interior of the set χ. We denote by h_e the length of the edge e. Additionally, C or c denotes a general positive constant independent of h_τ, p_τ, h_{τ_U}, p_{τ_U}, h_e, p_e, and p_{τ_U}.

Now, we define the hp finite element space $S^{p_1}(\mathscr{T}) \subset H^1(\Omega)$ and the hp discontinuous Galerkin finite element space $S^{p_2}(\mathscr{T}_U) \subset L^2(\Omega_U)$ by

$$\begin{aligned}
&S^{p_1}(\mathscr{T}) = \{v \in C(\Omega);\ v|_\tau \circ F_\tau \in \Pi_{p_\tau}(\widehat{\tau})\},\\
&S^{p_2}(\mathscr{T}_U) = \{v \in L^2(\Omega_U);\ v|_{\tau_U} \circ F_{\tau_U} \in \Pi_{p_{\tau_U}}(\widehat{\tau})\},
\end{aligned}$$

where we set

$$\Pi_k(\widehat{\tau}) = \begin{cases} P_k = \operatorname{span}\{x^i y^j : 0 \leqslant i+j \leqslant k\}, & \text{if } \widehat{\tau} = T,\\ Q_k = \operatorname{span}\{x^i y^j : 0 \leqslant i, j \leqslant k\}, & \text{if } \widehat{\tau} = S. \end{cases}$$

We also assume that the polynomial degree vector $p_1 = (p_\tau)_{\tau \in \mathscr{T}}$ satisfies:

$$\gamma^{-1} p_\tau \leqslant p_{\tau'} \leqslant \gamma p_\tau, \qquad \tau, \tau' \in \mathscr{T} \text{ with } \overline{\tau} \cap \overline{\tau}' \neq \emptyset. \tag{2.10}$$

Then we can construct the following finite element spaces

$$V_{hp} = V \cap S^{p_1}(\mathscr{T}), \quad K_{hp} = K \cap S^{p_2}(\mathscr{T}_U).$$

Then the hp finite element approximation of (2.2)–(2.3) is as follows:

$$\min_{u_{hp} \in K_{hp}} \{g(y_{hp}) + j(u_{hp})\}, \tag{2.11}$$

$$a(y(u_{hp}), w_{hp}) + (u_{hp} y_{hp}, w_{hp}) = (f, w_{hp}), \quad \forall w_{hp} \in V_{hp}. \tag{2.12}$$

It is well known that the optimal control problems (2.11)–(2.12) has a solution (y_{hp}, u_{hp}) and that if a pair $(y_{hp}, u_{hp}) \in V_{hp} \times K_{hp}$ is the solution of (2.11)–(2.12), then there is a co-state $p_{hp} \in V_{hp}$ such that the triplet (y_{hp}, p_{hp}, u_{hp}) satisfies the following optimality conditions:

$$a(y_{hp}, w_{hp}) + (u_{hp}y_{hp}, w_{hp}) = (f, w_{hp}), \quad \forall w_{hp} \in V_{hp} \subset V = H_0^1(\Omega), \tag{2.13}$$

$$a(q_{hp}, p_{hp}) + (u_{hp}p_{hp}, q_{hp}) = (g'(y_{hp}), q_{hp}), \quad \forall q_{hp} \in V_{hp} \subset V = H_0^1(\Omega), \tag{2.14}$$

$$(h'(u_{hp}) - y_{hp}p_{hp}, v_{hp} - u_{hp})_U \geqslant 0, \quad \forall v_{hp} \in K_{hp} \subset U = L^2(\Omega_U). \tag{2.15}$$

The following lemmas are important in deriving a posteriori error estimates of residual type [25].

LEMMA 2.1. *Let p_1 be an arbitrary polynomial degree distribution satisfies (2.10). Then there exists a linear operator $E^{hp} : H_0^1(\Omega) \to S^{p_1}(\mathscr{T}) \cap H_0^1(\Omega)$, and there exists a constant $C > 0$ depending solely on γ such that for every $v \in H_0^1(\Omega)$ and all elements $\tau \in \mathscr{T}$ and all edges $e \in \mathscr{E}(\mathscr{T})$*

$$\|v - E^{hp}v\|_{L^2(\tau)} + \frac{h_\tau}{p_\tau}\|\nabla(v - E^{hp}v)\|_{L^2(\tau)} \leqslant C\frac{h_\tau}{p_\tau}\|\nabla v\|_{L^2(w_\tau^1)},$$

$$\|v - E^{hp}v\|_{L^2(e)} \leqslant C\left(\frac{h_e}{p_e}\right)^{\frac{1}{2}}\|\nabla v\|_{L^2(w_e^1)}.$$

LEMMA 2.2. *There exists a constant $C > 0$ independent of v, h_{τ_U}, and p_{τ_U} and a mapping $\pi_{p_{\tau_U}}^{h_{\tau_U}} : H^1(\tau_U) \to \mathscr{P}_{p_{\tau_U}}(\tau_U)$ such that $\forall v \in H^1(\tau_U), \tau_U \in \mathscr{T}_U$ the following inequality is valid*

$$\|v - \pi_{p_{\tau_U}}^{h_{\tau_U}} v\|_{L^2(\tau_U)} \leqslant C\frac{h_{\tau_U}}{p_{\tau_U}}\|v\|_{H^1(\tau_U)}.$$

where we will write $v \in \mathscr{P}_{p_{\tau_U}}(\tau_U)$ if the following satisfied: $v|_{\tau_U} \circ F_{\tau_U} \in P_{p_{\tau_U}}(\widehat{\tau})$ if τ_U is a triangle; $v|_{\tau_U} \circ F_{\tau_U} \in Q_{p_{\tau_U}}(\widehat{\tau})$ if τ_U is a parallelogram.

Proof. For a proof we refer to the Lemma 4.5 in [1]. □

Let $H^*(\Omega_U, \mathscr{T}_U) = \{v : v|_{\tau_U} \in H^1(\tau_U), \forall \tau_U \in \mathscr{T}_U\}$, and then we can define the mapping that is useful in the estimate of the control, i.e., there exist a mapping I_U^{hp} : $H^*(\Omega_U, \mathscr{T}_U) \to S^{p_2}(\mathscr{T}_U)$ such that

$$I_U^{hp} v|_{\tau_U} = \pi_{p_{\tau_U}}^{h_{\tau_U}}(v|_{\tau_U}), \qquad \forall \tau_U \in \mathscr{T}_U. \tag{2.16}$$

3. A residual-based posteriori Error estimates

In this section, we will discuss hp a residual-based posteriori error estimates for the optimal control problems. Let $y(u)$ and $y_{hp}(u_{hp})$ are the solutions of (2.2)–(2.3) and (2.11)–(2.12) respectively. For simplicity of presentation, let

$$J(u) = g(y(u)) + j(u), \quad J_{hp}(u_{hp}) = g(y(u_{hp})) + j(u_{hp}).$$

Then the reduced optimal control problems of (2.2) and (2.11) read as

$$\min_{u\in K}\{J(u)\}, \tag{3.1}$$

and

$$\min_{u_{hp}\in K_{hp}}\{J_{hp}(u_{hp})\}, \tag{3.2}$$

respectively. It can be shown that

$$\begin{aligned}
&(J'(u),v)_U = (h'(u)-yp,v)_U,\\
&(J'_{hp}(u_{hp}),v)_U = (h'(u_{hp})-y_{hp}p_{hp},v)_U,\\
&(J'(u_{hp}),v)_U = (h'(u_{hp})-y(u_{hp})p(u_{hp}),v)_U,
\end{aligned}$$

where $p(u_{hp})$ is the solution of the auxiliary equations:

$$a(y(u_{hp}),w)+(u_{hp}y(u_{hp}),w) = (f,w), \quad \forall w\in V = H_0^1(\Omega), \tag{3.3}$$

$$a(q,p(u_{hp}))+(u_{hp}p(u_{hp}),q) = (g'(y(u_{hp})),q), \quad \forall q\in V = H_0^1(\Omega). \tag{3.4}$$

THEOREM 3.1. *Let u and u_{hp} be the solutions of (3.1) and (3.2) respectively. Assume that*

$$(J'(u)-J'(v),u-v)_U \geqslant c\|u-v\|^2_{L^2(\Omega_U)}, \quad \forall u,v\in U. \tag{3.5}$$

Moreover, we assume that $h'(u_{hp})-y_{hp}p_{hp}\in H^*(\Omega_U,\mathscr{T}_U)$. Then,

$$\|u-u_{hp}\|^2_{L^2(\Omega_U)} \leqslant C(\eta_1^2+\|y_{hp}-y(u_{hp})\|^2_{L^2(\Omega)}+\|p_{hp}-p(u_{hp})\|^2_{L^2(\Omega)}), \tag{3.6}$$

where p_{hp} and $p(u_{hp})$ are the solutions of the equations (2.14) and (3.4) respectively, and

$$\eta_1^2 = \sum_{\tau_U}\frac{h^2_{\tau_U}}{p^2_{\tau_U}}\int_{\tau_U}(\nabla(h'(u_{hp})-y_{hp}p_{hp}))^2.$$

Proof. By using (2.6), (2.15), and (3.5), we have

$$\begin{aligned}
&c\|u-u_{hp}\|^2_{L^2(\Omega_U)}\\
\leqslant&(J'(u),u-u_{hp})_U-(J'(u_{hp}),u-u_{hp})_U\\
\leqslant&-(J'(u_{hp}),u-u_{hp})_U\\
\leqslant&-(J'(u_{hp}),u-u_{hp})_U+(h'(u_{hp})-y_{hp}p_{hp},v_{hp}-u_{hp})_U\\
=&(J'_{hp}(u_{hp}),u_{hp}-u)_U+(J'_{hp}(u_{hp})-J'(u_{hp}),u-u_{hp})_U+(h'(u_{hp})-y_{hp}p_{hp},v_{hp}-u_{hp})_U\\
=&(h'(u_{hp})-y_{hp}p_{hp},u_{hp}-u)_U+(y(u_{hp})p(u_{hp})-y_{hp}p_{hp},u-u_{hp})_U\\
&+(h'(u_{hp})-y_{hp}p_{hp},v_{hp}-u_{hp})_U
\end{aligned}$$

$$
\begin{aligned}
&= (h'(u_{hp}) - y_{hp}p_{hp}, v_{hp} - u)_U + ((y(u_{hp}) - y_{hp})p(u_{hp}) + y_{hp}(p(u_{hp}) - p_{hp}), u - u_{hp})_U \\
&\leqslant (h'(u_{hp}) - y_{hp}p_{hp}, v_{hp} - u)_U + C\|y_{hp} - y(u_{hp})\|^2_{L^2(\Omega)} \\
&\quad + C\|p_{hp} - p(u_{hp})\|^2_{L^2(\Omega)} + \frac{c}{4}\|u - u_{hp}\|^2_{L^2(\Omega_U)}.
\end{aligned} \tag{3.7}
$$

Now, we introduce the $L^2(\Omega_U)-$projection of u into $S^{p_2}(\mathscr{T}_U)$, i.e., let $P_{hp}u \in S^{p_2}(\mathscr{T}_U)$ be the function defined by

$$
(u - P_{hp}u, w_{hp})_U = 0, \qquad \forall w_{hp} \in S^{p_2}(\mathscr{T}_U). \tag{3.8}
$$

Setting $w_{hp} = 1$ in (3.8), we have

$$
\int_{\Omega_U} P_{hp}u = \int_{\Omega_U} u \geqslant 0.
$$

So we have $P_{hp}u \in K_{hp}$. In order to use the scaling argument, we introduce the following notation:

$$
G|_{\tau_U} = h'(u_{hp}) - y_{hp}p_{hp},
$$

$$
\widehat{G}\,|_{\widehat{\tau}} = \widehat{G|_{\tau_U}} = (h'(u_{hp}) - y_{hp}p_{hp}) \circ F_{\tau_U} \in H^1(\widehat{\tau}), \qquad \overline{\widehat{G}\,|_{\widehat{\tau}}} = \int_{\widehat{\tau}} \widehat{G}\,|_{\widehat{\tau}} \Big/ \int_{\widehat{\tau}} 1.
$$

It follows easily from (3.8) that

$$
\|u - P_{hp}u\|_{L^2(\Omega_U)} \leqslant \|u - v_{hp}\|_{L^2(\Omega_U)}, \qquad \forall v_{hp} \in S^{p_2}(\mathscr{T}_U). \tag{3.9}
$$

Set $v_{hp} = P_{hp}u \in K_{hp}$, it follows from (3.8), (3.9), Lemma 2.2, Poincaré inequality, and scaling argument that

$$
\begin{aligned}
&(h'(u_{hp}) - y_{hp}p_{hp}, v_{hp} - u)_U \\
&= (h'(u_{hp}) - y_{hp}p_{hp}, P_{hp}u - u)_U \\
&= \Big(h'(u_{hp}) - y_{hp}p_{hp} - \overline{\widehat{G}\,|_{\widehat{\tau}}} - I_U^{hp}(h'(u_{hp}) - y_{hp}p_{hp} - \overline{\widehat{G}\,|_{\widehat{\tau}}}), P_{hp}u - u\Big)_U \\
&= \sum_{\tau_U} \Big(h'(u_{hp}) - y_{hp}p_{hp} - \overline{\widehat{G}\,|_{\widehat{\tau}}} - \pi_{p_{\tau_U}}^{h_{\tau_U}}(h'(u_{hp}) - y_{hp}p_{hp} - \overline{\widehat{G}\,|_{\widehat{\tau}}}), P_{hp}u - u\Big)_{\tau_U} \\
&\leqslant C\sum_{\tau_U} \left(\frac{h_{\tau_U}}{p_{\tau_U}} \left\|h'(u_{hp}) - y_{hp}p_{hp} - \overline{\widehat{G}\,|_{\widehat{\tau}}}\right\|_{H^1(\tau_U)} \cdot \|P_{hp}u - u\|_{L^2(\tau_U)}\right) \\
&\leqslant C\sum_{\tau_U} \frac{h^2_{\tau_U}}{p^2_{\tau_U}} \left\|h'(u_{hp}) - y_{hp}p_{hp} - \overline{\widehat{G}\,|_{\widehat{\tau}}}\right\|^2_{H^1(\tau_U)} + \frac{c}{4}\|P_{hp}u - u\|^2_{L^2(\Omega_U)} \\
&\leqslant C\sum_{\tau_U} \frac{h^2_{\tau_U}}{p^2_{\tau_U}} \|\nabla(h'(u_{hp}) - y_{hp}p_{hp})\|^2_{L^2(\tau_U)} + \frac{c}{4}\|u - u_{hp}\|^2_{L^2(\Omega_U)}.
\end{aligned} \tag{3.10}
$$

By using (3.7) and (3.10), we have

$$
\begin{aligned}
\|u - u_{hp}\|^2_{L^2(\Omega_U)} \leqslant{} & C\sum_{\tau_U} \frac{h^2_{\tau_U}}{p^2_{\tau_U}} \|\nabla(h'(u_{hp}) - y_{hp}p_{hp})\|^2_{L^2(\tau_U)} \\
& + C\|y_{hp} - y(u_{hp})\|^2_{L^2(\Omega)} + C\|p_{hp} - p(u_{hp})\|^2_{L^2(\Omega)}.
\end{aligned}
$$

The proof of Theorem 3.1 is completed. $\square$

THEOREM 3.2. *Let (y,p,u) and (y_{hp},p_{hp},u_{hp}) be the solutions of (2.4)–(2.6) and (2.13)–(2.15). Supposed all the conditions in Theorem 3.1 are valid. Moreover, assume that g' and h' are locally Lipschitz continuous in a neighborhood of y and u. Then,*

$$\|u-u_{hp}\|^2_{L^2(\Omega_U)}+\|y-y_{hp}\|^2_{H^1(\Omega)}+\|p-p_{hp}\|^2_{H^1(\Omega)}\leqslant C\sum_{i=1}^{3}\eta_i^2, \tag{3.11}$$

where

$$\eta_2^2=\sum_\tau\int_\tau\frac{h_\tau^2}{p_\tau^2}(g'(y_{hp})+\mathrm{div}(A^*\nabla p_{hp})-u_{hp}p_{hp})^2+\sum_e\int_e\frac{h_e}{p_e}[(A^*\nabla p_{hp}\cdot n)]^2,$$
$$\eta_3^2=\sum_\tau\int_\tau\frac{h_\tau^2}{p_\tau^2}(f+\mathrm{div}(A\nabla y_{hp})-u_{hp}y_{hp})^2+\sum_e\int_e\frac{h_e}{p_e}[(A\nabla y_{hp}\cdot n)]^2,$$

where e is a edge of an element τ, $[(A^*\nabla p_{hp}\cdot n)]$ and $[(A\nabla y_{hp}\cdot n)]$ are the A-normal derivative jumps over the interior edge e, defined by

$$[(A^*\nabla p_{hp}\cdot n)]_e=(A^*\nabla p_{hp}|_{\tau_e^1}-A^*\nabla p_{hp}|_{\tau_e^2})\cdot n,$$
$$[(A\nabla y_{hp}\cdot n)]_e=(A\nabla y_{hp}|_{\tau_e^1}-A\nabla y_{hp}|_{\tau_e^2})\cdot n,$$

where n is the unit normal vector on $e=\bar{\tau}_e^1\cap\bar{\tau}_e^2$ outwards τ_e^1. For ease of exposition, we let $[(A^*\nabla p_{hp}\cdot n)]_e=[(A\nabla y_{hp}\cdot n)]_e=0$ when $e\subset\partial\Omega$.

Proof. Firstly, let $e^p=p(u_{hp})-p_{hp}$ and E^{hp} be the interpolator defined in Lemma 2.1. It follows from (2.1), (2.14), (3.4), and Lemma 2.1 that

$$\begin{aligned}
c\|e^p\|^2_{H^1(\Omega)}\leqslant&(\nabla e^p,A^*\nabla(p(u_{hp})-p_{hp}))+(u_{hp}(p(u_{hp})-p_{hp}),e^p)\\
=&(\nabla(e^p-E^{hp}e^p),A^*\nabla(p(u_{hp})-p_{hp}))+(u_{hp}(p(u_{hp})-p_{hp}),e^p-E^{hp}e^p)\\
&+(\nabla E^{hp}e^p,A^*\nabla(p(u_{hp})-p_{hp}))+(u_{hp}(p(u_{hp})-p_{hp}),E^{hp}e^p)\\
=&\sum_\tau\int_\tau(g'(y_{hp})+\mathrm{div}(A^*\nabla p_{hp})-u_{hp}p_{hp})(e^p-E^{hp}e^p)\\
&-\sum_e\int_e[A^*\nabla p_{hp}\cdot n](e^p-E^{hp}e^p)ds+(g'(y(u_{hp}))-g'(y_{hp}),e^p)\\
\leqslant&C(\delta)\sum_\tau\frac{h_\tau^2}{p_\tau^2}\int_\tau(g'(y_{hp})+\mathrm{div}(A^*\nabla p_{hp})-u_{hp}p_{hp})^2\\
&+C(\delta)\sum_e\frac{h_e}{p_e}\int_e[A^*p_{hp}\cdot n]^2+C\delta\sum_\tau\frac{p_\tau^2}{h_\tau^2}\int_\tau|e^p-E^{hp}e^p|^2\\
&+C\delta\sum_e\int_e\frac{p_e}{h_e}|e^p-E^{hp}e^p|^2ds+C\|g'(y(u_{hp}))-g'(y_{hp})\|_{L^2(\Omega)}\cdot\|e^p\|_{L^2(\Omega)}
\end{aligned}$$

$$\leqslant C(\delta)\sum_\tau \frac{h_\tau^2}{p_\tau^2}\int_\tau (g'(y_{hp})+\mathrm{div}(A^*\nabla p_{hp})-u_{hp}p_{hp})^2$$
$$+C(\delta)\sum_e \frac{h_e}{p_e}\int_e [A^*p_{hp}\cdot n]^2 + C(\delta)\|y_{hp}-y(u_{hp})\|^2_{L^2(\Omega)} + C\delta\|e^p\|^2_{H^1(\Omega)}, \tag{3.12}$$

where we have used the embedding theorem $\|v\|_{L^4(\Omega)} \leqslant C\|v\|_{H^1(\Omega)}$ and the property $\|p_{hp}\|_{H^1(\Omega)} \leqslant C$. Then, let $\delta = \frac{c}{2C}$, we have

$$\|p_{hp}-p(u_{hp})\|^2_{H^1(\Omega)} \leqslant C\sum_\tau \frac{h_\tau^2}{p_\tau^2}\int_\tau (g'(y_{hp})+\mathrm{div}(A^*\nabla p_{hp})-u_{hp}p_{hp})^2$$
$$+C\sum_e \frac{h_e}{p_e}\int_e [A^*p_{hp}\cdot n]^2 + C\|y_{hp}-y(u_{hp})\|^2_{L^2(\Omega)}. \tag{3.13}$$

Similarly, let $e^y = y(u_{hp}) - y_{hp}$ and let E^{hp} be the interpolator defined in Lemma 2.1. It follows from (2.1), (2.13), (3.3), and Lemma 2.1 that

$$\begin{aligned}
c\|y(u_{hp})-y_{hp}\|^2_{H^1(\Omega)} &\leqslant (A\nabla(y(u_{hp})-y_{hp}),\nabla e^y) + (u_{hp}(y(u_{hp})-y_{hp}),e^y)\\
&= (A\nabla(y(u_{hp})-y_{hp}),\nabla(e^y-E^{hp}e^y))\\
&\quad + (u_{hp}(y(u_{hp})-y_{hp}),e^y-E^{hp}e^y)\\
&= \sum_\tau\int_\tau (f+\mathrm{div}(A\nabla y_{hp})-u_{hp}y_{hp})(e^y-E^{hp}e^y)\\
&\quad -\sum_\tau\int_{\partial\tau}(A\nabla y_{hp}\cdot n)(e^y-E^{hp}e^y)ds\\
&\leqslant C(\delta)\sum_\tau\frac{h_\tau^2}{p_\tau^2}\int_\tau (f+\mathrm{div}(A\nabla y_{hp})-u_{hp}y_{hp})^2\\
&\quad + C(\delta)\sum_e\frac{h_e}{p_e}\int_e [A\nabla y_{hp}\cdot n]^2 + C\delta\|e^y\|^2_{H^1(\Omega)}.
\end{aligned} \tag{3.14}$$

Hence,

$$\|y(u_{hp})-y_{hp}\|^2_{H^1(\Omega)} \leqslant C\sum_\tau\frac{h_\tau^2}{p_\tau^2}\int_\tau (f+\mathrm{div}(A\nabla y_{hp})-u_{hp}y_{hp})^2$$
$$+C\sum_e\frac{h_e}{p_e}\int_e [A\nabla y_{hp}\cdot n]^2. \tag{3.15}$$

It follows from (3.13), (3.15), and Theorem 3.1 that

$$\|u-u_{hp}\|^2_{L^2(\Omega_U)} \leqslant C\sum_{i=1}^3 \eta_i^2. \tag{3.16}$$

Note that

$$\|y-y_{hp}\|_{H^1(\Omega)} \leqslant \|y(u_{hp})-y_{hp}\|_{H^1(\Omega)} + \|y-y(u_{hp})\|_{H^1(\Omega)}, \tag{3.17}$$

$$\|p-p_{hp}\|_{H^1(\Omega)} \leqslant \|p(u_{hp})-p_{hp}\|_{H^1(\Omega)} + \|p-p(u_{hp})\|_{H^1(\Omega)}, \tag{3.18}$$

and

$$\|p-p(u_{hp})\|^2_{H^1(\Omega)} + \|y-y(u_{hp})\|^2_{H^1(\Omega)} \leqslant C\|u-u_{hp}\|^2_{L^2(\Omega_U)}. \tag{3.19}$$

Therefore, (3.11) follows from (3.13), (3.15), (3.16), and (3.17)–(3.19). □

Now, we will derive hp a posteriori lower error bounds for the optimal control problems governed by bilinear elliptic equations. To obtain the posteriori lower error bounds, we need the following polynomial inverse estimates [26].

LEMMA 3.1. *Let $-1<\alpha<\beta$, $\delta\in[0,1]$ and let $\Phi_{\hat{e}}(x)=x(1-x)$. Then there exist $C_1=C(\alpha,\beta)$, and $C_2=C(\delta)$ such that for all univariate polynomials ψ_k of degree k*

$$\int_0^1 \Phi_{\hat{e}}^{2\alpha}\psi_k^2(x)dx \leqslant C_1 k^{2(\beta-\alpha)}\int_0^1 \Phi_{\hat{e}}^{\beta}\psi_k^2(x)dx, \tag{3.20}$$

$$\int_0^1 \Phi_{\hat{e}}^{2\delta}(\psi_k'(x))^2dx \leqslant C_2 k^{2(2-\delta)}\int_0^1 \Phi_{\hat{e}}^{\delta}\psi_k^2(x)dx. \tag{3.21}$$

LEMMA 3.2. *Let $\widehat{\tau}=S$ or $\widehat{\tau}=T$ and let $\Phi_{\widehat{\tau}}=dist(x,\partial\widehat{\tau})$. Let $-1<\alpha<\beta$ and $\delta\in[0,1]$. Then there exist $C_3=C(\alpha,\beta)$ and $C_4=C(\delta)$ such that for all polynomials $\psi_k\in Q_k$*

$$\int_{\widehat{\tau}}(\Phi_{\widehat{\tau}})^{\alpha}\psi_k^2dxdy \leqslant C_3 k^{2(\beta-\alpha)}\int_{\widehat{\tau}}(\Phi_{\widehat{\tau}})^{\beta}\psi_k^2dxdy, \tag{3.22}$$

$$\int_{\widehat{\tau}}\Phi_{\widehat{\tau}}^{2\delta}|\nabla\psi_k|^2dxdy \leqslant C_4 k^{2(2-\delta)}\int_{\widehat{\tau}}(\Phi_{\widehat{\tau}})^{\delta}\psi_k^2dxdy. \tag{3.23}$$

LEMMA 3.3. *Let $\widehat{\tau}=S$ or $\widehat{\tau}=T$, $\alpha\in(\frac{1}{2},1]$. Set $\hat{e}=(0,1)\times\{0\}$ and let $\Phi_{\hat{e}}$ and $\Phi_{\widehat{\tau}}$ be given in Lemma 3.1 and Lemma 3.2 respectively. For every univariate polynomial $\psi\in P_k$ of degree k and every $\varepsilon\in(0,1]$, there exists a constant $C=C_\alpha>0$ and an extension $w_{\hat{e}}\in H^1(\widehat{\tau})$ such that*

$$w_{\hat{e}}|_{\hat{e}}=\psi\cdot\Phi_{\hat{e}}^{\alpha} \quad and \quad w_{\hat{e}}|_{\partial\widehat{\tau}\setminus\hat{e}}=0, \tag{3.24}$$

$$\|w_{\hat{e}}\|^2_{L^2(\widehat{\tau})} \leqslant C\varepsilon\|\psi\Phi_{\hat{e}}^{\frac{\alpha}{2}}\|^2_{L^2(\hat{e})}, \tag{3.25}$$

$$\|\nabla w_{\hat{e}}\|^2_{L^2(\widehat{\tau})} \leqslant C(\varepsilon k^{2(2-\alpha)}+\varepsilon^{-1})\|\psi\Phi_{\hat{e}}^{\frac{\alpha}{2}}\|^2_{L^2(\hat{e})}. \tag{3.26}$$

Noting that F_τ is the element map for the element τ and e is the image of the edge $\hat{e}$ under F_τ [25], we define Φ_τ and Φ_e as follows:

$$\Phi_\tau = c_\tau\Phi_{\widehat{\tau}}\circ F_\tau^{-1}, \quad \Phi_e = c_e\Phi_{\hat{e}}\circ F_\tau^{-1},$$

with scaling factors $c_\tau, c_e>0$ chosen that

$$\|\Phi_\tau\|_{L^\infty(\tau)}=1, \quad \|\Phi_e\|_{L^\infty(e)}=1. \tag{3.27}$$

For F_{τ_U}, we have the same definition as that of F_τ.

THEOREM 3.3. *Let (y,p,u) and (y_{hp},p_{hp},u_{hp}) be the solutions of (2.4)–(2.6) and (2.13)–(2.15). Assume that A is a constant matrix, $(h'(u_{hp})-y_{hp}p_{hp})|_{\tau_U}$ is a polynomial of degree p_{τ_U} for any $\tau_U \in \mathscr{T}_U$, the solution u satisfying $\int_{\Omega_U} udx > 0$, and g' and h' are locally Lipschitz continuous in a neighborhood of y and u. Then for any ε satisfying $0<\varepsilon<3/2$, we have*

$$\sum_{i=1}^{3}\hat{\eta}_i^2 \leqslant C(\|u-u_{hp}\|_{H^1(\Omega_U)}+\|y-y_{hp}\|^2_{H^1(\Omega)}+\|p-p_{hp}\|^2_{H^1(\Omega)}+\varpi^2), \tag{3.28}$$

where

$$\hat{\eta}_1^2 = \sum_{\tau_U}\int_{\tau_U}\frac{h^2_{\tau_U}}{p^4_{\tau_U}}(\nabla(h'(u_{hp})-y_{hp}p_{hp}))^2,$$

$$\hat{\eta}_2^2 = \sum_{\tau}\int_{\tau}\frac{h^2_{\tau}}{p^4_{\tau}}(g'(y_{hp})+\mathrm{div}(A^*\nabla p_{hp})-u_{hp}p_{hp})^2+\sum_{e}\int_{e}\frac{h_e}{p_e^{3+2\varepsilon}}[(A^*\nabla p_{hp})\cdot n]^2,$$

$$\hat{\eta}_3^2 = \sum_{\tau}\int_{\tau}\frac{h^2_{\tau}}{p^4_{\tau}}(f+\mathrm{div}(A\nabla y_{hp})-u_{hp}y_{hp})^2+\sum_{e}\int_{e}\frac{h_e}{p_e^{3+2\varepsilon}}[(A\nabla y_{hp}\cdot n)]^2,$$

and

$$\begin{aligned}\varpi^2 = &\sum_{\tau}\frac{h^2_{\tau}}{p_{\tau}^{3-2\varepsilon}}\|f-\bar{f}\|^2_{L^2(\tau)}+\sum_{\tau}\frac{h^2_{\tau}}{p_{\tau}^{3-2\varepsilon}}\|u_{hp}y_{hp}-\overline{u_{hp}y_{hp}}\|^2_{L^2(\tau)}\\ &+\sum_{\tau}\frac{h^2_{\tau}}{p_{\tau}^{3-2\varepsilon}}\|g'(y_{hp})-\overline{g'(y_{hp})}\|^2_{L^2(\tau)}.\end{aligned}$$

Proof. It follows from (2.6) and $\int_{\Omega_U} udx>0$ that

$$h'(u)-yp=0, \quad a.e. \text{ in } \Omega_U. \tag{3.29}$$

It follows from the polynomial inverse estimates (3.23), (3.29), and the affine map F_{τ_U} that

$$\begin{aligned}&h^2_{\tau_U}\|\nabla(h'(u_{hp})-y_{hp}p_{hp})\|^2_{L^2(\tau_U)}\\ \leqslant\, &C\frac{p^4_{\tau_U}}{h^2_{\tau_U}}h^2_{\tau_U}\|(h'(u_{hp})-y_{hp}p_{hp})\|^2_{L^2(\tau_U)}\\ =\, &Cp^4_{\tau_U}\|(h'(u_{hp})-h'(u)-y_{hp}p_{hp}+yp)\|^2_{L^2(\tau_U)}\\ =\, &Cp^4_{\tau_U}\|(h'(u_{hp})-h'(u)-(p_{hp}-p)y_{hp}-(y_{hp}-y)p\|^2_{L^2(\tau_U)}\\ \leqslant\, &Cp^4_{\tau_U}(\|h'(u_{hp})-h'(u)\|^2_{L^2(\tau_U)}+\|p_{hp}-p\|^2_{L^2(\tau_U)}+\|y_{hp}-y\|^2_{L^2(\tau_U)}).\end{aligned}$$

Then we have

$$\begin{aligned}&\frac{h^2_{\tau_U}}{p^4_{\tau_U}}\|\nabla(h'(u_h)-y_{hp}p_{hp})\|^2_{L^2(\tau_U)}\\ \leqslant\, &C(\|h'(u_{hp})-h'(u)\|^2_{L^2(\tau_U)}+\|p_{hp}-p\|^2_{L^2(\tau_U)}+\|y_{hp}-y\|^2_{L^2(\tau_U)}).\end{aligned} \tag{3.30}$$

It follows from (3.30) that

$$\begin{aligned}\hat{\eta}_1^2 &= \sum_{\tau_U} \frac{h_{\tau_U}^2}{p_{\tau_U}^4} \|\nabla(h'(u_{hp}) - u_{hp}p_{hp})\|_{L^2(\tau_U)}^2 \\ &\leqslant C \sum_{\tau_U} (\|h'(u_{hp}) - h'(u)\|_{L^2(\tau_U)}^2 + \|p_{hp} - p\|_{L^2(\tau_U)}^2 + \|y_{hp} - y\|_{L^2(\tau_U)}^2) \\ &\leqslant C(\|u - u_{hp}\|_{H^1(\Omega_U)}^2 + \|p - p_{hp}\|_{H^1(\Omega)}^2 + \|y - y_{hp}\|_{H^1(\Omega)}^2). \end{aligned} \tag{3.31}$$

To bound $\hat{\eta}_2^2$, we define $v_\tau = (\overline{g'(y_{hp})} + \mathrm{div}(A^*\nabla p_{hp}) - \overline{u_{hp}p_{hp}})\Phi_\tau^\alpha$, $\alpha \in (0,1]$. We use the trivial extension by zero on $\Omega\backslash\tau$ so that v_τ can be viewed as an element of $H_0^1(\Omega)$. Then we have

$$\begin{aligned} &\|v_\tau \Phi_\tau^{-\alpha/2}\|_{L^2(\tau)}^2 \\ &= \int_\tau (\overline{g'(y_{hp})} + \mathrm{div}(A^*\nabla p_{hp}) - \overline{u_{hp}p_{hp}}) v_\tau \\ &= \int_\tau (g'(y_{hp}) + \mathrm{div}(A^*\nabla p_{hp}) - u_{hp}p_{hp} - g'(y) - \mathrm{div}(A^*\nabla p) + up) v_\tau \\ &\quad + \int_\tau (\overline{g'(y_{hp})} - g'(y_{hp})) v_\tau + \int_\tau (u_{hp}p_{hp} - \overline{u_{hp}p_{hp}}) v_\tau \\ &= -\int_\tau (A\nabla v_\tau)\nabla(p_{hp} - p) + \int_\tau (up - u_{hp}p_{hp} + g'(y_{hp}) - g'(y)) v_\tau \\ &\quad + \int_\tau (\overline{g'(y_{hp})} - g'(y_{hp})) v_\tau + \int_\tau (u_{hp}p_{hp} - \overline{u_{hp}p_{hp}}) v_\tau \\ &\leqslant C|p_{hp} - p|_{H^1(\tau)} \cdot |v_\tau|_{H^1(\tau)} + \|(g'(y_{hp}) - g'(y))\Phi_\tau^{\alpha/2}\|_{L^2(\tau)} \cdot \|v_\tau \Phi_\tau^{-\alpha/2}\|_{L^2(\tau)} \\ &\quad + \|(up - u_{hp}p_{hp})\Phi_\tau^{\alpha/2}\|_{L^2(\tau)} \cdot \|v_\tau \Phi_\tau^{-\alpha/2}\|_{L^2(\tau)} \\ &\quad + \|(g'(y_{hp}) - \overline{g'(y_{hp})})\Phi_\tau^{\alpha/2}\|_{L^2(\tau)} \cdot \|v_\tau \Phi_\tau^{-\alpha/2}\|_{L^2(\tau)} \\ &\quad + \|(u_{hp}p_{hp} - \overline{u_{hp}p_{hp}})\Phi_\tau^{\alpha/2}\|_{L^2(\tau)} \cdot \|v_\tau \Phi_\tau^{-\alpha/2}\|_{L^2(\tau)}. \end{aligned} \tag{3.32}$$

To estimate $|v_\tau|_{H^1(\tau)}$, we use the inverse estimates (3.22) and (3.23) and the affine map F_τ. Then we have for $\alpha > 1/2$

$$\begin{aligned} |v_\tau|_{H^1(\tau)}^2 &\leqslant 2\int_\tau \Phi_\tau^{2\alpha} |\nabla(\overline{g'(y_{hp})} + \mathrm{div}(A^*\nabla p_{hp}) - \overline{u_{hp}p_{hp}})|^2 \\ &\quad + 2\int_\tau (\overline{g'(y_{hp})} + \mathrm{div}(A^*\nabla p_{hp}) - \overline{u_{hp}p_{hp}})^2 |\nabla \Phi_\tau^\alpha|^2 \\ &\leqslant C\frac{p_\tau^{2(2-\alpha)}}{h_\tau^2} \int_\tau \Phi_\tau^\alpha (\overline{g'(y_{hp})} + \mathrm{div}(A^*\nabla p_{hp}) - \overline{u_{hp}p_{hp}})^2 \\ &\quad + C\frac{1}{h_\tau^2} \int_\tau \Phi_\tau^{2(\alpha-1)} (\overline{g'(y_{hp})} + \mathrm{div}(A^*\nabla p_{hp}) - \overline{u_{hp}p_{hp}})^2 \end{aligned}$$

$$
\begin{aligned}
&\leqslant C\frac{p_\tau^{2(2-\alpha)}}{h_\tau^2}\int_\tau \Phi_\tau^\alpha(\overline{g'(y_{hp})}+\mathrm{div}(A^*\nabla p_{hp})-\overline{u_{hp}p_{hp}})^2\\
&\leqslant Cp_\tau^{2(1-\alpha)}\frac{p_\tau^2}{h_\tau^2}\|v_\tau\Phi_\tau^{-\alpha/2}\|_{L^2(\tau)}^2.
\end{aligned}
\tag{3.33}
$$

Thus it follows from (3.32), (3.33), and (3.27) that

$$
\begin{aligned}
&\|v_\tau\Phi_\tau^{-\alpha/2}\|_{L^2(\tau)}\\
&\leqslant C(p_\tau^{1-\alpha}\frac{p_\tau}{h_\tau}|p_{hp}-p|_{H^1(\tau)}+\|g'(y_{hp})-g'(y)\|_{L^2(\tau)}+\|up-u_{hp}p_{hp}\|_{L^2(\tau)}\\
&\quad+\|g'(y_{hp})-\overline{g'(y_{hp})}\|_{L^2(\tau)}+\|u_{hp}p_{hp}-\overline{u_{hp}p_{hp}}\|_{L^2(\tau)}).
\end{aligned}
\tag{3.34}
$$

Combining (3.22) and (3.34), we have for $\beta>1/2$

$$
\begin{aligned}
&\|\overline{g'(y_{hp})}+\mathrm{div}(A^*\nabla p_{hp})-\overline{u_{hp}p_{hp}}\|_{L^2(\tau)}\\
&\leqslant Cp_\tau^\beta\|(\overline{g'(y_{hp})}+\mathrm{div}(A^*\nabla p_{hp})-\overline{u_{hp}p_{hp}})\Phi_\tau^{\beta/2}\|_{L^2(\tau)}\\
&\leqslant Cp_\tau^\beta\Big(p_\tau^{1-\beta}\frac{p_\tau}{h_\tau}|p_{hp}-p|_{H^1(\tau)}+\|g'(y_{hp})-g'(y)\|_{L^2(\tau)}+\|up-u_{hp}p_{hp}\|_{L^2(\tau)}\\
&\quad+\|g'(y_{hp})-\overline{g'(y_{hp})}\|_{L^2(\tau)}+\|u_{hp}p_{hp}-\overline{u_{hp}p_{hp}}\|_{L^2(\tau)}\Big).
\end{aligned}
\tag{3.35}
$$

Setting $\beta=1/2+\varepsilon\ (0<\varepsilon<3/2)$ in the above result (3.35), we have

$$
\begin{aligned}
&\frac{h_\tau^2}{p_\tau^4}\|\overline{g'(y_{hp})}+\mathrm{div}(A^*\nabla p_{hp})-\overline{u_{hp}p_{hp}}\|_{L^2(\tau)}^2\\
&\leqslant C|p_{hp}-p|_{H^1(\tau)}^2+C\frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|g'(y_{hp})-g'(y)\|_{L^2(\tau)}^2+C\frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|up-u_{hp}p_{hp}\|_{L^2(\tau)}^2\\
&\quad+C\frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|g'(y_{hp})-\overline{g'(y_{hp})}\|_{L^2(\tau)}^2+C\frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|u_{hp}p_{hp}-\overline{u_{hp}p_{hp}}\|_{L^2(\tau)}^2.
\end{aligned}
\tag{3.36}
$$

It follows from the inequality (3.36) that

$$
\begin{aligned}
&\sum_\tau\frac{h_\tau^2}{p_\tau^4}\|g'(y_{hp})+\mathrm{div}(A^*\nabla p_{hp})-u_{hp}p_{hp}\|_{L^2(\tau)}^2\\
&\leqslant C\sum_\tau\frac{h_\tau^2}{p_\tau^4}\|\overline{g'(y_{hp})}+\mathrm{div}(A^*\nabla p_{hp})-\overline{u_{hp}p_{hp}}\|_{L^2(\tau)}^2\\
&\quad+C\sum_\tau\frac{h_\tau^2}{p_\tau^4}\|g'(y_{hp})-\overline{g'(y_{hp})}\|_{L^2(\tau)}^2+C\sum_\tau\frac{h_\tau^2}{p_\tau^4}\|u_{hp}p_{hp}-\overline{u_{hp}p_{hp}}\|_{L^2(\tau)}^2\\
&\leqslant C\sum_\tau|p_{hp}-p|_{H^1(\tau)}^2+C\sum_\tau\frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|g'(y_{hp})-g'(y)\|_{L^2(\tau)}^2\\
&\quad+C\sum_\tau\frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|up-u_{hp}p_{hp}\|_{L^2(\tau)}^2+C\sum_\tau\frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|g'(y_{hp})-\overline{g'(y_{hp})}\|_{L^2(\tau)}^2\\
&\quad+C\sum_\tau\frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|u_{hp}p_{hp}-\overline{u_{hp}p_{hp}}\|_{L^2(\tau)}^2
\end{aligned}
$$

$$\leqslant C(\|p_{hp}-p\|^2_{H^1(\Omega)}+\|g'(y_{hp})-g'(y)\|^2_{L^2(\Omega)}+\|up-u_{hp}p_{hp}\|^2_{L^2(\Omega)})$$
$$+C\sum_\tau \frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|u_{hp}p_{hp}-\overline{u_{hp}p_{hp}}\|^2_{L^2(\tau)}$$
$$+C\sum_\tau \frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|g'(y_{hp})-\overline{g'(y_{hp})}\|^2_{L^2(\tau)}. \tag{3.37}$$

Next, let $e=\partial\tau_e^1\cap\partial\tau_e^2$ and $\bar{\tau}_e=\overline{\tau}_e^1\cup\overline{\tau}_e^2$. We construct a function $w_e|_e=[(A^*\nabla p_{hp})\cdot n]\Phi_e^\alpha$ in Lemma 3.3 where $w_e|_{\tau_e^1}$ and $w_e|_{\tau_e^2}$ are defined as the affine transformation of $w_{\hat{e}}$ on the reference element. It is easy to see that the univariate polynomial ψ is corresponds to $[(A^*\nabla p_{hp})\cdot n]$. Then $w_e\in H_0^1(\tau_e)$, and we can use the trivial extension of w_e by zero on $\Omega\backslash\tau_e$ so that w_e can be viewed as an element of $H_0^1(\Omega)$. Then we obtain

$$\begin{aligned}
\|w_e\Phi_e^{-\alpha/2}\|^2_{L^2(e)} &= \|[(A^*\nabla p_{hp})\cdot n]\Phi_e^{\alpha/2}\|^2_{L^2(e)}\\
&=\int_e [(A^*\nabla p_{hp})\cdot n]w_e\\
&=\int_{\tau_e}(A^*\nabla p_{hp})\cdot\nabla w_e+\int_{\tau_e}\mathrm{div}(A^*\nabla p_{hp})w_e\\
&=\int_{\tau_e}(A^*\nabla(p_{hp}-p))\cdot\nabla w_e+\int_{\tau_e}(g'(y)+\mathrm{div}(A^*\nabla p_{hp})-up)w_e\\
&=\int_{\tau_e}(A\nabla w_e)\cdot\nabla(p_{hp}-p)+\int_{\tau_e}(g'(y_{hp})+\mathrm{div}(A^*\nabla p_{hp})-u_{hp}p_{hp})w_e\\
&\quad+\int_{\tau_e}(u_{hp}p_{hp}-g'(y_{hp})-up+g'(y))w_e\\
&\leqslant C|p_{hp}-p|_{H^1(\tau_e)}\cdot|w_e|_{H^1(\tau_e)}+\|u_{hp}p_{hp}-up\|_{L^2(\tau_e)}\cdot\|w_e\|_{L^2(\tau_e)}\\
&\quad+\|g'(y_{hp})+\mathrm{div}(A^*\nabla p_{hp})-u_{hp}p_{hp}\|_{L^2(\tau_e)}\cdot\|w_e\|_{L^2(\tau_e)}\\
&\quad+\|g'(y)-g'(y_{hp})\|_{L^2(\tau_e)}\cdot\|w_e\|_{L^2(\tau_e)}.
\end{aligned}\tag{3.38}$$

For the case $\alpha\in(1/2,1]$, it follows from (3.25) and (3.26) that

$$|w_e|^2_{H^1(\tau_e)}\leqslant C\frac{1}{h_\tau}(\varepsilon p_\tau^{2(2-\alpha)}+\varepsilon^{-1})\|[(A^*\nabla p_{hp})\cdot n]\Phi_e^{\alpha/2}\|^2_{L^2(e)}, \tag{3.39}$$
$$\|w_e\|^2_{L^2(\tau_e)}\leqslant Ch_\tau\varepsilon\|[(A^*\nabla p_{hp})\cdot n]\Phi_e^{\alpha/2}\|^2_{L^2(e)}. \tag{3.40}$$

Then it follows from (3.38), (3.39), and (3.40) that

$$\begin{aligned}
&\|[(A^*\nabla p_{hp})\cdot n]\Phi_e^{\alpha/2}\|^2_{L^2(e)}\\
&\leqslant C\frac{1}{h_\tau}(\varepsilon p_\tau^{2(2-\alpha)}+\varepsilon^{-1})|p_{hp}-p|^2_{H^1(\tau_e)}+Ch_\tau\varepsilon\|u_{hp}p_{hp}-up\|^2_{L^2(\tau_e)}\\
&\quad+Ch_\tau\varepsilon\|g'(y)-g'(y_{hp})\|^2_{L^2(\tau_e)}+Ch_\tau\varepsilon\|g'(y_{hp})+\mathrm{div}(A^*\nabla p_{hp})-u_{hp}p_{hp}\|^2_{L^2(\tau_e)}.
\end{aligned}\tag{3.41}$$

It follows from (3.36) and (3.41) that for $\beta > 1/2$

$$
\begin{aligned}
&\|[(A^*\nabla p_{hp})\cdot n]\|^2_{L^2(e)}\\
&\leqslant Cp_\tau^{2\beta}\|[(A^*\nabla p_{hp})\cdot n]\Phi_e^{\beta/2}\|^2_{L^2(e)}\\
&\leqslant Cp_\tau^{2\beta}\frac{1}{h_\tau}(\varepsilon p_\tau^{2(2-\beta)}+\varepsilon^{-1})|p_{hp}-p|^2_{H^1(\tau_e)}+Cp_\tau^{2\beta}h_\tau\varepsilon\|u_{hp}p_{hp}-up\|^2_{L^2(\tau_e)}\\
&\quad+Cp_\tau^{2\beta}h_\tau\varepsilon\|g'(y)-g'(y_{hp})\|^2_{L^2(\tau_e)}+Cp_\tau^{2\beta}h_\tau\varepsilon\|g'(y_{hp})+\mathrm{div}(A^*\nabla p_{hp})-u_{hp}p_{hp}\|^2_{L^2(\tau_e)}\\
&\leqslant Cp_\tau^{2\beta}\frac{1}{h_\tau}(\varepsilon p_\tau^{2(2-\beta)}+\varepsilon^{-1})|p_{hp}-p|^2_{H^1(\tau_e)}+Cp_\tau^{2\beta}h_\tau\varepsilon\|u_{hp}p_{hp}-up\|^2_{L^2(\tau_e)}\\
&\quad+Cp_\tau^{2\beta}h_\tau\varepsilon\|g'(y)-g'(y_{hp})\|^2_{L^2(\tau_e)}+Cp_\tau^{2\beta}h_\tau\varepsilon\|\overline{g'(y_{hp})}+\mathrm{div}(A^*\nabla p_{hp})-\overline{u_{hp}p_{hp}}\|^2_{L^2(\tau_e)}\\
&\quad+Cp_\tau^{2\beta}h_\tau\varepsilon\|u_{hp}p_{hp}-\overline{u_{hp}p_{hp}}\|^2_{L^2(\tau_e)}+Cp_\tau^{2\beta}h_\tau\varepsilon\|g'(y_{hp})-\overline{g'(y_{hp})}\|^2_{L^2(\tau_e)}\\
&\leqslant Cp_\tau^{2\beta}\frac{1}{h_\tau}(\varepsilon p_\tau^{2(2-\beta)}+\varepsilon^{-1})|p_{hp}-p|^2_{H^1(\tau_e)}+Cp_\tau^{4+2\beta}\frac{1}{h_\tau}\varepsilon|p_{hp}-p|^2_{H^1(\tau_e)}\\
&\quad+Cp_\tau^{1+2\varepsilon+2\beta}h_\tau\varepsilon\|u_{hp}p_{hp}-up\|^2_{L^2(\tau_e)}+Cp_\tau^{1+2\varepsilon+2\beta}h_\tau\varepsilon\|g'(y)-g'(y_{hp})\|^2_{L^2(\tau_e)}\\
&\quad+Cp_\tau^{1+2\varepsilon+2\beta}h_\tau\varepsilon\|u_{hp}p_{hp}-\overline{u_{hp}p_{hp}}\|^2_{L^2(\tau_e)}\\
&\quad+Cp_\tau^{1+2\varepsilon+2\beta}h_\tau\varepsilon\|g'(y_{hp})-\overline{g'(y_{hp})}\|^2_{L^2(\tau_e)}.
\end{aligned}\tag{3.42}
$$

Setting $\varepsilon = 1/p_\tau^2$ and $\beta = 1/2+\varepsilon$ $(0<\varepsilon<3/2)$ in (3.42), we have

$$
\begin{aligned}
&\|[(A^*\nabla p_{hp})\cdot n]\|^2_{L^2(e)}\\
&\leqslant C\frac{p_\tau^{3+2\varepsilon}}{h_\tau}|p_{hp}-p|^2_{H^1(\tau_e)}+Cp_\tau^{4\varepsilon}h_\tau\|u_{hp}p_{hp}-up\|^2_{L^2(\tau_e)}+Cp_\tau^{4\varepsilon}h_\tau\|g'(y)-g'(y_{hp})\|^2_{L^2(\tau_e)}\\
&\quad+Cp_\tau^{4\varepsilon}h_\tau\|u_{hp}p_{hp}-\overline{u_{hp}p_{hp}}\|^2_{L^2(\tau_e)}+Cp_\tau^{4\varepsilon}h_\tau\|g'(y_{hp})-\overline{g'(y_{hp})}\|^2_{L^2(\tau_e)}.
\end{aligned}\tag{3.43}
$$

Then it follows from (3.43) that

$$
\begin{aligned}
&\sum_e\frac{h_e}{p_e^{3+2\varepsilon}}\|[(A^*\nabla p_{hp})\cdot n]\|^2_{L^2(e)}\\
&\leqslant C\sum_e|p_{hp}-p|^2_{H^1(\tau_e)}+C\sum_e\frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|u_{hp}p_{hp}-up\|^2_{L^2(\tau_e)}\\
&\quad+C\sum_e\frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|g'(y)-g'(y_{hp})\|^2_{L^2(\tau_e)}+C\sum_e\frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|g'(y_{hp})-\overline{g'(y_{hp})}\|^2_{L^2(\tau_e)}\\
&\quad+C\sum_e\frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|u_{hp}p_{hp}-\overline{u_{hp}p_{hp}}\|^2_{L^2(\tau_e)}\\
&\leqslant C(\|p-p_{hp}\|^2_{H^1(\Omega)}+\|u_{hp}p_{hp}-up\|^2_{L^2(\Omega)}+\|g'(y)-g'(y_{hp})\|^2_{L^2(\Omega)})\\
&\quad+C\sum_\tau\frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|g'(y_{hp})-\overline{g'(y_{hp})}\|^2_{L^2(\tau)}+C\sum_\tau\frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|u_{hp}p_{hp}-\overline{u_{hp}p_{hp}}\|^2_{L^2(\tau)}.
\end{aligned}\tag{3.44}
$$

Therefore, it follows from (3.44) that

$$\begin{aligned}\hat{\eta}_2^2 &= \sum_\tau \frac{h_\tau^2}{p_\tau^4}\|g'(y_{hp})+\mathrm{div}(A^*\nabla p_{hp})-u_{hp}p_{hp}\|_{L^2(\tau)}^2+\sum_e \frac{h_e}{p_e^{3+2\varepsilon}}\|[(A^*\nabla p_{hp})\cdot n]\|_{L^2(e)}^2\\&\leqslant C(\|p-p_{hp}\|_{H^1(\Omega)}^2+\|u_{hp}p_{hp}-up\|_{L^2(\Omega)}^2+\|g'(y)-g'(y_{hp})\|_{L^2(\Omega)}^2)\\&\quad+C\sum_\tau \frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|g'(y_{hp})-\overline{g'(y_{hp})}\|_{L^2(\tau)}^2+C\sum_\tau \frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|u_{hp}p_{hp}-\overline{u_{hp}p_{hp}}\|_{L^2(\tau)}^2.\end{aligned} \tag{3.45}$$

Note that

$$\begin{aligned}&\|u_{hp}p_{hp}-up\|_{L^2(\Omega)}^2\\&\leqslant C\|u_{hp}\|_{L^4(\Omega)}^2\cdot\|p-p_{hp}\|_{L^4(\Omega)}^2+C\|p\|_{L^4(\Omega)}^2\cdot\|u-u_{hp}\|_{L^4(\Omega_U)}^2\\&\leqslant C\|p-p_{hp}\|_{H^1(\Omega)}^2+C\|u-u_{hp}\|_{H^1(\Omega_U)}^2.\end{aligned} \tag{3.46}$$

Therefore, it follows from (3.45) and (3.46) that

$$\begin{aligned}\hat{\eta}_2^2 &\leqslant C(\|p-p_{hp}\|_{H^1(\Omega)}^2+\|y-y_{hp}\|_{H^1(\Omega)}^2+\|u-u_{hp}\|_{H^1(\Omega_U)}^2)\\&\quad+C\sum_\tau \frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|g'(y_{hp})-\overline{g'(y_{hp})}\|_{L^2(\tau)}^2+C\sum_\tau \frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|u_{hp}p_{hp}-\overline{u_{hp}p_{hp}}\|_{L^2(\tau)}^2.\end{aligned} \tag{3.47}$$

Similarly,

$$\begin{aligned}\hat{\eta}_3^2 &= \sum_\tau \frac{h_\tau^2}{p_\tau^4}\|f+\mathrm{div}(A\nabla y_{hp})-u_{hp}y_{hp}\|_{L^2(\tau)}^2+\sum_e \frac{h_e}{p_e^{3+2\varepsilon}}\|[(A\nabla y_{hp}\cdot n)]\|_{L^2(e)}^2\\&\leqslant C(\|p-p_{hp}\|_{H^1(\Omega)}^2+\|y-y_{hp}\|_{H^1(\Omega)}^2+\|u-u_{hp}\|_{H^1(\Omega_U)}^2)+C\sum_\tau \frac{h_\tau^2}{p_\tau^{3-2\varepsilon}}\|f-\bar{f}\|_{L^2(\tau)}^2.\end{aligned} \tag{3.48}$$

Then, (3.28) follows from (3.31), (3.47), and (3.48). The proof of Theorem 3.3 is completed. □

4. Conclusion and future works

In this paper, we use the hp finite element approximation for both the state and the co-state and the hp discontinuous Galerkin finite element approximation for the control. And then we derive a posteriori error estimates for the optimal control problem governed by bilinear elliptic equations in L^2-H^1 norms. To our best knowledge in the context of optimal control problems, these posteriori error estimates for the bilinear optimal control problems are new.

In future, we shall consider the hp mixed finite element method for optimal control problems. Furthermore, we shall consider a posteriori error estimates and superconvergence of the hp mixed finite element solutions for optimal control problems.

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Zuliang Lu
Key Laboratory for Nonlinear Science and System Structure
Chongqing Three Gorges University
Chongqing 404000, P. R. China
and
Research Center for Mathematics and Economics
Tianjin University of Finance and Economics
Tianjin 300222, P. R. China
e-mail: zulianglux@126.com