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INEQUALITIES ON THE RICCI CURVATURE

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Abstract. We improve Chen-Ricci inequalities for a Lagrangian submanifold M^n of dimension n $(n \geqslant 2)$ in a $2n$-dimensional complex space form $\widetilde{M}^{2n}(4c)$ of constant holomorphic sectional curvature $4c$ with a semi-symmetric metric connection and a Legendrian submanifold M^n in a Sasakian space form $\widetilde{M}^{2n+1}(c)$ of constant φ-sectional curvature c with a semi-symmetric metric connection, respectively.

1. Introduction

The Riemannian invariants are the intrinsic characteristics of a Riemannian manifold. The main intrinsic invariants include the sectional curvature, the Ricci curvature and the scalar curvature. The main extrinsic invariant is the squared mean curvature $\|H\|^2$.

In [6] B. Y. Chen proved a geometrical inequality for Lagrangian submanifolds in complex space forms involving the Ricci curvature *Ric* and the squared mean curvature $\|H\|^2$. This inequality is known as *Chen-Ricci inequality*. Afterwards Chen-Ricci inequalities for special classes of submanifolds in different space forms were obtained (see, for instance, F. Malek and M. B. K. Balgeshir [12]).

Recently, in [9], Chen-Ricci inequality was improved and the author completely characterized Lagrangian submanifolds in complex space forms satisfying the equality. In [10] an improved Chen-Ricci inequality for Lagrangian submanifolds in quaternion space forms was obtained. In [17] the present author and I. N. Radulescu improved Chen-Ricci inequality for Kaehlerian slant submanifolds in complex space forms. In [19] an improved Chen-Ricci inequality for Legendrian submanifolds in Sasakian space forms was obtained by I. Mihai and I. N. Radulescu. In [29] M. M. Tripathi improved Chen-Ricci inequality for curvature-like tensors.

On the other hand, in [11], H. A. Hayden introduced the notion of a semi-symmetric metric connection on a Riemannian manifold. In [30] K. Yano studied some properties of a Riemannian manifold endowed with a semi-symmetric metric connection. In [20] Z. Nakao studied submanifolds of a Riemannian manifold with semi-symmetric connections. In [15] and [16] the present author and C. Ozgur studied B. Y. Chen inequalities for submanifolds of real space forms, complex space forms and Sasakian space forms with semi-symmetric metric connections, respectively.

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Motivated by the above studies, in the present paper we improve Chen-Ricci inequalities for a Lagrangian submanifold M^n of dimension n $(n \geqslant 2)$ in a complex space form $\widetilde{M}^{2n}(4c)$ of constant holomorphic sectional curvature c with a semi-symmetric metric connection and a Legendrian submanifold M^n in a Sasakian space form $\widetilde{M}^{2n+1}(c)$ of constant φ-sectional curvature c with a semi-symmetric metric connection, respectively, by using similar methods as in [9].

2. Preliminaries

Let $\widetilde{M}$ be a Riemannian manifold and $\widetilde{\nabla}$ a linear connection on $\widetilde{M}$. If the torsion tensor $\widetilde{T}$ of $\widetilde{\nabla}$ satisfies

$$\widetilde{T}\left(\widetilde{X},\widetilde{Y}\right) = \omega(\widetilde{Y})\widetilde{X} - \omega(\widetilde{X})\widetilde{Y}$$

for a 1-form ω, then the connection $\widetilde{\nabla}$ is called a *semi-symmetric connection*. Let g be a Riemannian metric on $\widetilde{M}$. If $\widetilde{\nabla}g = 0$, then $\widetilde{\nabla}$ is called a *semi-symmetric metric connection* on $\widetilde{M}$.

A semi-symmetric metric connection $\widetilde{\nabla}$ on $\widetilde{M}$ is given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} + \omega(\widetilde{Y})\widetilde{X} - g(\widetilde{X},\widetilde{Y})P,$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on $\widetilde{M}$, where $\overset{\circ}{\widetilde{\nabla}}$ denotes the Levi-Civita connection with respect to the Riemannian metric g and P is a vector field defined by $g(P,\widetilde{X}) = \omega(\widetilde{X})$, for any vector field $\widetilde{X}$ [30]. Now let $\widetilde{M}$ be a Riemannian manifold endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection denoted by $\overset{\circ}{\widetilde{\nabla}}$. Then the curvature tensor $\widetilde{R}$, given by $\widetilde{R}(x,y,z,w) = g(\widetilde{R}(x,y)z,w)$, with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ on $\widetilde{M}$ can be written as (see [30])

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = \overset{\circ}{\widetilde{R}}(X,Y,Z,W) &- \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) \\ &- \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z),\end{aligned} \tag{1}$$

for any vector fields $X,Y,Z,W \in \chi(M)$, where α is a $(0,2)$-tensor field defined by

$$\alpha(X,Y) = \left(\overset{\circ}{\widetilde{\nabla}}_X \omega\right) Y - \omega(X)\omega(Y) + \frac{1}{2}\omega(P)g(X,Y), \quad \forall X,Y \in \chi(M).$$

Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold $\widetilde{M}^{n+p}$. On the submanifold M^n we consider the induced semi-symmetric metric connection denoted by ∇ and the induced Levi-Civita connection denoted by $\overset{\circ}{\nabla}$.

Let $\widetilde{R}$ be the curvature tensor of $\widetilde{M}^{n+p}$ with respect to $\widetilde{\nabla}$ and $\overset{\circ}{\widetilde{R}}$ the curvature tensor of $\widetilde{M}^{n+p}$ with respect to $\overset{\circ}{\widetilde{\nabla}}$. We also denote by R and $\overset{\circ}{R}$ the curvature tensors of ∇ and $\overset{\circ}{\nabla}$, respectively, on M^n.

The Gauss formulas with respect to ∇, respectively $\overset{\circ}{\nabla}$, can be written as:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad X,Y \in \chi(M^n),$$
$$\overset{\circ}{\widetilde{\nabla}}_X Y = \overset{\circ}{\nabla}_X Y + \overset{\circ}{h}(X,Y), \quad X,Y \in \chi(M^n),$$

where $\overset{\circ}{h}$ is the second fundamental form of M^n in $\widetilde{M}^{n+p}$ and h is a $(0,2)$-tensor on M^n. According to the formula (7) from [20], h is also symmetric and $h = \overset{\circ}{h}$ if and only if P is tangent to M^n. The Gauss equation for the submanifold M^n into an $(n+p)$-dimensional Riemannian manifold $\widetilde{M}^{n+p}$ is

$$\overset{\circ}{\widetilde{R}}(X,Y,Z,W) = \overset{\circ}{R}(X,Y,Z,W) + g(\overset{\circ}{h}(X,Z),\overset{\circ}{h}(Y,W)) - g(\overset{\circ}{h}(X,W),\overset{\circ}{h}(Y,Z)). \quad (2)$$

From [20], the Gauss equation with respect to the semi-symmetric metric connection is

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(Y,Z),h(X,W)). \quad (3)$$

One denotes by $\overset{\circ}{H}$ the mean curvature vector of M^n in $\widetilde{M}^{n+p}$.

On the other hand, we recall the following results from [9]:

LEMMA 2.1. *Let $f_1(x_1,x_2,\ldots,x_n)$ be a function on $\mathbb{R}^n$ defined by:*

$$f_1(x_1,x_2,\ldots,x_n) = x_1 \sum_{j=2}^{n} x_j - \sum_{j=2}^{n} x_j^2.$$

If $x_1 + x_2 + \ldots + x_n = 2na$, then we have

$$f_1(x_1,x_2,\ldots,x_n) \leqslant \frac{n-1}{4n}(x_1 + x_2 + \ldots + x_n)^2,$$

with the equality sign holding if and only if $\frac{1}{n+1}x_1 = x_2 = \ldots = x_n = a$.

LEMMA 2.2. *Let $f_2(x_1,x_2,\ldots,x_n)$ be a function on $\mathbb{R}^n$ defined by:*

$$f_2(x_1,x_2,\ldots,x_n) = x_1 \sum_{j=2}^{n} x_j - x_1^2.$$

If $x_1 + x_2 + \ldots + x_n = 4a$, then we have

$$f_2(x_1,x_2,\ldots,x_n) \leqslant \frac{1}{8}(x_1 + x_2 + \ldots + x_n)^2,$$

with the equality sign holding if and only if $x_1 = a$ and $x_2 + \ldots + x_n = 3a$.

3. An improved Chen-Ricci inequality for submanifolds of complex space forms with a semi-symmetric metric connection

Let $\widetilde{M}^{2m}$ be a Kaehler manifold and J the canonical almost complex structure. The sectional curvature of $\widetilde{M}^{2m}$ in the direction of an invariant 2-plane section by J is called the *holomorphic sectional curvature*. If the holomorphic sectional curvature is constant $4c$ for all plane sections π of $T_x\widetilde{M}^{2m}$ invariant by J for any $x \in \widetilde{M}^{2m}$, then $\widetilde{M}^{2m}$ is called a *complex space form* and is denoted by $\widetilde{M}^{2m}(4c)$. The curvature tensor $\overset{\circ}{\widetilde{R}}$ with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ on $\widetilde{M}^{2m}(4c)$ is given by (see [8])

$$\begin{aligned}\overset{\circ}{\widetilde{R}}(X,Y,Z,W) = c[&g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ &-g(JX,Z)g(JY,W) + g(JX,W)g(JY,Z) - 2g(X,JY)g(Z,JW)].\end{aligned} \tag{4}$$

If $\widetilde{M}^{2m}(4c)$ is a complex space form of constant holomorphic sectional curvature $4c$ with a semi-symmetric metric connection $\widetilde{\nabla}$, then from (1) and (4), the curvature tensor $\widetilde{R}$ of $\widetilde{M}^{2m}(4c)$ can be expressed as

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = c[&g(X,W)g(Y,Z) - g(X,Z)g(Y,W) - g(JX,Z)g(JY,W) \\ &+g(JX,W)g(JY,Z) - 2g(X,JY)g(Z,JW)] - \alpha(Y,Z)g(X,W) \\ &+\alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z).\end{aligned} \tag{5}$$

Let $M^n, n \geqslant 2$, be an n-dimensional submanifold of a $2m$-dimensional complex space form $\widetilde{M}^{2m}(4c)$ of constant holomorphic sectional curvature $4c$. If $J(T_pM^n) \subset T_p^{\perp}M^n$, then M^n is called an *a totally real submanifold* of $\widetilde{M}^{2m}$. For a totally real submanifold of a Kaehlerian manifold it is known that (see [31])

$$\overset{\circ}{A}_{JX}\, Y = \overset{\circ}{A}_{JY}\, X, \quad X,Y \in T_pM,$$

or equivalently,

$$\overset{\circ}{h}{}^k_{ij} = \overset{\circ}{h}{}^j_{ik} = \overset{\circ}{h}{}^i_{jk}, \quad \forall i,j,k = 1,\dots,n, \tag{6}$$

where $\overset{\circ}{A}$ is the shape operator with respect to $\overset{\circ}{\widetilde{\nabla}}$ and

$$\overset{\circ}{h}{}^k_{ij} = g(\overset{\circ}{h}(e_i,e_j),Je_k), \quad i,j,k = 1,\dots,n.$$

A Lagrangian submanifold is a totally real submanifold of maximum dimension [8].

THEOREM 3.1. *Let M^n be a Lagrangian submanifold of dimension n $(n \geqslant 2)$ in a $2n$-dimensional complex space form $\widetilde{M}^{2n}(4c)$ of constant holomorphic sectional curvature $4c$ with a semi-symmetric metric connection such that the vector field P is tangent to M^n. Then for any unit tangent vector X to M^n we have*

$$Ric(X) + (n-2)\alpha(X,X) + tr\alpha \leqslant (n-1)\left(c + \frac{n}{4}\|H\|^2\right). \tag{7}$$

The equality sign of (7) *holds identically if and only if either*
(*i*) M^n *is totally geodesic, or*
(*ii*) $n=2$, *and* M^2 *is a* H*-umbilical Lagrangian surface with* $\lambda = 3\mu$.

Proof. Since P is tangent to M^n, we have $h = \overset{\circ}{h}$ and $\overset{\circ}{H} = H$. For a given point $p \in M^n$ and a given unit vector $X \in T_pM^n$, we choose an orthonormal basis $\{e_1 = X, e_2, \ldots, e_n\} \subset T_pM^n$ and

$$\{e_{n+1} = Je_1, \ldots, e_{2n} = Je_n\} \subset T_p^{\perp}M.$$

Now we put in (3) $X = W = e_j$ and $Y = Z = e_1$, for $j = 2,..,n$, and by (5) it follows that

$$\begin{aligned} R(e_j,e_1,e_1,e_j) = {} & c[g(e_1,e_1)g(e_j,e_j) - g^2(e_j,e_1)] + g(h(e_1,e_1),h(e_j,e_j)) \\ & -g(h(e_1,e_j),h(e_1,e_j)) - \alpha(e_1,e_1)g(e_j,e_j) + \alpha(e_j,e_1)g(e_1,e_j) \\ & -\alpha(e_j,e_j)g(e_1,e_1) + \alpha(e_1,e_j)g(e_j,e_1). \end{aligned}$$

By summing after $j = \overline{2,n}$, we get

$$Ric(X) = (n-1)c + \sum_{r=1}^{n}\sum_{j=2}^{n}\left[h_{11}^r h_{jj}^r - (h_{1j}^r)^2\right] - (n-2)\alpha(X,X) - tr\alpha.$$

It follows that

$$\begin{aligned} & Ric(X) - (n-1)c + (n-2)\alpha(X,X) + tr\alpha \\ & = \sum_{r=1}^{n}\sum_{j=2}^{n}\left[h_{11}^r h_{jj}^r - (h_{1j}^r)^2\right] \\ & \leqslant \sum_{r=1}^{n}\sum_{j=2}^{n} h_{11}^r h_{jj}^r - \sum_{j=2}^{n}(h_{1j}^1)^2 - \sum_{j=2}^{n}(h_{1j}^j)^2. \end{aligned} \tag{8}$$

Since M^n is a Lagrangian submanifold, we have the relation (6) and

$$\begin{aligned} & Ric(X) - (n-1)c + (n-2)\alpha(X,X) + tr\alpha \\ & \leqslant \sum_{r=1}^{n}\sum_{j=2}^{n} h_{11}^r h_{jj}^r - \sum_{j=2}^{n}(h_{11}^j)^2 - \sum_{j=2}^{n}(h_{jj}^1)^2. \end{aligned} \tag{9}$$

Now we put

$$f_1(h_{11}^1, h_{22}^1, \ldots, h_{nn}^1) = h_{11}^1 \sum_{j=2}^{n} h_{jj}^1 - \sum_{j=2}^{n}(h_{jj}^1)^2$$

and

$$f_r(h_{11}^r, h_{22}^r, \ldots, h_{nn}^r) = h_{11}^r \sum_{j=2}^{n} h_{jj}^r - (h_{11}^r)^2, \ \ \forall r = \overline{2,n}.$$

Since $nH^1 = h^1_{11} + h^1_{22} + \ldots + h^1_{nn}$, we obtain by using Lemma 2.1 that

$$f_1(h^1_{11}, h^1_{22}, \ldots, h^1_{nn}) \leqslant \frac{n-1}{4n}(nH^1)^2 = \frac{n(n-1)}{4}(H^1)^2. \tag{10}$$

By applying Lemma 2.2 for $2 \leqslant r \leqslant n$, we get

$$f_r(h^r_{11}, h^r_{22}, \ldots, h^r_{nn}) \leqslant \frac{1}{8}(nH^r)^2 = \frac{n^2}{8}(H^r)^2 \leqslant \frac{n(n-1)}{4}(H^r)^2. \tag{11}$$

From (9), (10) and (11), we obtain

$$Ric(X) - (n-1)c + (n-2)\alpha(X,X) + tr\alpha \leqslant \frac{n(n-1)}{4}\sum_{r=1}^{n}(H^r)^2 = \frac{n(n-1)}{4}\|H\|^2.$$

Thus we have

$$Ric(X) \leqslant (n-1)c - (n-2)\alpha(X,X) - tr\alpha + \frac{n(n-1)}{4}\|H\|^2, \tag{12}$$

which implies (7).

Next, we shall study the equality case.

Case 1. For $n \geqslant 3$, we choose Je_1 parallel to H. Then we have $H^r = 0$, for $r \geqslant 2$. Thus, by Lemma 2.2, we get

$$h^1_{1j} = h^j_{11} = \frac{nH^j}{4} = 0, \ \ \forall j \geqslant 2,$$

and

$$h^1_{jk} = 0, \ \ \forall j,k \geqslant 2, \ j \neq k.$$

From Lemma 2.1, we have $h^1_{11} = (n+1)a$ and $h^1_{jj} = a$, $\forall j \geqslant 2$, with $a = \frac{H^1}{2}$.

In (8) we computed $Ric(X) = Ric(e_1)$. Similarly, by computing $Ric(e_2)$ and using the equality, we get

$$h^r_{2j} = h^2_{jr} = 0, \ \ \forall r \neq 2, \ \ j \neq 2, \ \ r \neq j.$$

Then we obtain

$$\frac{h^2_{11}}{n+1} = h^2_{22} = \ldots = h^2_{nn} = \frac{H^2}{2} = 0.$$

The argument is also true for matrices $\left(h^r_{jk}\right)$ because the equality holds for all unit tangent vectors; so, $h^2_{2j} = h^j_{22} = \frac{H^j}{2} = 0, \ \ \forall j \geqslant 3$.

The matrix $\left(h^2_{jk}\right)$ (respectively the matrix $\left(h^r_{jk}\right)$) has only two possible nonzero entries $h^2_{12} = h^2_{21} = h^1_{22} = \frac{H^1}{2}$ (respectively $h^r_{1r} = h^r_{r1} = h^1_{rr} = \frac{H^1}{2}$, $\forall r \geqslant 3$).

Now, after putting $X = Z = e_2$ and $Y = W = e_j$, $j = 2, \ldots, n$ in (3) we obtain

$$\widetilde{R}(e_2, e_j, e_2, e_j) = R(e_2, e_j, e_2, e_j) - \left(\frac{H^1}{2}\right)^2, \ \forall j \geqslant 3.$$

If we put $X = Z = e_2$ and $Y = W = e_1$ in (3), we get

$$\widetilde{R}(e_2,e_1,e_2,e_1) = R(e_2,e_1,e_2,e_1) - (n+1)\left(\frac{H^1}{2}\right)^2 + \left(\frac{H^1}{2}\right)^2.$$

After combining the last two relations, we find

$$Ric(e_2) - (n-1)c + (n-2)\alpha(e_2,e_2) + tr\alpha = 2(n-1)\left(\frac{H^1}{2}\right)^2.$$

On the other hand, the equality case of (7) implies that

$$Ric(e_2) - (n-1)c + (n-2)\alpha(X,X) + tr\alpha = \frac{n(n-1)}{4}\|H\|^2 = n(n-1)\left(\frac{H^1}{2}\right)^2.$$

Since $n \neq 1,2$, by the last 2 equations we find $H^1 = 0$. Thus, $\left(h^r_{jk}\right)$ are all zero, i.e. M is a totally geodesic submanifold in $\widetilde{M}(4c)$.

Case 2. $n = 2$.

If M^2 is not totally geodesic, then $h(e_1,e_1) = \lambda e_3$, $h(e_2,e_2) = \mu e_3$, $h(e_1,e_2) = \mu e_4$, with $\lambda = 3\mu$. Such a surface is said to be a H-umbilical surface. □

4. An improved Chen-Ricci inequality for submanifolds of Sasakian space forms with a semi-symmetric metric connection

A $(2m+1)$-dimensional Riemannian manifold $(\widetilde{M}^{2m+1}, g)$ has an *almost contact metric structure* if it admits a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η satisfying:

$$\begin{aligned} \varphi^2 X &= -X + \eta(X)\xi, \eta(\xi) = 1 \\ g(\varphi X, \varphi Y) &= g(X,Y) - \eta(X)\eta(Y), \\ g\,(X,\xi) &= \eta(X), \end{aligned}$$

for any vector fields X,Y on $T\widetilde{M}^{2m+1}$. Let Φ denote the fundamental 2-form in $\widetilde{M}^{2m+1}$, given by $\Phi(X,Y) = g(X,\phi Y)$, for all X,Y on $T\widetilde{M}^n$. If $\Phi = d\eta$, then $\widetilde{M}^{2m+1}$ is called a *contact metric manifold*. The structure of $\widetilde{M}^{2m+1}$ is called *normal* if

$$[\phi,\phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ. A *Sasakian manifold* is a normal contact metric manifold.

A plane section π in $T_p\widetilde{M}^{2m+1}$ is called a φ-*section* if it is spanned by X and φX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature of a φ-section is called a φ-*sectional curvature*. A Sasakian manifold with constant φ-sectional curvature c is said to be a *Sasakian space form* and is denoted by $\widetilde{M}^{2m+1}(c)$.

The curvature tensor $\overset{\circ}{\widetilde{R}}$ with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ on $\widetilde{M}^{2m+1}(c)$ is expressed by

$$\begin{aligned}\overset{\circ}{\widetilde{R}}(X,Y,Z,W) = &\frac{c+3}{4}[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)]\\ &+\frac{c-1}{4}[\eta(X)\eta(Z)g(Y,W)-\eta(Y)\eta(Z)g(X,W)\\ &+\eta(Y)\eta(W)g(X,Z)-\eta(X)\eta(W)g(Y,Z)+g(X,\varphi Z)g(\varphi Y,W)\\ &-g(Y,\varphi Z)g(\varphi X,W)+2g(X,\varphi Y)g(\varphi Z,W)],\end{aligned}$$

for vector fields X,Y,Z,W on $\widetilde{M}^{2m+1}(c)$.

If $\widetilde{M}^{2m+1}(c)$ is a $(2m+1)$-dimensional Sasakian space form of constant φ-sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$, from (1) it follows that the curvature tensor $\widetilde{R}$ of $\widetilde{M}^{2m+1}$ can be expressed as

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = &\frac{c+3}{4}[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)]\\ &+\frac{c-1}{4}[\eta(X)\eta(Z)g(Y,W)-\eta(Y)\eta(Z)g(X,W)\\ &+\eta(Y)\eta(W)g(X,Z)-\eta(X)\eta(W)g(Y,Z)+g(X,\varphi Z)g(\varphi Y,W)\\ &-g(Y,\varphi Z)g(\varphi X,W)+2g(X,\varphi Y)g(\varphi Z,W)]-\alpha(Y,Z)g(X,W)\\ &+\alpha(X,Z)g(Y,W)-\alpha(X,W)g(Y,Z)+\alpha(Y,W)g(X,Z).\end{aligned} \tag{13}$$

A submanifold M^n of a Sasakian manifold $\widetilde{M}^{2m+1}$ normal to ξ is called a *C-totally real submanifold*. On such a submanifold, φ maps any tangent vector to M^n at $p\in M^n$ into the normal space $T_p^{\perp}M^n$. In particular, if $n=m$, i. e. M^n has maximum dimension, then it is a *Legendrian submanifold*. For a Legendrian submanifold M^n, if $\{e_1,\dots.e_n\}$ is an orthonormal basis of T_pM, we may choose an orthonormal basis of $T_p^{\perp}M^n$ of the form $\{e_{n+1}=\varphi e_1,\dots,e_{2n}=\varphi e_n, e_{2n+1}=\xi\}$. One has (see [31])

$$\overset{\circ}{A}_{\varphi X}Y=\overset{\circ}{A}_{\varphi Y}X,\quad X,Y\in T_pM^n,$$

or equivalently,

$$\overset{\circ}{h}{}^k_{ij}=\overset{\circ}{h}{}^j_{ik}=\overset{\circ}{h}{}^i_{jk},\quad \forall i,j,k=1,\dots,n, \tag{14}$$

where $\overset{\circ}{A}$ is the corresponding shape operator and

$$\overset{\circ}{h}{}^k_{ij}=g(\overset{\circ}{h}(e_i,e_j),\varphi e_k),\quad i,j,k=1,\dots,n.$$

THEOREM 4.1. *Let M^n be an n-dimensional Legendrian submanifold $(n\geqslant 2)$ in a Sasakian space form $\widetilde{M}^{2n+1}(c)$ of constant φ-sectional curvature c with a semi-symmetric metric connection such that the vector field P is tangent to M^n. Then for any unit tangent vector X to M^n we have*

$$Ric(X)+(n-2)\alpha(X,X)+tr\alpha\leqslant\frac{n-1}{4}\left(c+3+n\|H\|^2\right). \tag{15}$$

The equality sign of (15) *holds identically if and only if either*
(i) M^n *is totally geodesic, or*
(ii) $n=2$*, and* M^2 *is a* H*-umbilical Legendrian surface with* $\lambda = 3\mu$*.*

Proof. Since P is tangent to M^n, we have $h = \overset{\circ}{h}$ and $H = \overset{\circ}{H}$. For a given point $p \in M^n$ and a given unit vector $X \in T_pM^n$, we choose an orthonormal basis $\{e_1 = X, e_2, \ldots, e_n\} \subset T_pM^n$ and

$$\{e_{n+1} = \varphi e_1, \ldots, e_{2n} = \varphi e_n, e_{2n+1} = \xi\} \subset T_p^{\perp}M^n.$$

Now we put in (3) and (13) $X = W = e_j$ and $Y = Z = e_1$, for $j = 2,..,n$; it follows that

$$\begin{aligned} R(e_j,e_1,e_1,e_j) = &\frac{c+3}{4}[g(e_1,e_1)g(e_j,e_j) - g^2(e_j,e_1)] + g(h(e_1,e_1),h(e_j,e_j)) \\ &- g(h(e_1,e_j),h(e_1,e_j)) - \alpha(e_1,e_1)g(e_j,e_j) + \alpha(e_j,e_1)g(e_1,e_j) \\ &- \alpha(e_j,e_j)g(e_1,e_1) + \alpha(e_1,e_j)g(e_j,e_1). \end{aligned}$$

By summing after $j = \overline{2,n}$, we get

$$Ric(X) = (n-1)\frac{c+3}{4} + \sum_{r=1}^{n}\sum_{j=2}^{n}\left[h_{11}^{r}h_{jj}^{r} - (h_{1j}^{r})^2\right] - (n-2)\alpha(X,X) - tr\alpha.$$

It follows that

$$\begin{aligned} &Ric(X) - (n-1)\frac{c+3}{4} + (n-2)\alpha(X,X) + tr\alpha \qquad (16) \\ &= \sum_{r=1}^{n}\sum_{j=2}^{n}\left[h_{11}^{r}h_{jj}^{r} - (h_{1j}^{r})^2\right] \\ &\leqslant \sum_{r=1}^{n}\sum_{j=2}^{n} h_{11}^{r}h_{jj}^{r} - \sum_{j=2}^{n}(h_{1j}^{1})^2 - \sum_{j=2}^{n}(h_{1j}^{j})^2. \end{aligned}$$

Since M^n is a Legendrian submanifold, we have the relation (14) and

$$\begin{aligned} &Ric(X) - (n-1)\frac{c+3}{4} + (n-2)\alpha(X,X) + tr\alpha \qquad (17) \\ &\leqslant \sum_{r=1}^{n}\sum_{j=2}^{n} h_{11}^{r}h_{jj}^{r} - \sum_{j=2}^{n}(h_{11}^{j})^2 - \sum_{j=2}^{n}(h_{jj}^{1})^2. \end{aligned}$$

Now we put

$$f_1(h_{11}^1, h_{22}^1, \ldots, h_{nn}^1) = h_{11}^1\sum_{j=2}^{n} h_{jj}^1 - \sum_{j=2}^{n}(h_{jj}^1)^2$$

and

$$f_r(h_{11}^r, h_{22}^r, \ldots, h_{nn}^r) = h_{11}^r\sum_{j=2}^{n} h_{jj}^r - (h_{11}^r)^2, \ \ \forall r = \overline{2,n}.$$

Since $nH^1 = h^1_{11} + h^1_{22} + \ldots + h^1_{nn}$, we obtain by using Lemma 2.1 that

$$f_1(h^1_{11}, h^1_{22}, \ldots, h^1_{nn}) \leqslant \frac{n-1}{4n}(nH^1)^2 = \frac{n(n-1)}{4}(H^1)^2. \tag{18}$$

By applying Lemma 2.2 for $2 \leqslant r \leqslant n$, we get

$$f_r(h^r_{11}, h^r_{22}, \ldots, h^r_{nn}) \leqslant \frac{1}{8}(nH^r)^2 = \frac{n^2}{8}(H^r)^2 \leqslant \frac{n(n-1)}{4}(H^r)^2. \tag{19}$$

From (17), (18) and (19) we obtain

$$Ric(X) - (n-1)\frac{c+3}{4} + (n-2)\alpha(X,X) + tr\alpha \leqslant \frac{n(n-1)}{4}\sum_{r=1}^{n}(H^r)^2 = \frac{n(n-1)}{4}\|H\|^2.$$

Thus we have

$$Ric(X) \leqslant (n-1)\frac{c+3}{4} - (n-2)\alpha(X,X) - tr\alpha + \frac{n(n-1)}{4}\|H\|^2,$$

which implies (15).

Next, we shall study the equality case. For $n \geqslant 3$, we choose ϕe_1 parallel to H. Then we have $H^r = 0$, for $r \geqslant 2$. Thus, by Lemma 2.2, we get

$$h^1_{1j} = h^j_{11} = \frac{nH^j}{4} = 0, \ \forall j \geqslant 2$$

and

$$h^1_{jk} = 0, \ \forall j,k \geqslant 2, \ j \neq k.$$

From Lemma 2.1 we have $h^1_{11} = (n+1)a$ and $h^1_{jj} = a$, $\forall j \geqslant 2$, with $a = \frac{H^1}{2}$.

In (16) we computed $Ric(X) = Ric(e_1)$. Similarly, by computing $Ric(e_2)$ and using the equality, we get

$$h^r_{2j} = h^2_{jr} = 0, \ \forall r \neq 2, \ j \neq 2, \ r \neq j.$$

Then we obtain

$$\frac{h^2_{11}}{n+1} = h^2_{22} = \ldots = h^2_{nn} = \frac{H^2}{2} = 0.$$

The argument is also true for matrices $\left(h^r_{jk}\right)$ because the equality holds for all unit tangent vectors; so, $h^2_{2j} = h^j_{22} = \frac{H^j}{2} = 0, \ \forall j \geqslant 3$.

The matrix $\left(h^2_{jk}\right)$ (respectively the matrix $\left(h^r_{jk}\right)$) has only two possible nonzero entries $h^2_{12} = h^2_{21} = h^1_{22} = \frac{H^1}{2}$ (respectively $h^r_{1r} = h^r_{r1} = h^1_{rr} = \frac{H^1}{2}$, $\forall r \geqslant 3$).

Now, after putting $X = Z = e_2$ and $Y = W = e_j$, $j = 2, \ldots, n$ in (3) we obtain

$$\widetilde{R}(e_2, e_j, e_2, e_j) = R(e_2, e_j, e_2, e_j) - \left(\frac{H^1}{2}\right)^2, \ \forall j \geqslant 3.$$

If we put $X = Z = e_2$ and $Y = W = e_1$ in (3), we get

$$\widetilde{R}(e_2,e_1,e_2,e_1) = R(e_2,e_1,e_2,e_1) - (n+1)\left(\frac{H^1}{2}\right)^2 + \left(\frac{H^1}{2}\right)^2.$$

After combining the last two relations, we find

$$Ric(e_2) - (n-1)\frac{c+3}{4} + (n-2)\alpha(e_2,e_2) + tr\alpha = 2(n-1)\left(\frac{H^1}{2}\right)^2.$$

On the other hand, the equality case of (15) implies that

$$Ric(e_2) - (n-1)\frac{c+3}{4} + (n-2)\alpha(e_2,e_2) + tr\alpha = \frac{n(n-1)}{4}\|H\|^2 = n(n-1)\left(\frac{H^1}{2}\right)^2.$$

Since $n \neq 1,2$, by the last 2 equations we find $H^1 = 0$. Thus, $\left(h^r_{jk}\right)$ are all zero, i.e., M^n is a totally geodesic submanifold in $\widetilde{M}^{2n+1}(c)$.

Now assume that $n = 2$. If M^2 is not totally geodesic, one has

$$h(e_1,e_1) = \lambda e_3,\ h(e_2,e_2) = \mu e_3,\ h(e_1,e_2) = \mu e_4,$$

with $\lambda = 3\mu = \frac{3H^1}{2}$, i.e., M^2 is H-umbilical. This gives case (ii) of the theorem. □

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