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SOME NEW DISCRETE FRACTIONAL INEQUALITIES AND THEIR APPLICATIONS IN FRACTIONAL DIFFERENCE EQUATIONS

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Abstract. In this paper, some new Gronwall-Bellman type discrete fractional difference inequalities and fractional sum inequalities are established. Based on the theory of discrete fractional calculus, explicit bounds for unknown functions concerned are presented. These inequalities can be used as a handy tool in the qualitative analysis of solutions of discrete fractional difference equations. As for applications, we apply the presented results to research boundedness, uniqueness, and continuous dependence on the initial value for the solutions of certain initial value problems of fractional difference equations.

1. Introduction

It is well known that Gronwall-Bellman type inequalities play important roles in the research of qualitative as well as quantitative properties of solutions of differential equations, difference equations, dynamic equations and so on. The main merit of Gronwall-Bellman type inequalities lies in that such inequalities can provide explicit bounds for unknown functions concerned. In the last few decades, many authors have paid much attention to research Gronwall-Bellman type inequalities, and many Gronwall-Bellman type differential inequalities, difference inequalities, and dynamic inequalities on time scales have been established so far in the literature. For example, in [1–3], some Gronwall-Bellman type differential inequalities have been presented, which have been widely used in the qualitative and quantitative analysis of solutions of various differential equations. In [4–9], a series of retarded Gronwall-Bellman type differential inequalities have been presented, which are generalizations of the earlier inequalities, and can be used in the research of retarded differential equations. In [10–12], some discrete Gronwall-Bellman type inequalities have been established, which can be used as a handy tool in the research of solutions of difference equations. In [13–16], the authors have investigated some Gronwall-Bellman type inequalities on time scales, which unify the Gronwall-Bellman type differential inequalities and the corresponding discrete inequalities, and can be used in the research of dynamic equations on time scales. Then in [17–20], the authors presented some new more generalized Gronwall-Bellman type inequalities on time scales than those in [13–16]. Recently, with the

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development of the theory of fractional differential equations, some new Gronwall-Bellman type inequalities suitable for the qualitative analysis of solutions of fractional differential equations have been presented (for example, see [21–25]).

In these inequalities, we notice that few inequalities have been oriented to the qualitative analysis of solutions of discrete fractional difference equations arising in the theory of discrete fractional calculus. For recent results on this direction, we refer the reader to [26–27].

Motivated by the above analysis, in this paper, we establish some new Gronwall-Bellman type discrete fractional difference inequalities and fractional sum inequalities, which can be used as a handy tool in the qualitative analysis of solutions of discrete fractional difference equations. Based on these inequalities, explicit bounds for unknown functions are presented.

Some important definitions and conclusions in discrete fractional calculus are listed as follows [28]. For the convenience, we denote $\mathbb{N}_t = \{t, t+1, t+2, \ldots\}$, and in the next of this paper, $\sum_{s=m_0}^{m_1} f(s) = 0$ provided $m_0 > m_1$.

DEFINITION 1. Let $\upsilon > 0$, $\sigma(s) = s+1$, and the function f is defined for $s = a$ mod 1. Then the $\upsilon-$th fractional sum of f is defined by

$$\Delta^{-\upsilon} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\upsilon} (t-\sigma(s))^{(\upsilon-1)} f(s),$$

where $t^{(\upsilon)} = \dfrac{\Gamma(t+1)}{\Gamma(t+1-\upsilon)}$, $\Delta^{-\upsilon} f$ is defined for $t = a+\upsilon$ mod 1, and $\Delta^{-\upsilon}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\upsilon}$.

DEFINITION 2. Let $\mu > 0$, and $m-1 < \mu < m$, where m is a positive integer. Then the $\mu-$th fractional difference of f is defined by

$$\Delta^{\mu} f(t) = \Delta^{m-\upsilon} f(t) = \Delta^{m}(\Delta^{-\upsilon} f(t)),$$

where $-\upsilon = \mu - m$.

THEOREM 3. [28, Theorem 1.1] *Let f be a real-valued function defined on $\mathbb{N}_a$, and μ, $\upsilon > 0$. Then the following equalities hold:*

$$\Delta^{-\upsilon}[\Delta^{-\mu} f(t)] = \Delta^{-(\upsilon+\mu)} f(t) = \Delta^{-\mu}[\Delta^{-\upsilon} f(t)].$$

THEOREM 4. [28, Theorem 2.1] *Let f be a real-valued function defined on $\mathbb{N}_a$, and $\upsilon > 0$. Then the following equalities hold:*

$$\Delta^{-\upsilon}\Delta f(t) = \Delta\Delta^{-\upsilon} f(t) - \frac{(t-a)^{(\upsilon-1)}}{\Gamma(\upsilon)} f(a).$$

For other important properties on the discrete fractional calculus, we refer the reader to [28–30].

The next of this paper is organized as follows. In Section 2, we present the main results, in which new Gronwall-Bellman type discrete fractional difference inequalities and fractional sum inequalities are established. In Section 3, we present some examples, and apply the inequalities established to research boundedness, uniqueness, and continuous dependence on the initial value for the solutions of certain initial value problems of fractional difference equations. In Section 4, some conclusions are presented.

2. Main results

THEOREM 5. *Assume* $0<\alpha\leqslant 1$, $u(n)$ *is a nonnegative function defined on* $\mathbb{N}_{\alpha-1}$, *and* a, b *are nonnegative functions defined on* $\mathbb{N}_0$. *If the following inequality holds:*

$$\Delta^{\alpha}u(k)\leqslant a(k)u(k+\alpha-1)+b(k),\quad k\in\mathbb{N}_0, \tag{1}$$

then for $n\in\mathbb{N}_{\alpha}$, *one has*

$$\begin{aligned}u(n)\leqslant{}&\frac{n^{(\alpha-1)}}{\Gamma(\alpha)}A+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}b(s)\\&+A\left\{d(n-\alpha,n)\frac{(n-1)^{(\alpha-1)}}{\Gamma(\alpha)}+\sum_{s=0}^{n-\alpha-1}|d(s,n)-d(s,n-1)|\frac{(s+\alpha-1)^{(\alpha-1)}}{\Gamma(\alpha)}\right\}\\&+A\sum_{p=\alpha}^{n-1}\left\{\left[d(p-\alpha,p)\frac{(p-1)^{(\alpha-1)}}{\Gamma(\alpha)}+\sum_{s=0}^{p-\alpha-1}|d(s,p)-d(s,p-1)|\frac{(s+\alpha-1)^{(\alpha-1)}}{\Gamma(\alpha)}\right]\right.\\&\left.\times\prod_{\xi=p+1}^{n}[1+d(\xi-\alpha,\xi)+\sum_{s=0}^{\xi-\alpha-1}|d(s,\xi)-d(s,\xi-1)|]\right\}.\end{aligned}\tag{2}$$

where $A=\Delta^{\alpha-1}u(k)|_{k=0}$, $d(s,n)=\dfrac{1}{\Gamma(\alpha)}a(s)(n-s-1)^{(\alpha-1)}$.

Proof. Denote

$$\Delta^{\alpha}u(k)=f(k+\alpha-1,u(k+\alpha-1)),\quad k\in\mathbb{N}_0. \tag{3}$$

Then calculating discrete $\alpha-$th fractional sum $\Delta^{-\alpha}$ on both sides of (3) yields that for $n\in\mathbb{N}_{\alpha}$,

$$\begin{aligned}\Delta^{-\alpha}\Delta^{\alpha}u(n)&=\Delta^{-\alpha}f(n+\alpha-1,u(n+\alpha-1))\\&=\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}f(s+\alpha-1,u(s+\alpha-1)).\end{aligned}$$

By setting $A=\Delta^{\alpha-1}u(k)|_{k=0}$, $d(s,n)=\dfrac{1}{\Gamma(\alpha)}a(s)(n-s-1)^{(\alpha-1)}$, using Theorems 3 and 4 one can obtain that

$$
\begin{aligned}
\Delta^{-\alpha}\Delta^{\alpha}u(n) = \Delta^{-\alpha}\Delta\Delta^{-(1-\alpha)}u(n) = \Delta\Delta^{-\alpha}\Delta^{-(1-\alpha)}u(n) - \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}A \\
= \Delta\Delta^{-1}u(n) - \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}A = u(n) - \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}A,
\end{aligned}
$$

and furthermore,

$$
\begin{aligned}
u(n) &= \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}A + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}f(s+\alpha-1, u(s+\alpha-1)) \\
&= \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}A + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}\Delta^{\alpha}u(s) \\
&\leqslant \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}A + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}[a(s)u(s+\alpha-1)+b(s)] \\
&= \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}A + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}b(s) + \sum_{s=0}^{n-\alpha}d(s,n)u(s+\alpha-1) \\
&= e(n) + \sum_{s=0}^{n-\alpha}d(s,n)u(s+\alpha-1), \qquad (4)
\end{aligned}
$$

where $e(n) = \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}A + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}b(s)$.

By denoting $v(n) = \sum_{s=0}^{n-\alpha}d(s,n)u(s+\alpha-1)$, one has

$$
u(n) \leqslant e(n) + v(n),\ n \in \mathbb{N}_{\alpha}, \qquad (5)
$$

and furthermore, for $n \in \mathbb{N}_{\alpha}$,

$$
\begin{aligned}
v(n) - v(n-1) &= d(n-\alpha,n)u(n-1) + \sum_{s=0}^{n-\alpha-1}[d(s,n)-d(s,n-1)]u(s+\alpha-1) \\
&\leqslant d(n-\alpha,n)[e(n-1)+v(n-1)] + \sum_{s=0}^{n-\alpha-1}|d(s,n)-d(s,n-1)|e(s+\alpha-1) \\
&\quad + \sum_{s=0}^{n-\alpha-1}|d(s,n)-d(s,n-1)|v(n-1),
\end{aligned}
$$

which is rewritten by

$$
\begin{aligned}
&v(n) - \{1 + d(n-\alpha,n) + \sum_{s=0}^{n-\alpha-1}|d(s,n)-d(s,n-1)|\}v(n-1) \\
&\leqslant d(n-\alpha,n)e(n-1) + \sum_{s=0}^{n-\alpha-1}|d(s,n)-d(s,n-1)|e(s+\alpha-1). \qquad (6)
\end{aligned}
$$

For $n > \alpha$, substituting n with p in (6), multiplying on both sides by $\prod_{\xi=p+1}^{n}\{1+d(\xi-\alpha,\xi)+\sum_{s=0}^{\xi-\alpha-1}|d(s,\xi)-d(s,\xi-1)|\}$, a summation with respect to p from α to $n-1$ together with (6), and using $v(\alpha-1)=0$, yields that

$$\begin{aligned}
v(n) \leqslant\ & d(n-\alpha,n)e(n-1)+\sum_{s=0}^{n-\alpha-1}|d(s,n)-d(s,n-1)|e(s+\alpha-1)\\
&+\sum_{p=\alpha}^{n-1}\{[d(p-\alpha,p)e(p-1)+\sum_{s=0}^{p-\alpha-1}|d(s,p)-d(s,p-1)|e(s+\alpha-1)]\\
&\times\prod_{\xi=p+1}^{n}[1+d(\xi-\alpha,\xi)+\sum_{s=0}^{\xi-\alpha-1}|d(s,\xi)-d(s,\xi-1)|]\}. \qquad (7)
\end{aligned}$$

Note that (7) also holds for $n=\alpha$. So (7) holds in fact for $n\in\mathbb{N}_\alpha$. Combining (5) and (7) we can get the desired result. □

THEOREM 6. *Assume $0<\alpha\leqslant 1$, $u(n)$ is a nonnegative function defined on $\mathbb{N}_{\alpha-1}$, and a, b are nonnegative functions defined on $\mathbb{N}_0$. If the following inequality holds:*

$$\Delta^\alpha u(k)\leqslant a(k)[u(k+\alpha-1)+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{k-1}(k+\alpha-s-2)^{(\alpha-1)}u(s+\alpha-1)]+b(k),\ k\in\mathbb{N}_0, \qquad (8)$$

then for $n\in\mathbb{N}_\alpha$, one has

$$\begin{aligned}
u(n)\leqslant\ & \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}A+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}\{a(s)[\frac{(s+\alpha-1)^{(\alpha-1)}}{\Gamma(\alpha)}\widetilde{A}\\
&+\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{s-1}(s+\alpha-\xi-2)^{(\alpha-1)}b(\xi)+\widetilde{A}q(s+\alpha-1)]+b(s)\}. \qquad (9)
\end{aligned}$$

where

$$\begin{cases}
A=\Delta^{\alpha-1}u(k)|_{k=0},\\
\widetilde{A}=\Delta^{\alpha-1}u(k)|_{k=0}+\Delta^{-1}u(k+\alpha-1)|_{k=0},\\
q(n)=\widetilde{d}(n-\alpha,n)\dfrac{(n-1)^{(\alpha-1)}}{\Gamma(\alpha)}+\sum_{s=0}^{n-\alpha-1}|\widetilde{d}(s,n)-\widetilde{d}(s,n-1)|\dfrac{(s+\alpha-1)^{(\alpha-1)}}{\Gamma(\alpha)}\\
\qquad+\sum_{p=\alpha}^{n-1}\Big\{\Big[\widetilde{d}(p-\alpha,p)\dfrac{(p-1)^{(\alpha-1)}}{\Gamma(\alpha)}+\sum_{s=0}^{p-\alpha-1}|\widetilde{d}(s,p)-\widetilde{d}(s,p-1)|\dfrac{(s+\alpha-1)^{(\alpha-1)}}{\Gamma(\alpha)}\Big]\\
\qquad\times\prod_{\xi=p+1}^{n}[1+\widetilde{d}(\xi-\alpha,\xi)+\sum_{s=0}^{\xi-\alpha-1}|\widetilde{d}(s,\xi)-\widetilde{d}(s,\xi-1)|]\Big\},\\
\widetilde{d}(s,n)=\dfrac{1}{\Gamma(\alpha)}[a(s)+1](n-s-1)^{(\alpha-1)}.
\end{cases} \qquad (10)$$

Proof. Denote

$$z(n)=u(n)+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}u(s+\alpha-1),\ n\in\mathbb{N}_\alpha.$$

Then for $k\in\mathbb{N}_0$, one has

$$\Delta^\alpha u(k)\leqslant a(k)z(k+\alpha-1)+b(k), \tag{11}$$

and

$$u(k+\alpha-1)\leqslant z(k+\alpha-1).$$

Furthermore, it holds that

$$\Delta^\alpha z(k)=\Delta^\alpha u(k)+u(k+\alpha-1)\leqslant [a(k)+1]z(k+\alpha-1)+b(k), \tag{12}$$

and

$$\Delta^{\alpha-1}z(k)=\Delta^{-1}\Delta\Delta^{-(1-\alpha)}z(k)=\Delta^{\alpha-1}u(k)+\Delta^{-1}u(k+\alpha-1).$$

Applying Theorem 5 to (12) yields that for $n\in\mathbb{N}_\alpha$,

$$z(n)\leqslant\frac{n^{(\alpha-1)}}{\Gamma(\alpha)}\widetilde{A}+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}b(s)+\widetilde{A}q(n). \tag{13}$$

where $q(n)$, $\widetilde{A}$, $\widetilde{d}(s,n)$ are defined as in (10). So by a combination of (11) and (13) we obtain that

$$\begin{aligned}\Delta^\alpha u(k)\leqslant a(k)\left[\frac{(k+\alpha-1)^{(\alpha-1)}}{\Gamma(\alpha)}\widetilde{A}+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{k-1}(k+\alpha-s-2)^{(\alpha-1)}b(s)+\widetilde{A}q(k+\alpha-1)\right]\\ +b(k)\triangleq\widetilde{b}(k),\quad k\in\mathbb{N}_0.\end{aligned} \tag{14}$$

Applying Theorem 5 again to (14) yields that for $n\in\mathbb{N}_\alpha$,

$$u(n)\leqslant\frac{n^{(\alpha-1)}}{\Gamma(\alpha)}A+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}\widetilde{b}(s), \tag{15}$$

where $A=\Delta^{\alpha-1}u(k)|_{k=0}$ is defined as in (10). The desired inequality (9) can be obtained after substituting the expression of $\widetilde{b}(s)$ into (15). □

Next we consider two discrete fractional sum inequalities based on Theorems 5 and 6.

LEMMA 7. [31] *Assume that $a\geqslant 0$, $p\geqslant q\geqslant 0$, and $p\neq 0$, then for any $K>0$,*

$$a^{\frac{q}{p}}\leqslant\frac{q}{p}K^{\frac{q-p}{p}}a+\frac{p-q}{p}K^{\frac{q}{p}}.$$

THEOREM 8. *Assume* $0 < \alpha \leqslant 1$, $u(n)$, $a(n)$ *are nonnegative functions defined on* $\mathbb{N}_{\alpha-1}$*, and* b, c *are nonnegative functions defined on* $\mathbb{N}_0$, m_1, m_2, m_3 *are constants with* $m_1 \geqslant m_2 \geqslant m_3 > 0$*. If for* $n \in \mathbb{N}_\alpha$*, the following inequality holds:*

$$u^{m_1}(n) \leqslant a(n)+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}[b(s)u^{m_2}(s+\alpha-1)+c(s)u^{m_3}(s+\alpha-1)], \quad (16)$$

then for $n \in \mathbb{N}_\alpha$*, one has*

$$\begin{aligned} u(n) \leqslant \{a(n) &+ \widehat{e}(n) + \widehat{d}(n-\alpha,n)\widehat{e}(n-1) + \sum_{s=0}^{n-\alpha-1}|\widehat{d}(s,n)-\widehat{d}(s,n-1)|\widehat{e}(s+\alpha-1) \\ &+ \sum_{p=\alpha}^{n-1}\{[\widehat{d}(p-\alpha,p)\widehat{e}(p-1) + \sum_{s=0}^{p-\alpha-1}|\widehat{d}(s,p)-\widehat{d}(s,p-1)|\widehat{e}(s+\alpha-1)] \\ &\times \prod_{\xi=p+1}^{n}[1+\widehat{d}(\xi-\alpha,\xi)+\sum_{s=0}^{\xi-\alpha-1}|\widehat{d}(s,\xi)-\widehat{d}(s,\xi-1)|]\}\}^{\frac{1}{m_1}}, \end{aligned} \quad (17)$$

where

$$\begin{cases} \widehat{d}(s,n) = \dfrac{1}{\Gamma(\alpha)}(n-s-1)^{(\alpha-1)}\Big[\dfrac{m_2}{m_1}K^{\frac{m_2-m_1}{m_1}}b(s)+\dfrac{m_3}{m_1}K^{\frac{m_3-m_1}{m_1}}c(s)\Big], \\ \widehat{e}(n) = \dfrac{1}{\Gamma(\alpha)}\sum\limits_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}\Big\{b(s)\Big[\dfrac{m_2}{m_1}K^{\frac{m_2-m_1}{m_1}}a(s+\alpha-1)+\dfrac{m_1-m_2}{m_1}K^{\frac{m_2}{m_1}}\Big] \\ \qquad +c(s)\Big[\dfrac{m_3}{m_1}K^{\frac{m_3-m_1}{m_1}}a(s+\alpha-1)+\dfrac{m_1-m_3}{m_1}K^{\frac{m_3}{m_1}}\Big]\Big\}, \end{cases} \quad (18)$$

and $K > 0$ *is an arbitrary constant.*

Proof. Denote

$$v(n) = \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}[b(s)u^{m_2}(s+\alpha-1)+c(s)u^{m_3}(s+\alpha-1)].$$

Then

$$u^{m_1}(n) \leqslant a(n)+v(n),\ n \in \mathbb{N}_\alpha. \quad (19)$$

and it holds that

$$\begin{aligned} v(n) &\leqslant \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}\{b(s)[a(s+\alpha-1)+v(s+\alpha-1)]^{\frac{m_2}{m_1}} \\ &\quad + c(s)[a(s+\alpha-1)+v(s+\alpha-1)]^{\frac{m_3}{m_1}}\} \\ &\leqslant \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}\Big\{b(s)\Big[\frac{m_2}{m_1}K^{\frac{m_2-m_1}{m_1}}(a(s+\alpha-1)+v(s+\alpha-1)) \\ &\quad +\frac{m_1-m_2}{m_1}K^{\frac{m_2}{m_1}}\Big]+c(s)\Big[\frac{m_3}{m_1}K^{\frac{m_3-m_1}{m_1}}(a(s+\alpha-1)+v(s+\alpha-1))+\frac{m_1-m_3}{m_1}K^{\frac{m_3}{m_1}}\Big]\Big\} \\ &= \widehat{e}(n)+\sum_{s=0}^{n-\alpha}\widehat{d}(s,n)v(s+\alpha-1), \end{aligned} \quad (20)$$

where $\widehat{e}(n)$, $\widehat{d}(s,n)$ are defined as in (18), and $K > 0$ is an arbitrary constant.

Since the structure of (20) is similar to that of (4), following a similar manner to the process of (4) to (7) one can deduce that

$$\begin{aligned} v(n) \leqslant\ & \widehat{e}(n) + \widehat{d}(n-\alpha,n)\widehat{e}(n-1) + \sum_{s=0}^{n-\alpha-1} |\widehat{d}(s,n) - \widehat{d}(s,n-1)|\widehat{e}(s+\alpha-1) \\ & + \sum_{p=\alpha}^{n-1} \{[\widehat{d}(p-\alpha,p)\widehat{e}(p-1) + \sum_{s=0}^{p-\alpha-1} |\widehat{d}(s,p) - \widehat{d}(s,p-1)|\widehat{e}(s+\alpha-1)] \\ & \times \prod_{\xi=p+1}^{n} [1 + \widehat{d}(\xi-\alpha,\xi) + \sum_{s=0}^{\xi-\alpha-1} |\widehat{d}(s,\xi) - \widehat{d}(s,\xi-1)|]\},\ n \in \mathbb{N}_\alpha. \end{aligned} \tag{21}$$

Combining (19) and (21) we can obtain the desired inequality (17). □

THEOREM 9. *Under the conditions of Theorem 6, furthermore, assume* $0 < l_1 \leqslant 1$. *If the following inequality holds:*

$$\Delta^\alpha u(k) \leqslant a(k)[u^{l_1}(k+\alpha-1) + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{k-1} (k+\alpha-s-2)^{(\alpha-1)} u^{l_1}(s+\alpha-1)] + b(k),\ k \in \mathbb{N}_0, \tag{22}$$

then for $n \in \mathbb{N}_\alpha$, *one has*

$$\begin{aligned} u(n) \leqslant\ & \frac{n^{(\alpha-1)}}{\Gamma(\alpha)} A + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{n-\alpha} (n-s-1)^{(\alpha-1)} \Big\{ \overline{a}(s) \Big[\frac{(s+\alpha-1)^{(\alpha-1)}}{\Gamma(\alpha)} \overline{A} \\ & + \frac{1}{\Gamma(\alpha)} \sum_{\xi=0}^{s-1} (s+\alpha-\xi-2)^{(\alpha-1)} \overline{b}(\xi) + \overline{A}\overline{q}(s+\alpha-1) \Big] + \overline{b}(s) \Big\}. \end{aligned} \tag{23}$$

where

$$\begin{cases} A = \Delta^{\alpha-1} u(k)|_{k=0}, \\ \overline{A} = \Delta^{\alpha-1} u(k)|_{k=0} + \Delta^{-1} u(k+\alpha-1)|_{k=0}, \\ \overline{q}(n) = \overline{d}(n-\alpha,n)\dfrac{(n-1)^{(\alpha-1)}}{\Gamma(\alpha)} + \sum\limits_{s=0}^{n-\alpha-1} |\overline{d}(s,n) - \overline{d}(s,n-1)| \dfrac{(s+\alpha-1)^{(\alpha-1)}}{\Gamma(\alpha)} \\ \qquad + \sum\limits_{p=\alpha}^{n-1} \Big\{ \Big[\overline{d}(p-\alpha,p) \dfrac{(p-1)^{(\alpha-1)}}{\Gamma(\alpha)} + \sum\limits_{s=0}^{p-\alpha-1} |\overline{d}(s,p) - \overline{d}(s,p-1)| \dfrac{(s+\alpha-1)^{(\alpha-1)}}{\Gamma(\alpha)} \Big] \\ \qquad \times \prod\limits_{\xi=p+1}^{n} \Big[1 + \overline{d}(\xi-\alpha,\xi) + \sum\limits_{s=0}^{\xi-\alpha-1} |\overline{d}(s,\xi) - \overline{d}(s,\xi-1)| \Big] \Big\}, \\ \overline{d}(s,n) = \dfrac{1}{\Gamma(\alpha)} [\overline{a}(s) + 1](n-s-1)^{(\alpha-1)}, \\ \overline{a}(k) = l_1 K^{l_1-1} a(k), \\ \overline{b}(k) = b(k) + a(k) K^{l_1} (1-l_1) [1 + \dfrac{1}{\Gamma(\alpha)} \sum\limits_{s=0}^{k-1} (k+\alpha-s-2)^{(\alpha-1)}], \end{cases} \tag{24}$$

and $K > 0$ is an arbitrary constant.

Proof. By (22) and Lemma 7 one has

$$\begin{aligned}\Delta^{\alpha}u(k) &\leqslant a(k)[u^{l_1}(k+\alpha-1)+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{k-1}(k+\alpha-s-2)^{(\alpha-1)}u^{l_1}(s+\alpha-1)]+b(k)\\ &\leqslant a(k)\{[l_1K^{l_1-1}u(k+\alpha-1)+(1-l_1)K^{l_1}]\\ &\quad+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{k-1}(k+\alpha-s-2)^{(\alpha-1)}[l_1K^{l_1-1}u(s+\alpha-1)+(1-l_1)K^{l_1}]\}+b(k)\\ &=\overline{a}(k)\{u(k+\alpha-1)+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{k-1}(k+\alpha-s-2)^{(\alpha-1)}u(s+\alpha-1)\}+\overline{b}(k),\end{aligned} \tag{25}$$

where $\overline{a}(k)$, $\overline{b}(k)$ are defined as in (24), and $K > 0$ is an arbitrary constant.

Applying Theorem 6 to (25) yields the desired result. □

REMARK 1. *If $m_1 = m_2 = 1$, $m_3 < 1$, then Theorem 8 becomes the discrete version of the fractional differential inequality (17) in [22, Theorem 4].* If $m_1 = m_2 = 1$, $m_3 = 0$, then Theorem 8 becomes the discrete version of the fractional differential inequalities in [21, Theorem 1] and [24, Theorem 3] with $g(t)$, $b(t) \equiv 1$ there. Moreover, *we note that the main inequalities in Theorems 5, 6 and 9 are essentially different from the main results in [25]*, and are discrete Gronwall-Bellman type inequalities of new forms so far in the literature.

3. Applications

In this section, we apply the inequalities established above to research boundedness, uniqueness, and continuous dependence on the initial value for the solution to a fractional difference equation.

EXAMPLE 1. Consider the following IVP of fractional difference equation [28, Eqs. (3.1)–(3.2)]:

$$\begin{cases}\Delta^{\alpha}u(k)=L(k+\alpha-1,u(k+\alpha-1)), & k\in\mathbb{N}_0,\\ \Delta^{\alpha-1}u(k)|_{k=0}=a_0,\end{cases} \tag{26}$$

where $0<\alpha<1$, $u(n)$ is an unknown function defined on $\mathbb{N}_{\alpha-1}$, $L:\ \mathbb{N}_{\alpha-1}\times\mathbb{R}\to\mathbb{R}$.

THEOREM 10. *For the IVP (26), if $|L(k+\alpha-1,u(k+\alpha-1))|\leqslant g(k)|u(k+\alpha-1)|$, where g is a nonnegative function defined on $\mathbb{N}_0$, then we have the following estimate for $u(n)$:*

$$
\begin{aligned}
|u(n)| \leqslant & \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}|a_0|+\widehat{e}(n)+\widehat{d}(n-\alpha,n)\widehat{e}(n-1)+\sum_{s=0}^{n-\alpha-1}|\widehat{d}(s,n)-\widehat{d}(s,n-1)|\widehat{e}(s+\alpha-1) \\
&+\sum_{p=\alpha}^{n-1}\{[\widehat{d}(p-\alpha,p)\widehat{e}(p-1)+\sum_{s=0}^{p-\alpha-1}|\widehat{d}(s,p)-\widehat{d}(s,p-1)|\widehat{e}(s+\alpha-1)] \\
&\times\prod_{\xi=p+1}^{n}[1+\widehat{d}(\xi-\alpha,\xi)+\sum_{s=0}^{\xi-\alpha-1}|\widehat{d}(s,\xi)-\widehat{d}(s,\xi-1)|]\}\triangleq I(n),\ n\in\mathbb{N}_\alpha, \quad (27)
\end{aligned}
$$

where

$$
\begin{cases}
\widehat{d}(s,n)=\dfrac{1}{\Gamma(\alpha)}(n-s-1)^{(\alpha-1)}g(s), \\
\widehat{e}(n)=\dfrac{|a_0|}{[\Gamma(\alpha)]^2}\sum\limits_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}g(s)(s+\alpha-1)^{(\alpha-1)},
\end{cases}
$$

Proof. By [28, Eq. (3.4)], the equivalent discrete fractional sum equation of the IVP (26) can be denoted as follows:

$$
u(n)=\frac{n^{(\alpha-1)}}{\Gamma(\alpha)}a_0+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}L(s+\alpha-1,u(s+\alpha-1)).
$$

So

$$
\begin{aligned}
|u(n)| &\leqslant \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}|a_0|+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}|L(s+\alpha-1,u(s+\alpha-1))| \\
&\leqslant \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}|a_0|+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}g(s)|u(s+\alpha-1)|. \quad (28)
\end{aligned}
$$

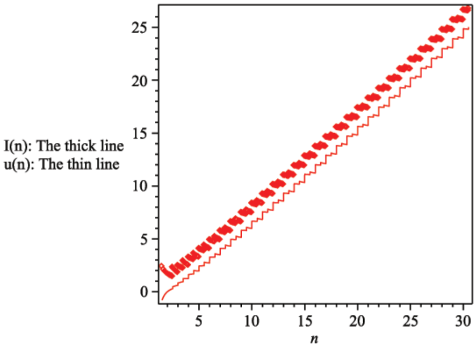

Fig 1. The solution u(n) and its bound I(n) in (26) with α=0.5, g(k)=1, $a_0 = 1$

Figure 1: *The solution $u(n)$ and its bound $I(n)$ in (26) with $\alpha = 0.5,\ g(k) = 1,\ a_0 = 1$.*

Then a suitable application of Theorem 8 (with $m_1 = m_2 = 1,\ c(s) \equiv 1$) to (28) yields the desired result.

In Figure 1, the bound $I(n)$ for the function $u(n)$ is demonstrated with $\alpha = \frac{1}{2}$, $g(k) \equiv 1$, $a_0 = 1$. □

THEOREM 11. *For the IVP (26), if* $|L(k+\alpha-1,u)-L(k+\alpha-1,v)| \leqslant g(k)|u-v|$*, where* g *is a nonnegative function defined on* $\mathbb{N}_0$*, then the IVP (26) has at most one solution.*

Proof. Suppose the IVP (26) has two solutions $u_1(n),\ u_2(n)$. Then we have

$$u_1(n) = \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}a_0 + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}L(s+\alpha-1,u(s+\alpha-1)), \tag{29}$$

and

$$u_2(n) = \frac{n^{(\alpha-1)}}{\Gamma(\alpha)}a_0 + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}L(s+\alpha-1,u(s+\alpha-1)). \tag{30}$$

Furthermore,

$$\begin{aligned} u_1(n)-u_2(n) = \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}[&L(s+\alpha-1,u_1(s+\alpha-1)) \\ &-L(s+\alpha-1,u_2(s+\alpha-1))], \end{aligned} \tag{31}$$

which implies

$$\begin{aligned} |u_1(n)-u_2(n)| &\leqslant \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}|L(s+\alpha-1,u_1(s+\alpha-1)) \\ &\quad -L(s+\alpha-1,u_2(s+\alpha-1))| \\ &\leqslant \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}g(s)|u_1(s+\alpha-1)-u_2(s+\alpha-1)|, \end{aligned} \tag{32}$$

Treating $|u_1(n)-u_2(n)|$ as one unity, by applying Theorem 8 to (32) one can deduce that $|u_1(n)-u_2(n)| \leqslant 0$. So $u_1(n) \equiv u_2(n)$, and the proof is complete. □

Now we research the continuous dependence on the initial value for the solution of the IVP (26).

THEOREM 12. *Let* $u(n)$ *be the solution of the IVP (26), and* $\overline{u}(n)$ *be the solution of the following IVP:*

$$\begin{cases} \Delta^{\alpha}\overline{u}(k) = L(k+\alpha-1,\overline{u}(k+\alpha-1)),\ k=0,1,2..., \\ \Delta^{\alpha-1}\overline{u}(k)|_{k=0} = \overline{a}_0, \end{cases} \tag{33}$$

If $|a(0)-\overline{a}(0)|<\varepsilon$*, where* ε *is an arbitrarily small constant, and* $|L(k+\alpha-1,u)-L(k+\alpha-1,v)|\leqslant g(k)|u-v|$*, where* g *is a nonnegative function defined on* $\mathbb{N}_0$*, then we have*

$$|u(n)-\overline{u}(n)|\leqslant \varepsilon Q(n), \tag{34}$$

where

$$\begin{aligned}Q(n)=\Bigg\{&\frac{n^{(\alpha-1)}}{\Gamma(\alpha)}+\widetilde{\widetilde{e}}(n)+\widetilde{\widetilde{d}}(n-\alpha,n)\widetilde{\widetilde{e}}(n-1)+\sum_{s=0}^{n-\alpha-1}|\widetilde{\widetilde{d}}(s,n)-\widetilde{\widetilde{d}}(s,n-1)|\widetilde{\widetilde{e}}(s+\alpha-1)\\&+\sum_{p=\alpha}^{n-1}\{[\widetilde{\widetilde{d}}(p-\alpha,p)\widetilde{\widetilde{e}}(p-1)+\sum_{s=0}^{p-\alpha-1}|\widetilde{\widetilde{d}}(s,p)-\widetilde{\widetilde{d}}(s,p-1)|\widetilde{\widetilde{e}}(s+\alpha-1)]\\&\times\prod_{\xi=p+1}^{n}[1+\widetilde{\widetilde{d}}(\xi-\alpha,\xi)+\sum_{s=0}^{\xi-\alpha-1}|\widetilde{\widetilde{d}}(s,\xi)-\widetilde{\widetilde{d}}(s,\xi-1)|]\}\Bigg\},\end{aligned}$$

and

$$\begin{cases}\widetilde{\widetilde{d}}(s,n)=\dfrac{1}{\Gamma(\alpha)}(n-s-1)^{(\alpha-1)}g(s),\\ \widetilde{\widetilde{e}}(n)=\dfrac{1}{[\Gamma(\alpha)]^2}\displaystyle\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}g(s)(s+\alpha-1)^{(\alpha-1)},\end{cases}$$

Proof. Similar to Theorem 10, we can obtain the equivalent discrete fractional sum equation of the IVP (33) as follows:

$$\overline{u}(n)=\frac{n^{(\alpha-1)}}{\Gamma(\alpha)}\overline{a}_0+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}L(s+\alpha-1,\overline{u}(s+\alpha-1)). \tag{35}$$

So we have

$$\begin{aligned}u(n)-\overline{u}(n)=&\frac{n^{(\alpha-1)}}{\Gamma(\alpha)}|a_0-\overline{a}_0|+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}\\&\times[L(s+\alpha-1,u(s+\alpha-1))-L(s+\alpha-1,\overline{u}(s+\alpha-1))].\end{aligned}\tag{36}$$

Furthermore,

$$|u(n)-\overline{u}(n)|\leqslant\frac{n^{(\alpha-1)}}{\Gamma(\alpha)}\varepsilon+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{n-\alpha}(n-s-1)^{(\alpha-1)}g(s)|u_1(s+\alpha-1)-u_2(s+\alpha-1)|. \tag{37}$$

Applying Theorem 8 to (37), after some basic computation we can deduce the desired result (34). □

EXAMPLE 2. Consider the following IVP of fractional difference equation:

$$\begin{cases}\Delta^{\frac{1}{2}}u^3(k)=g_1(k)u^3(k+\alpha-1)+g_2(k)u(k+\alpha-1), & k\in\mathbb{N}_0,\\ \Delta^{-\frac{1}{2}}u^3(k)|_{k=0}=u_0,\end{cases}\tag{38}$$

where $u(n)$ is an unknown function defined on $\mathbb{N}_{-\frac{1}{2}}$.

THEOREM 13. *For the IVP (26), if g_1, g_2 are nonnegative functions defined on $\mathbb{N}_0$, then we have the following estimate for $u(n)$:*

$$|u(n)| \leqslant \Bigg\{ \frac{n^{(-\frac{1}{2})}}{\Gamma\left(\frac{1}{2}\right)}|u_0| + \widehat{e}(n) + \widehat{d}\Big(n-\frac{1}{2},n\Big)\widehat{e}(n-1) + \sum_{s=0}^{n-\frac{3}{2}} |\widehat{d}(s,n)-\widehat{d}(s,n-1)|\widehat{e}\Big(s-\frac{1}{2}\Big)$$
$$+ \sum_{p=\frac{1}{2}}^{n-1} \Bigg\{ \Bigg[\widehat{d}\Big(p-\frac{1}{2},p\Big)\widehat{e}(p-1) + \sum_{s=0}^{p-\frac{3}{2}} |\widehat{d}(s,p)-\widehat{d}(s,p-1)|\widehat{e}\Big(s-\frac{1}{2}\Big)\Bigg]$$
$$\times \prod_{\xi=p+1}^{n} \Bigg[1+\widehat{d}\Big(\xi-\frac{1}{2},\xi\Big) + \sum_{s=0}^{\xi-\frac{3}{2}} |\widehat{d}(s,\xi)-\widehat{d}(s,\xi-1)|\Bigg]\Bigg\}\Bigg\}^{\frac{1}{3}} \triangleq \theta(n),\ n\in\mathbb{N}_{\frac{1}{2}}, \tag{39}$$

where

$$\begin{cases} \widehat{d}(s,n) = \dfrac{1}{\Gamma\left(\frac{1}{2}\right)}(n-s-1)^{(-\frac{1}{2})}[g_1(s)+\frac{1}{3}K^{-\frac{2}{3}}g_2(s)], \\ \widehat{e}(n) = \dfrac{1}{\Gamma\left(\frac{1}{2}\right)}\displaystyle\sum_{s=0}^{n-\frac{1}{2}}(n-s-1)^{(-\frac{1}{2})}\Bigg\{g_1(s)\frac{\left(s-\frac{1}{2}\right)^{(-\frac{1}{2})}}{\Gamma\left(\frac{1}{2}\right)}|u_0| \\ \qquad + g_2(s)\Bigg[\frac{1}{3}K^{-\frac{2}{3}}\frac{\left(s-\frac{1}{2}\right)^{(-\frac{1}{2})}}{\Gamma\left(\frac{1}{2}\right)}|u_0| + \frac{2}{3}K^{\frac{1}{3}}\Bigg]\Bigg\}, \end{cases} \tag{40}$$

and $K>0$ is an arbitrary constant.

Proof. Similar to Theorem 10, the equivalent discrete fractional sum equation of the IVP (38) can be denoted as follows:

$$u^3(n) = \frac{n^{(-\frac{1}{2})}}{\Gamma\left(\frac{1}{2}\right)}u_0 + \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sum_{s=0}^{n-\frac{1}{2}}(n-s-1)^{(-\frac{1}{2})}\Bigg[g_1(s)u^3\Big(s-\frac{1}{2}\Big) + g_2(s)u\Big(s-\frac{1}{2}\Big)\Bigg].$$

So

$$|u(n)|^3 \leqslant \frac{n^{(-\frac{1}{2})}}{\Gamma\left(\frac{1}{2}\right)}|u_0| + \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sum_{s=0}^{n-\frac{1}{2}}(n-s-1)^{(-\frac{1}{2})}\Bigg|g_1(s)u^3\Big(s-\frac{1}{2}\Big) + g_2(s)u\Big(s-\frac{1}{2}\Big)\Bigg|$$
$$\leqslant \frac{n^{(-\frac{1}{2})}}{\Gamma\left(\frac{1}{2}\right)}|u_0| + \frac{1}{\Gamma\left(\frac{1}{2}\right)}\sum_{s=0}^{n-\frac{1}{2}}(n-s-1)^{(-\frac{1}{2})}\Bigg[g_1(s)\Big|u\Big(s-\frac{1}{2}\Big)\Big|^3 + g_2(s)\Big|u\Big(s-\frac{1}{2}\Big)\Big|\Bigg]. \tag{41}$$

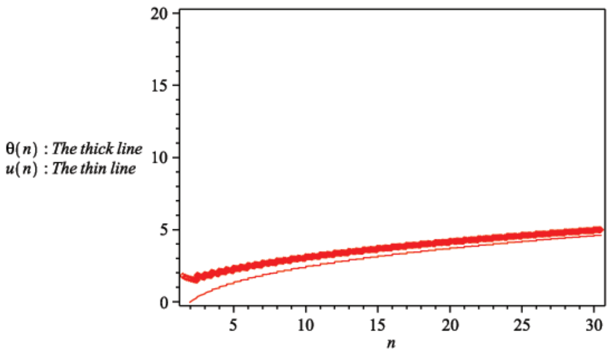

Fig 2. The solution u(n) and its bound θ(n) in (39) with $u_0 = 1, g_1(k) = g_2(k) = 1$,

$K=\frac{\Gamma(s+0.5)}{\Gamma(s+1)\Gamma(0.5)}$

Figure 2: *The solution $u(n)$ and its bound $\theta(n)$ in (39) with* $u_0 = 1$, $g_1(k) = g_2(k) = 1$, $K = \frac{\Gamma(s+0.5)}{\Gamma(s+1)\Gamma(0.5)}$.

Applying Theorem 8 (with $m_1 = m_2 = 3$, $m_3 = 1$) to (41) yields the desired result.

In Figure 2, the bound $\theta(n)$ for the function $u(n)$ is demonstrated with $g_1(k) = g_2(k) \equiv 1$, $u_0 = 1$, $K = 1$. $\square$

EXAMPLE 3. Consider the following fractional sum-difference equation:

$$\Delta^{\frac{1}{3}}|u(k)| = u\left(k-\frac{2}{3}\right) + \frac{1}{\Gamma\left(\frac{1}{3}\right)}\sum_{s=0}^{k-1}\left(k-s-\frac{5}{3}\right)^{\left(-\frac{2}{3}\right)} u\left(s-\frac{2}{3}\right), \quad k \in \mathbb{N}_0. \tag{42}$$

where $u(n)$ is an unknown function defined on $\mathbb{N}_{-\frac{2}{3}}$.

THEOREM 14. *For (42), we have the following estimate for $u(n)$:*

$$u(n) \leqslant \frac{n^{\left(-\frac{2}{3}\right)}}{\Gamma\left(\frac{1}{3}\right)}A + \frac{1}{\Gamma\left(\frac{1}{3}\right)}\sum_{s=0}^{n-\frac{1}{3}}(n-s-1)^{\left(-\frac{2}{3}\right)}\left\{\left[\frac{\left(s-\frac{2}{3}\right)^{\left(-\frac{2}{3}\right)}}{\Gamma\left(\frac{1}{3}\right)}\widetilde{A} + \widetilde{A}q\left(s-\frac{2}{3}\right)\right]\right\}, \tag{43}$$

where

$$
\begin{cases}
A=\Delta^{-\frac{2}{3}}u(k)|_{k=0},\\
\widetilde{A}=\Delta^{-\frac{2}{3}}u(k)|_{k=0}+\Delta^{-1}u(k-\frac{2}{3})|_{k=0},\\
q(n)=\widetilde{d}(n-\frac{1}{3},n)\dfrac{(n-1)^{(-\frac{2}{3})}}{\Gamma\left(\frac{1}{3}\right)}+\sum\limits_{s=0}^{n-\frac{4}{3}}|\widetilde{d}(s,n)-\widetilde{d}(s,n-1)|\dfrac{\left(s-\frac{2}{3}\right)^{(-\frac{2}{3})}}{\Gamma\left(\frac{1}{3}\right)}\\
\qquad+\sum\limits_{p=\frac{1}{3}}^{n-1}\left\{\left[\widetilde{d}(p-\frac{1}{3},p)\dfrac{(p-1)^{(-\frac{2}{3})}}{\Gamma\left(\frac{1}{3}\right)}+\sum\limits_{s=0}^{p-\frac{4}{3}}|\widetilde{d}(s,p)-\widetilde{d}(s,p-1)|\dfrac{\left(s-\frac{2}{3}\right)^{(-\frac{2}{3})}}{\Gamma\left(\frac{1}{3}\right)}\right]\right.\\
\qquad\left.\times\prod\limits_{\xi=p+1}^{n}[1+\widetilde{d}(\xi-\frac{1}{3},\xi)+\sum\limits_{s=0}^{\xi-\frac{4}{3}}|\widetilde{d}(s,\xi)-\widetilde{d}(s,\xi-1)|]\right\},\\
\widetilde{d}(s,n)=\dfrac{2}{\Gamma\left(\frac{1}{3}\right)}(n-s-1)^{(-\frac{2}{3})}.
\end{cases}
\tag{44}
$$

Proof. By (42) one has:

$$
\begin{aligned}
\Delta^{\frac{1}{3}}|u(k)| &\leqslant \left|u\left(k-\frac{2}{3}\right)+\frac{1}{\Gamma\left(\frac{1}{3}\right)}\sum_{s=0}^{k-1}\left(k-s-\frac{5}{3}\right)^{(-\frac{2}{3})}u\left(s-\frac{2}{3}\right)\right|\\
&\leqslant \left|u\left(k-\frac{2}{3}\right)\right|+\frac{1}{\Gamma\left(\frac{1}{3}\right)}\sum_{s=0}^{k-1}\left(k-s-\frac{5}{3}\right)^{(-\frac{2}{3})}\left|u\left(s-\frac{2}{3}\right)\right|.
\end{aligned}
\tag{45}
$$

Applying Theorem 6 (with $a(k)\equiv 1$, $a(k)\equiv 0$) to (45) yields the desired result. □

4. Conclusions

In this paper, based on the theory of discrete fractional calculus, we have presented some new Gronwall-Bellman type discrete fractional difference inequalities and fractional sum inequalities, which can be seen as the discrete version of Gronwall-Bellman type fractional differential inequalities. These inequalities can be used to provide explicit estimates for solutions of unknown functions of discrete fractional difference equations, and can also be used in the analysis of uniqueness and continuous dependence on the initial value for the solutions. For demonstrating the validity of the presented results, we apply them to research several initial value problems of fractional difference equations.

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