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# OPTIMAL ONE–PARAMETER MEAN BOUNDS FOR THE CONVEX COMBINATION OF ARITHMETIC AND LOGARITHMIC MEANS

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*Abstract.* We find the greatest value $p_1 = p_1(\alpha)$ and the least value $p_2 = p_2(\alpha)$ such that the double inequality $J_{p_1}(a,b) < \alpha A(a,b) + (1-\alpha)L(a,b) < J_{p_2}(a,b)$ holds for any $\alpha \in (0,1)$ and all $a,b > 0$ with $a \neq b$. Here, $A(a,b)$, $L(a,b)$ and $J_p(a,b)$ denote the arithmetic, logarithmic and $p$-th one-parameter means of two positive numbers $a$ and $b$, respectively.

## 1. Introduction

For $p \in \mathbb{R}$ the $p$-th one-parameter mean $J_p(a,b)$, arithmetic mean $A(a,b)$ and logarithmic mean $L(a,b)$ of two positive real numbers $a$ and $b$ with $a \neq b$ are defined by

$$J_p(a,b) = \begin{cases} \frac{p(a^{p+1}-b^{p+1})}{(p+1)(a^p-b^p)}, & a \neq b,\ p \neq 0, -1, \\ \frac{a-b}{\log a - \log b}, & a \neq b,\ p = 0, \\ \frac{ab(\log a - \log b)}{a-b}, & a \neq b,\ p = -1, \end{cases} \tag{1.1}$$

$A(a,b) = (a+b)/2$ and $L(a,b) = (a-b)/(\log a - \log b)$, respectively.

Recently, the one-parameter, arithmetic and logarithmic means have been the subject of intensive research. In particular, many remarkable inequalities for these means can be found in the literature [1–6]. It might be surprising that the logarithmic mean has applications in physics, economics, and even in meteorology [7–9]. In [7] the authors study a variant of Jensen's functional equation involving the logarithmic mean, which appears in a heat conduction problem.

It is well-known that the one-parameter mean $J_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$. Many mean values are the special case of the one-parameter mean, for example

$J_1(a,b) = (a+b)/2 = A(a,b)$, the arithmetic mean,

$J_{1/2}(a,b) = (a+\sqrt{ab}+b)/3 = He(a,b)$, the Heronian mean,

$J_{-1/2}(a,b) = \sqrt{ab} = G(a,b)$, the geometric mean

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and

$$J_{-2}(a,b) = 2ab/(a+b) = H(a,b), \qquad \text{the harmonic mean.}$$

For $r \in \mathbb{R}$ the power mean $M_r(a,b)$ of order $r$ of two positive numbers $a$ and $b$ is defined by

$$M_r(a,b) = \begin{cases} (\frac{a^r+b^r}{2})^{1/r}, & r \neq 0, \\ \sqrt{ab}, & r = 0. \end{cases} \tag{1.2}$$

The main properties of the power mean are given in [10].

Gao and Niu [5] presented the best possible parameters $p = p(\alpha,\beta)$, $q = q(\alpha,\beta)$, $s_1 = s_1(\alpha,\beta)$ and $s_2 = s_2(\alpha,\beta)$ such that the double inequalities

$$J_p(a,b) \leqslant A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) \leqslant J_q(a,b)$$

$$G_{s_1}(a,b) \leqslant A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) \leqslant G_{s_2}(a,b)$$

hold for all $a,b > 0$ and $\alpha,\beta > 0$ with $\alpha+\beta < 1$, where $G_s(a,b) = [(a^s+b^s)/(a+b)]^{1/(s-1)}$ $(s \neq 1)$ is the Gini mean.

In [6], the authors found the optimal upper and lower one-parameter mean bounds for the Second Seiffert mean $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$.

Xia, Chu and Wang [11] answered the question: for any $\alpha \in (0,1)$, what are the greatest value $p$ and the least value $q$, such that the double inequality $M_p(a,b) < \alpha A(a,b) + (1-\alpha)L(a,b) < M_q(a,b)$ for all $a,b > 0$ with $a \neq b$.

The purpose of this paper is to find the greatest value $p_1 = p_1(\alpha)$ and the least value $p_2 = p_2(\alpha)$ such that the double inequality $J_{p_1}(a,b) < \alpha A(a,b) + (1-\alpha)L(a,b) < J_{p_2}(a,b)$ holds for any $\alpha \in (0,1)$ and all $a,b > 0$ with $a \neq b$.

## 2. Lemmas

In order to establish our main result we need several lemmas, which we present in this section.

LEMMA 2.1. *If* $\alpha \in (0,1)$ *and* $x \in (1,\infty)$, *then*

$$(1-\alpha)x^{\alpha+1} - (1+\alpha)x^{\alpha} + (1+\alpha)x - (1-\alpha) > 0. \tag{2.1}$$

*Proof.* Let

$$f(x) = (1-\alpha)x^{\alpha+1} - (1+\alpha)x^{\alpha} + (1+\alpha)x - (1-\alpha). \tag{2.2}$$

Then simple computations lead to

$$f(1) = 0, \tag{2.3}$$

$$f'(x) = (1-\alpha)(1+\alpha)x^{\alpha} - \alpha(1+\alpha)x^{\alpha-1} + (1+\alpha), \tag{2.4}$$

$$f'(1) = 2(1-\alpha^2) > 0 \tag{2.5}$$

and

$$f''(x) = \alpha(1-\alpha)(1+\alpha)x^{\alpha-2}(x+1) > 0. \tag{2.6}$$

From (2.5) and (2.6) we clearly see that $f(x)$ is strictly increasing in [0,1]. Therefore, Lemma 2.1 follows from (2.3) and the monotonicity of $f(x)$. □

LEMMA 2.2. *If* $\alpha \in (0,1)$ *and* $x \in (1,\infty)$, *then*

$$\begin{aligned}&(1-\alpha)^2x^{2\alpha+2}+2(1-\alpha)(1+\alpha)x^{2\alpha+1}+(1+\alpha)^2x^{2\alpha}\\&-2(1-\alpha)(1+\alpha)x^{\alpha+2}-4(1+\alpha^2)x^{\alpha+1}-2(1-\alpha)(1+\alpha)x^{\alpha}\\&+(1+\alpha)^2x^2+2(1+\alpha)(1-\alpha)x+(1-\alpha)^2>0.\end{aligned} \tag{2.7}$$

*Proof.* Let

$$\begin{aligned}g(x) = {}&(1-\alpha)^2x^{2\alpha+2}+2(1-\alpha)(1+\alpha)x^{2\alpha+1}+(1+\alpha)^2x^{2\alpha}\\&-2(1-\alpha)(1+\alpha)x^{\alpha+2}-4(1+\alpha^2)x^{\alpha+1}-2(1-\alpha)\\&\times(1+\alpha)x^{\alpha}+(1+\alpha)^2x^2+2(1+\alpha)(1-\alpha)x\\&+(1-\alpha)^2,\end{aligned} \tag{2.8}$$

$g_1(x) = g'(x)/[2(1+\alpha)]$, $g_2(x) = g_1{}''(x)/[\alpha(1-\alpha)x^{\alpha-3}]$, $g_3(x) = g_2'(x)/2$, $g_4(x) = g_3{}'(x)/(1+\alpha)$ and $g_5(x) = g_4{}'(x)/[\alpha(1-\alpha)x^{\alpha-3}]$. Then simple computations lead to

$$g(1) = 0, \tag{2.9}$$

$$\begin{aligned}g_1(x) = {}&(1-\alpha)^2x^{2\alpha+1}+(1-\alpha)(1+2\alpha)x^{2\alpha}+\alpha(1+\alpha)x^{2\alpha-1}\\&-(1-\alpha)(2+\alpha)x^{\alpha+1}-2(1+\alpha^2)x^{\alpha}-\alpha(1-\alpha)x^{\alpha-1}\\&+(1+\alpha)x+(1-\alpha),\end{aligned}$$

$$g_1(1) = 0, \tag{2.10}$$

$$\begin{aligned}g_1{}'(x) = {}&(1-\alpha)^2(2\alpha+1)x^{2\alpha}+2\alpha(1-\alpha)(2\alpha+1)x^{2\alpha-1}\\&+\alpha(1+\alpha)(2\alpha-1)x^{2\alpha-2}-(1-\alpha)(1+\alpha)(2+\alpha)x^{\alpha}\\&-2\alpha(1+\alpha^2)x^{\alpha-1}+\alpha(1-\alpha)^2x^{\alpha-2}+(1+\alpha),\end{aligned}$$

$$g_1{}'(1) = 0, \tag{2.11}$$

$$\begin{aligned}g_2(x) = {}&2(1-\alpha)(2\alpha+1)x^{\alpha+2}+2(2\alpha+1)(2\alpha-1)x^{\alpha+1}\\&-2(1+\alpha)(2\alpha-1)x^{\alpha}-(1+\alpha)(2+\alpha)x^2\\&+2(1+\alpha^2)x-(1-\alpha)(2-\alpha),\end{aligned}$$

$$g_2(1)=0, \tag{2.12}$$

$$\begin{aligned} g_3(x) = &(1-\alpha)(2\alpha+1)(\alpha+2)x^{\alpha+1}+(\alpha+1)(2\alpha+1)(2\alpha-1)x^{\alpha} \\ &-\alpha(1+\alpha)(2\alpha-1)x^{\alpha-1}-(1+\alpha)(2+\alpha)x+(1+\alpha^2), \end{aligned}$$

$$g_3(1)=0, \tag{2.13}$$

$$\begin{aligned} g_4(x) = &(1-\alpha)(2\alpha+1)(\alpha+2)x^{\alpha}+\alpha(2\alpha+1)(2\alpha-1)x^{\alpha-1} \\ &-\alpha(\alpha-1)(2\alpha-1)x^{\alpha-2}-2-\alpha, \end{aligned}$$

$$g_4(1)=0 \tag{2.14}$$

and

$$g_5(x)=(2\alpha+1)(\alpha+2)x^2-(2\alpha+1)(2\alpha-1)x-(2-\alpha)(2\alpha-1). \tag{2.15}$$

From (2.15) we know that $g_5'(x)=2(2\alpha+1)(\alpha+2)x+(1+2\alpha)(1-2\alpha) > g_5'(1)=10\alpha+5>0$ for $x\in[1,\infty)$, which implies that $g_5(x)$ is strictly increasing in $[1,\infty)$. Hence $g_5(x)>g_5(1)=5>0$ for $x\in[1,\infty)$, and $g_4(x)$ is strictly increasing in $[1,\infty)$.

Therefore, Lemma 2.2 follows from (2.8)–(2.14) and the monotonicity of $g_4(x)$. □

LEMMA 2.3. *If $\alpha\in(0,1)$, $p=\alpha/(2-\alpha)$ and $x\in(1,\infty)$, then*

$$\begin{aligned} &-2(1-\alpha)x^{2p}-\alpha x^{2p-1}+2p(1-\alpha)x^{p+1}-2(2\alpha-\alpha p-2)x^p \\ &+2p(1-\alpha)x^{p-1}-\alpha x-2(1-\alpha)>0. \end{aligned} \tag{2.16}$$

*Proof.* Let

$$\begin{aligned} h(x) = &-2(1-\alpha)x^{2p}-\alpha x^{2p-1}+2p(1-\alpha)x^{p+1} \\ &-2(2\alpha-\alpha p-2)x^p+2p(1-\alpha)x^{p-1} \\ &-\alpha x-2(1-\alpha), \end{aligned} \tag{2.17}$$

$h_1(x)=h''(x)/(2x^{p-3})$ and $h_2(x)={h_1}'''(x)/[p(1-p)x^{p-3}]$. Then simple computations lead to

$$h(1)=0, \tag{2.18}$$

$$\begin{aligned} h'(x) = &-4p(1-\alpha)x^{2p-1}-\alpha(2p-1)x^{2p-2}+2(1-\alpha)p(p+1)x^p \\ &-2p(2\alpha-\alpha p-2)x^{p-1}+2(1-\alpha)p(p-1)x^{p-2}-\alpha, \end{aligned}$$

$$h'(1)=0, \tag{2.19}$$

$$\begin{aligned} h_1(x) = &-2p(1-\alpha)(2p-1)x^{p+1}-\alpha(2p-1)(p-1)x^p \\ &+(1-\alpha)p^2(p+1)x^2-p(2\alpha-\alpha p-2)(p-1)x \\ &+(1-\alpha)p(p-1)(p-2), \end{aligned}$$

$$h_1(1)=0, \tag{2.20}$$

$$\begin{aligned} h_1{}'(x) = &-2p(1-\alpha)(2p-1)(p+1)x^p-\alpha p(2p-1)(p-1)x^{p-1} \\ &+2(1-\alpha)p^2(p+1)x-p(2\alpha-\alpha p-2)(p-1), \end{aligned}$$

$$h_1{}'(1)=0, \tag{2.21}$$

$$\begin{aligned} h_1{}''(x) = &-2p^2(1-\alpha)(2p-1)(p+1)x^{p-1}-\alpha p(2p-1)(p-1)^2x^{p-2} \\ &+2(1-\alpha)p^2(p+1), \end{aligned}$$

$$h_1{}''(1)=\frac{12\alpha^2(1-\alpha)^2}{(2-\alpha)^3}>0, \tag{2.22}$$

$$\lim_{x\to+\infty} h_1{}''(x)=2(1-\alpha)p^2(p+1)>0, \tag{2.23}$$

and

$$h_2(x)=(2p-1)[2p(1-\alpha)(p+1)x+\alpha(1-p)(2-p)]. \tag{2.24}$$

We divide the proof into two cases.

*Case* 1. If $\alpha\in(0,2/3]$, then $p\in(0,1/2]$. From (2.24) we clearly see that $h_1''(x)$ is strictly decreasing in $[1,\infty)$, then (2.23) leads to

$$h_1{}''(x)>\lim_{x\to+\infty} h_1{}''(x)>0 \tag{2.25}$$

for $x\in[1,\infty)$. Inequality (2.25) implies that $h_1{}'(x)$ is strictly increasing in $[1,\infty)$. Therefore, Lemma 2.3 follows from (2.17)–(2.21) and the monotonicity of $h_1{}'(x)$.

*Case* 2. If $\alpha\in(2/3,1)$, then $p\in(1/2,1)$. From (2.24) we clearly see that $h_1{}''(x)$ is strictly increasing in $[1,\infty)$, then (2.22) leads to

$$h_1{}''(x)>h_1{}''(1)>0 \tag{2.26}$$

for $x\in(1,\infty)$. Inequality (2.26) implies that $h_1{}'(x)$ is strictly increasing in $[1,\infty)$. Therefore, Lemma 2.3 follows from (2.17)–(2.21) and the monotonicity of $h_1{}'(x)$. □

## 3. Main result

THEOREM 3.1. *Inequality*

$$J_{\frac{\alpha}{2-\alpha}}(a,b) < \alpha A(a,b) + (1-\alpha)L(a,b) < J_\alpha(a,b)$$

*holds for any* $\alpha \in (0,1)$ *and all* $a,b>0$ *with* $a \neq b$, *and* $J_{\frac{\alpha}{2-\alpha}}(a,b)$ *and* $J_\alpha(a,b)$ *are the best possible lower and upper one-parameter mean bounds for the sum* $\alpha A(a,b) + (1-\alpha)L(a,b)$, *respectively.*

*Proof.* At first, we prove that

$$\alpha A(a,b) + (1-\alpha)L(a,b) < J_\alpha(a,b) \tag{3.1}$$

for any $\alpha \in (0,1)$ and all $a,b>0$ with $a \neq b$.

Without loss of generality, we assume that $a>b$. Let $x = a/b > 1$, then from (1.1) we get

$$\begin{aligned} & J_\alpha(a,b) - \alpha A(a,b) - (1-\alpha)L(a,b) \\ = & \frac{\alpha(x^{\alpha+1}-1)}{(1+\alpha)(x^\alpha-1)} - \frac{\alpha(1+x)}{2} - \frac{(1-\alpha)(x-1)}{\log x} \\ = & \frac{\alpha[(1-\alpha)x^{\alpha+1}-(1+\alpha)x^\alpha+(1+\alpha)x-(1-\alpha)]}{2(1+\alpha)(x^\alpha-1)\log x} F(x), \end{aligned} \tag{3.2}$$

where

$$F(x) = \log x - \frac{2(1-\alpha)(1+\alpha)(x^{\alpha+1}-x^\alpha-x+1)}{\alpha[(1-\alpha)x^{\alpha+1}-(1+\alpha)x^\alpha+(1+\alpha)x-(1-\alpha)]}. \tag{3.3}$$

Simple computations lead to

$$\lim_{x\to 1^+} F(x) = 0, \tag{3.4}$$

$$\begin{aligned} F'(x) &= \frac{1}{x} + \frac{4(1-\alpha)(1+\alpha)(x^{2\alpha}-x^{\alpha+1}-x^{\alpha-1}+1)}{[(1-\alpha)x^{\alpha+1}-(1+\alpha)x^\alpha+(1+\alpha)x-(1-\alpha)]^2} \\ &= \frac{F_1(x)}{x[(1-\alpha)x^{\alpha+1}-(1+\alpha)x^\alpha+(1+\alpha)x-(1-\alpha)]^2}, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} F_1(x) = & (1-\alpha)^2x^{2\alpha+2} + 2(1-\alpha)(1+\alpha)x^{2\alpha+1} + (1+\alpha)^2x^{2\alpha} \\ & -2(1-\alpha)(1+\alpha)x^{\alpha+2} - 4(1+\alpha^2)x^{\alpha+1} - 2(1-\alpha) \\ & \times(1+\alpha)x^\alpha + (1+\alpha)^2x^2 + 2(1+\alpha)(1-\alpha)x + (1-\alpha)^2. \end{aligned} \tag{3.6}$$

It follows from (3.6) and Lemma 2.2 that

$$F_1(x) > 0 \tag{3.7}$$

for $x \in (1,\infty)$ and $\alpha \in (0,1)$. Therefore, inequality (3.1) follows from (3.2)–(3.5) and (3.7) together with Lemma 2.1.

Next, we prove that

$$\alpha A(a,b) + (1-\alpha)L(a,b) > J_{\frac{\alpha}{2-\alpha}}(a,b) \tag{3.8}$$

for $\alpha \in (0,1)$ and all $a,b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $p = \alpha/(2-\alpha)$ and $x = a/b > 1$, then from (1.1) we have

$$\begin{aligned} &\alpha A(a,b) + (1-\alpha)L(a,b) - J_p(a,b) \\ =& \alpha\frac{1+x}{2} + (1-\alpha)\frac{x-1}{\log x} - \frac{p(x^{p+1}-1)}{(p+1)(x^p-1)} \\ =& \frac{\alpha(x-x^p)}{2(x^p-1)\log x}\left[-\log x + \frac{2(1-\alpha)(x-1)(x^p-1)}{\alpha(x-x^p)}\right]. \end{aligned} \tag{3.9}$$

Let

$$G(x) = -\log x + \frac{2(1-\alpha)(x-1)(x^p-1)}{\alpha(x-x^p)}. \tag{3.10}$$

Then simple computations lead to

$$\lim_{x\to 1^+} G(x) = 0, \tag{3.11}$$

$$\begin{aligned} G'(x) &= -\frac{1}{x} - \frac{2(1-\alpha)[x^{2p} - px^{p+1} + 2(p-1)x^p - px^{p-1} + 1]}{\alpha(x-x^p)^2} \\ &= \frac{G_1(x)}{\alpha(x^p-x)^2}, \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} G_1(x) = & -2(1-\alpha)x^{2p} - \alpha x^{2p-1} + 2p(1-\alpha)x^{p+1} \\ & -2(2\alpha - \alpha p - 2)x^p + 2(1-\alpha)px^{p-1} \\ & -\alpha x - 2(1-\alpha). \end{aligned} \tag{3.13}$$

It follows from (3.13) and Lemma 2.3 that

$$G_1(x) > 0 \tag{3.14}$$

for $x \in (1,\infty)$ and $\alpha \in (0,1)$. Therefore, inequality (3.8) follows from (3.9)–(3.12) and (3.14).

At last, we prove that $J_{\frac{\alpha}{2-\alpha}}(a,b)$ and $J_\alpha(a,b)$ are the best possible lower and upper one-parameter mean bounds for the sum $\alpha A(a,b) + (1-\alpha)L(a,b)$, respectively.

For any $0<\varepsilon<\alpha$ and $x>0$, from (1.1) one has

$$\begin{aligned} &\alpha A(1+x,1)+(1-\alpha)L(1+x,1)-J_{\alpha-\varepsilon}(1+x,1)\\ =&\ \alpha\big(1+\frac{x}{2}\big)+\frac{(1-\alpha)x}{\log(1+x)}-\frac{(\alpha-\varepsilon)[(1+x)^{\alpha-\varepsilon+1}-1]}{(\alpha-\varepsilon+1)[(1+x)^{\alpha-\varepsilon}-1]}\\ =&\ \frac{H(x)}{(\alpha-\varepsilon+1)[(1+x)^{\alpha-\varepsilon}-1]\log(1+x)} \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &\lim_{x\to+\infty}\frac{J_{\frac{\alpha}{2-\alpha}+\varepsilon}(1,x)}{\alpha A(1,x)+(1-\alpha)L(1,x)}\\ =&\ \frac{\frac{\alpha+(2-\alpha)\varepsilon}{2+(2-\alpha)\varepsilon}\cdot\frac{x^{\frac{\alpha}{2-\alpha}+\varepsilon}-\frac{1}{x}}{x^{\frac{\alpha}{2-\alpha}+\varepsilon}-1}}{\frac{\alpha(1+\frac{1}{x})}{2}+\frac{(1-\alpha)(1-\frac{1}{x})}{\log x}}\\ =&\ \frac{2\alpha+2(2-\alpha)\varepsilon}{2\alpha+\alpha(2-\alpha)\varepsilon}>1, \end{aligned} \tag{3.16}$$

where $H(x)=\alpha(1+\frac{x}{2})(\alpha-\varepsilon+1)[(1+x)^{\alpha-\varepsilon}-1]\log(1+x)+(1-\alpha)x(\alpha-\varepsilon+1)[(1+x)^{\alpha-\varepsilon}-1]-(\alpha-\varepsilon)[(1+x)^{\alpha-\varepsilon+1}-1]\log(1+x)$.

Let $x\to 0$, making use of the Taylor extension one has

$$H(x)=\frac{\varepsilon(\alpha-\varepsilon)(\alpha-\varepsilon+1)}{12}x^4+o(x^4). \tag{3.17}$$

From (3.15)–(3.17) we clearly see that for any $0<\varepsilon<\alpha$, there exist $\delta=\delta(\alpha,\varepsilon)>0$ and $X=X(\alpha,\varepsilon)>1$, such that $\alpha A(1+x,1)+(1-\alpha)L(1+x,1)>J_{\alpha-\varepsilon}(1+x,1)$ for $x\in(0,\delta)$ and $\alpha A(x,1)+(1-\alpha)L(x,1)<J_{\frac{\alpha}{2-\alpha}+\varepsilon}(x,1)$ for $x\in(X,+\infty)$. $\square$

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## REFERENCES

[1] M.-Y. SHI, Y.-M. CHU AND Y.-P. JIANG, *Optimal inequalities among various means of two arguments*, Abstr. Appl. Anal., 2009, Article ID 694394, 10 pages.

[2] Y.-M. CHU AND W.-F. XIA, *Two optimal double inequalities between power mean and logarithmic mean*, Comput. Math. Appl., 2010, **60** (2): 83–89.

[3] Y.-M. CHU, Y.-F. QIU, M.-K. WANG AND G.-D. WANG, *The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert's mean*, J. Inequal. Appl., 2010, Article ID 436457, 7 pages.

[4] H. ALZER AND S.-L. QIU, *Inequalities for means in two variables*, Arch. Math., 2003, **80** (2): 201–215.

[5] H.-Y. GAO AND W.-J. NIU, *Sharp inequalities related to one-parameter and Gini mean*, J. Math. Inequal., 2012, **6** (4): 545–555.
[6] H.-N. HU, G.-Y. TU AND Y.-M. CHU, *Opitimal bounds for Seiffert mean in terms of one-parameter means*, J. Appl. Math., 2012, Article ID 917120, 7 pages.
[7] P. KAHLIG AND J. MATKOWSKI, *Functional equations involving the logarithmic mean*, Z. Angew. Math. Mech., 1996, **76** (7): 385–390.
[8] A. O. PITTENGER, *The logarithmic mean in n variables*, Amer. Math. Monthly, 1985, **92** (2): 99–104.
[9] G. PÓLYA AND G. SZEGÖ, *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, Princeton, 1951.
[10] P. S. BULLEN, D. S. MITRINOVIĆ AND P. M. VASIĆ, *Means and Their Inequalities*, D. Reidel Pubishing Co., Dordrecht, 1988.
[11] W.-F. XIA, Y.-M. CHU AND G.-D. WANG, *The optimal upper and lower power mean bounds for a convex combination of the arithmetic and logarithmic means*, Abstr. Appl. Anal., 2010, Article ID 604804, 9 pages.

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