Journal of
Mathematical
Inequalities
Volume 9, Number 3 (2015), 847–872 doi:10.7153/jmi-09-70

GENERALIZATION OF MAJORIZATION THEOREM

M. ADIL KHAN, N. LATIF AND J. PEČARIĆ

(*Communicated by A. Aglić Aljinović*)

Abstract. We give generalization of majorization theorem for the class of n-convex functions by using Taylor's formula and Green function. We use inequalities for the Čebyšev functional to obtain bounds for the identities related to generalizations of majorization inequalities. We present mean value theorems and n-exponential convexity for the functional obtained from the generalized majorization inequalities. At the end we discuss the results for particular families of function and give means.

1. Introduction and preliminaries

For fixed $m \geqslant 2$ let

$$\mathbf{x} = (x_1, \dots, x_m), \ \mathbf{y} = (y_1, \dots, y_m)$$

denote two real m-tuples. Let

$$x_{[1]} \geqslant x_{[2]} \geqslant \dots \geqslant x_{[m]}, \quad y_{[1]} \geqslant y_{[2]} \geqslant \dots \geqslant y_{[m]},$$

$$x_{(1)} \leqslant x_{(2)} \leqslant \dots \leqslant x_{(m)}, \quad y_{(1)} \leqslant y_{(2)} \leqslant \dots \leqslant y_{(m)}$$

be their ordered components.

DEFINITION 1. [15, p. 319] $\mathbf{x}$ is said to majorize $\mathbf{y}$ (or $\mathbf{y}$ is said to be majorized by $\mathbf{x}$), in symbol, $\mathbf{x} \succ \mathbf{y}$, if

$$\sum_{i=1}^{l} y_{[i]} \leqslant \sum_{i=1}^{l} x_{[i]} \tag{1}$$

holds for $l = 1, 2, \dots, m-1$ and

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i.$$

Note that (1) is equivalent to

$$\sum_{i=m-l+1}^{m} y_{(i)} \leqslant \sum_{i=m-l+1}^{m} x_{(i)}$$

holds for $l = 1, 2, \dots, m-1$.

Mathematics subject classification (2010): 26D15, 26D20, 26D99.
Keywords and phrases: Majorization theorem, Taylor's formula, Cebysev functional, n-exponentially convex function, Stolarsky type means.
The research of the third author has been fully supported by Croatian Science Foundation under the project 5435.

The following theorem is well-known as the majorization theorem given by Marshall and Olkin [13, p. 14] (see also [15, p. 320]):

THEOREM 1. *Let* $\mathbf{x} = (x_1, \dots, x_m)$, $\mathbf{y} = (y_1, \dots, y_m)$ *be two* m*-tuples such that* x_i, $y_i \in [a,b]$ $(i = 1, \dots, m)$. *Then*

$$\sum_{i=1}^{m} \phi(y_i) \leqslant \sum_{i=1}^{m} \phi(x_i) \tag{2}$$

holds for every continuous convex function $\phi : [a,b] \to \mathbb{R}$ *if and only if* $\mathbf{x} \succ \mathbf{y}$ *holds.*

The following theorem can be regarded as a weighted version of Theorem 1 and is proved by Fuchs in [8] ([13, p. 580], [15, p. 323]):

THEOREM 2. *Let* $\mathbf{x} = (x_1, \dots, x_m)$, $\mathbf{y} = (y_1, \dots, y_m)$ *be two decreasing real* m*-tuples with* x_i, $y_i \in [a,b]$ $(i = 1, \dots, m)$ *and* $\mathbf{w} = (w_1, w_2, \dots, w_m)$ *be a real* m*-tuple such that*

$$\sum_{i=1}^{l} w_i y_i \leqslant \sum_{i=1}^{l} w_i x_i \text{ for } l = 1, \dots, m-1, \tag{3}$$

and

$$\sum_{i=1}^{m} w_i y_i = \sum_{i=1}^{m} w_i x_i. \tag{4}$$

Then for every continuous convex function $\phi : [a,b] \to \mathbb{R}$, *we have*

$$\sum_{i=1}^{m} w_i \phi(y_i) \leqslant \sum_{i=1}^{m} w_i \phi(x_i). \tag{5}$$

The following integral version of Theorem 2 is a simple consequence of Theorem 12.14 in [17] (see also [15, p. 328]):

THEOREM 3. *Let* $x, y : [a,b] \to [\alpha, \beta]$ *be decreasing and* $w : [a,b] \to \mathbb{R}$ *be continuous functions. If*

$$\int_a^{\nu} w(t) y(t)\, dt \leqslant \int_a^{\nu} w(t) x(t)\, dt \text{ for every } \nu \in [a,b], \tag{6}$$

and

$$\int_a^{b} w(t) y(t)\, dt = \int_a^{b} w(t) x(t)\, dt \tag{7}$$

hold, then for every continuous convex function $\phi : [\alpha, \beta] \to \mathbb{R}$, *we have*

$$\int_a^{b} w(t) \phi(y(t))\, dt \leqslant \int_a^{b} w(t) \phi(x(t))\, dt. \tag{8}$$

For other integral version and generalization of majorization theorem see [13, p. 583], [1, 2, 3, 4, 5, 11, 12].

Consider the Green function G defined on $[\alpha,\beta]\times[\alpha,\beta]$ by

$$G(t,s)=\begin{cases}\frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \alpha\leqslant s\leqslant t;\\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t\leqslant s\leqslant\beta.\end{cases} \tag{9}$$

The function G is convex in s, it is symmetric, so it is also convex in t. The function G is continuous in s and continuous in t.

For any function $\phi:[\alpha,\beta]\to\mathbb{R}$, $\phi\in C^2([\alpha,\beta])$, we can easily show by integrating by parts that the following is valid

$$\phi(x)=\frac{\beta-x}{\beta-\alpha}\phi(\alpha)+\frac{x-\alpha}{\beta-\alpha}\phi(\beta)+\int_\alpha^\beta G(x,s)\phi''(s)ds, \tag{10}$$

where the function G is defined as above in (9) ([19]).

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

THEOREM 4. *Let n be a positive integer and $\phi:[\alpha,\beta]\to\mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous, then for all $x\in[\alpha,\beta]$ the Taylor's formula at the point $c\in[\alpha,\beta]$ is*

$$\phi(x)=T_{n-1}(\phi;c,x)+R_{n-1}(\phi;c,x) \tag{11}$$

where $T_{n-1}(\phi;c,x)$ is Taylor's polynomial of degree $n-1$, i.e.

$$T_{n-1}(\phi;c,x)=\sum_{k=0}^{n-1}\frac{\phi^{(k)}(c)}{k!}(x-c)^k$$

and the remainder is given by

$$R_{n-1}(\phi;c,x)=\frac{1}{(n-1)!}\int_c^x\phi^{(n)}(t)(x-t)^{n-1}dt.$$

In order to recall the definition and basic result of n-convex function, firs we write the definition of divided difference.

DEFINITION 2. [15, p. 15] Let ϕ be a real-valued function defined on $[\alpha,\beta]$. The divided difference of order n of the function ϕ at distinct points $[\alpha,\beta]$ is defined recursively by

$$\phi[x_i]=\phi(x_i),\quad (i=0,\dots,n)$$

and

$$\phi[x_0,\dots,x_n]=\frac{\phi[x_1,\dots,x_n]-\phi[x_0,\dots,x_{n-1}]}{x_n-x_0}.$$

The value $\phi[x_0,\dots,x_n]$ is independent of the order of the points $x_0,\dots,x_n$.

The definition may be extended to include the case that some (or all) the points coincide. Assuming that $\phi^{(j-1)}(x)$ exists, we define

$$\phi\underbrace{[x,\ldots,x]}_{j-times} = \frac{\phi^{(j-1)}(x)}{(j-1)!}. \tag{12}$$

DEFINITION 3. [15, p. 15] A function $\phi:[\alpha,\beta]\to\mathbb{R}$ is said to be n-convex, $n\geqslant 0$, on $[\alpha,\beta]$ if and only if for all choices of $(n+1)$ distinct points $x_0,\ldots,x_n\in[\alpha,\beta]$, the nth order divided difference is non negative that is

$$\phi[x_0,x_1,\ldots,x_n]\geqslant 0$$

THEOREM 5. [15, p. 16] *Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be a function such that $\phi^{(n)}$ exists, then ϕ is n-convex if and only if $\phi^{(n)}\geqslant 0$.*

In this paper we utilize Taylor's theorem with the integral remainder and Green function and establish generalization of majorization theorem for the class of n-convex functions. We use inequalities for the Čebyšev functional to obtain bounds for the identities related to generalizations of majorization inequalities. We present mean value theorems and n-exponential convexity for the functional obtained from the generalized majorization inequalities which leads to exponential convexity and log-convexity for these functionals. Finally, we discuss the results for particular families of function and give classes of Cauchy type means and prove their monotonicity.

2. Main results

We begin this section with the proof of some identities related to generalizations of Majorization inequality.

THEOREM 6. *Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n\geqslant 3$ and let $\mathbf{w}=(w_1,\ldots,w_m)$, $\mathbf{x}=(x_1,\ldots,x_m)$ and $\mathbf{y}=(y_1,\ldots,y_m)$ be m-tuples such that $x_i,\ y_i\in[\alpha,\beta]$, $w_i\in\mathbb{R}$ $(i=1,\ldots,m)$ and G be the Green function as defined in (9). Then*

$$\begin{aligned}\sum_{i=1}^{m} w_i\,\phi(x_i)-\sum_{i=1}^{m} w_i\,\phi(y_i) &= \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^{m} w_i\,(x_i-y_i)\\ &+\sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_\alpha^\beta\left(\sum_{i=1}^{m} w_iG(x_i,s)-\sum_{i=1}^{m} w_iG(y_i,s)\right)(s-\alpha)^k ds\\ &+\frac{1}{(n-3)!}\int_\alpha^\beta\left(\int_t^\beta\left(\sum_{i=1}^{m} w_iG(x_i,s)-\sum_{i=1}^{m} w_iG(y_i,s)\right)(s-t)^{n-3}ds\right)\phi^{(n)}(t)dt.\end{aligned} \tag{13}$$

and

$$\sum_{i=1}^{m} w_i\,\phi(x_i) - \sum_{i=1}^{m} w_i\,\phi(y_i) = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^{m} w_i\,(x_i-y_i)$$
$$+\sum_{k=0}^{n-3}\frac{(-1)^k\phi^{(k+2)}(\beta)}{k!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m} w_iG(x_i,s)-\sum_{i=1}^{m} w_iG(y_i,s)\right)(\beta-s)^k ds$$
$$-\frac{1}{(n-3)!}\int_{\alpha}^{\beta}\left(\int_{\alpha}^{t}\left(\sum_{i=1}^{m} w_iG(x_i,s)-\sum_{i=1}^{m} w_iG(y_i,s)\right)(s-t)^{n-3}ds\right)\phi^{(n)}(t)dt. \tag{14}$$

Proof. Using (10) in $\sum_{i=1}^{m} w_i\,\phi(x_i) - \sum_{i=1}^{m} w_i\,\phi(y_i)$ we have

$$\sum_{i=1}^{m} w_i\,\phi(x_i) - \sum_{i=1}^{m} w_i\,\phi(y_i)$$
$$= \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^{m} w_i\,(x_i-y_i) + \int_{\alpha}^{\beta}\left[\sum_{i=1}^{m} w_iG(x_i,s)-\sum_{i=1}^{m} w_iG(y_i,s)\right]\phi''(s)ds. \tag{15}$$

Now applying Taylor's formula (11) on the function ϕ'' at the point α and replacing n by $n-2$ $(n \geqslant 3)$ we have

$$\phi''(s) = \sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}(s-\alpha)^k + \frac{1}{(n-3)!}\int_{\alpha}^{s}\phi^{(n)}(t)(s-t)^{n-3}dt \tag{16}$$

And similarly Taylor's formula for ϕ'' at the point β and replacing n by $n-2$ $(n \geqslant 3)$ we have

$$\phi''(s) = \sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\beta)}{k!}(s-\beta)^k - \frac{1}{(n-3)!}\int_{s}^{\beta}\phi^{(n)}(t)(s-t)^{n-3}dt \tag{17}$$

Using (16) in (15) we get

$$\sum_{i=1}^{m} w_i\,\phi(x_i) - \sum_{i=1}^{m} w_i\,\phi(y_i) = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^{m} w_i\,(x_i-y_i)$$
$$+\sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m} w_iG(x_i,s)-\sum_{i=1}^{m} w_iG(y_i,s)\right)(s-\alpha)^k ds$$
$$+\frac{1}{(n-3)!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m} w_iG(x_i,s)-\sum_{i=1}^{m} w_iG(y_i,s)\right)\left(\int_{\alpha}^{s}\phi^{(n)}(t)(s-t)^{n-3}dt\right)ds. \tag{18}$$

By applying Fubini's theorem in the last term of (18) we obtain (13).

Similarly using (17) in (15) and applying Fubini's theorem we obtain (14). □

Integral version of the above theorem can be stated as:

THEOREM 7. *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geqslant 3$ and let and $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions and G be the Green function as defined in (9). Then*

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$+\sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_\alpha^\beta\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^k ds$$
$$+\frac{1}{(n-3)!}\int_\alpha^\beta\left(\int_t^\beta\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-t)^{n-3}ds\right)\phi^{(n)}(t)dt \tag{19}$$

and

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$+\sum_{k=0}^{n-3}\frac{(-1)^k\phi^{(k+2)}(\beta)}{k!}\int_\alpha^\beta\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(\beta-s)^k ds$$
$$-\frac{1}{(n-3)!}\int_\alpha^\beta\left(\int_\alpha^t\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-t)^{n-3}ds\right)\phi^{(n)}(t)dt. \tag{20}$$

In the following theorem we obtain generalizations of majorization inequality for n-convex functions.

THEOREM 8. *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geqslant 3$ and let $\mathbf{w} = (w_1,\ldots,w_n)$, $\boldsymbol{x} = (x_1,\ldots,x_m)$ and $\mathbf{y} = (y_1,\ldots,y_m)$ be m-tuples such that x_i, $y_i \in [\alpha,\beta]$, $w_i \in \mathbb{R}$ $(i = 1,\ldots,m)$ and G be the Green function as defined in (9).*

(i) If ϕ is n-convex and

$$\int_t^\beta\left(\sum_{i=1}^m w_iG(x_i,s)-\sum_{i=1}^m w_iG(y_i,s)\right)(s-t)^{n-3}ds \geqslant 0, \quad t\in[\alpha,\beta], \tag{21}$$

then

$$\sum_{i=1}^m w_i\,\phi\,(x_i)-\sum_{i=1}^m w_i\,\phi\,(y_i)-\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^m w_i\,(x_i-y_i)$$
$$\geqslant\sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_\alpha^\beta\left(\sum_{i=1}^m w_iG(x_i,s)-\sum_{i=1}^m w_iG(y_i,s)\right)(s-\alpha)^k ds. \tag{22}$$

(ii) If ϕ is n-convex and

$$\int_{\alpha}^{t}\left(\sum_{i=1}^{m} w_i G(x_i,s)-\sum_{i=1}^{m} w_i G(y_i,s)\right)(s-t)^{n-3}ds \leqslant 0, \quad t\in[\alpha,\beta], \tag{23}$$

then

$$\sum_{i=1}^{m} w_i\,\phi\,(x_i)-\sum_{i=1}^{m} w_i\,\phi\,(y_i)-\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^{m} w_i\,(x_i-y_i)$$
$$\geqslant \sum_{k=0}^{n-3}\frac{(-1)^k\phi^{(k+2)}(\beta)}{k!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m} w_i G(x_i,s)-\sum_{i=1}^{m} w_i G(y_i,s)\right)(\beta-s)^k ds. \tag{24}$$

Proof. Since the function ϕ is n-convex, therefore without loss of generality we can assume that ϕ is n-times differentiable and $\phi^{(n)}\geqslant 0$ see [15, p. 16 and p. 293]. Hence, we can apply Theorem 6 to obtain (22) and (24) respectively. □

Integral version of the above theorem can be stated as:

THEOREM 9. *Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n\geqslant 3$ and let $x,y:[a,b]\to[\alpha,\beta]$, $w:[a,b]\to\mathbb{R}$ be continuous functions and G be the Green function as defined in (9). Then*

(i) If ϕ is n-convex and

$$\int_{t}^{\beta}\left(\int_{a}^{b} w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-t)^{n-3}ds \geqslant 0, \quad t\in[\alpha,\beta], \tag{25}$$

then

$$\int_a^b w(\tau)\phi(x(\tau))d\tau-\int_a^b w(\tau)\phi(y(\tau))d\tau-\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$\geqslant \sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_{\alpha}^{\beta}\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^k ds \tag{26}$$

(ii) If ϕ is n-convex and

$$\int_{\alpha}^{t}\left(\int_{a}^{b} w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-t)^{n-3}ds \leqslant 0, \quad t\in[\alpha,\beta], \tag{27}$$

then

$$\int_a^b w(\tau)\phi(x(\tau))d\tau-\int_a^b w(\tau)\phi(y(\tau))d\tau-\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$\geqslant \sum_{k=0}^{n-3}\frac{(-1)^k\phi^{(k+2)}(\beta)}{k!}\int_{\alpha}^{\beta}\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(\beta-s)^k ds \tag{28}$$

The following generalization of majorization theorem holds.

THEOREM 10. *Let $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geqslant 3$ and $\boldsymbol{x} = (x_1, \ldots, x_m)$, $\boldsymbol{y} = (y_1, \ldots, y_m)$ be two m-tuples such that $\mathbf{y} \prec \mathbf{x}$ with x_i, $y_i \in [\alpha, \beta]$ $(i = 1, \ldots, m)$ and G be the Green function as defined in (9).*

(i) If ϕ is n-convex, then

$$\sum_{i=1}^{m} \phi(x_i) - \sum_{i=1}^{m} \phi(y_i) \geqslant \sum_{k=0}^{n-3} \frac{\phi^{(k+2)}(\alpha)}{k!} \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} G(x_i, s) - \sum_{i=1}^{m} G(y_i, s) \right) (s-\alpha)^k ds. \tag{29}$$

Moreover if $\phi^{(j)}(\alpha) \geqslant 0$ for $j = 2, \ldots, n-1$, then the right hand side of (29) will be non negative, that is (2) holds.

(ii) If n is even and ϕ is n-convex, then

$$\sum_{i=1}^{m} \phi(x_i) - \sum_{i=1}^{m} \phi(y_i) \geqslant \sum_{k=0}^{n-3} \frac{(-1)^k \phi^{(k+2)}(\beta)}{k!} \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} G(x_i, s) - \sum_{i=1}^{m} G(y_i, s) \right) (\beta - s)^k ds. \tag{30}$$

Moreover if $\phi^{(j)}(\beta) \geqslant 0$ for $j = 2, 4, \ldots, n-2$ and $\phi^{(j)}(\beta) \leqslant 0$ for $j = 3, 5, \ldots, n-1$, then the right hand side of (30) will be non negative, that is (2) holds.

(iii) If n is odd and ϕ is n-convex, then

$$\sum_{i=1}^{m} \phi(x_i) - \sum_{i=1}^{m} \phi(y_i) \leqslant \sum_{k=0}^{n-3} \frac{(-1)^k \phi^{(k+2)}(\beta)}{k!} \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} G(x_i, s) - \sum_{i=1}^{m} G(y_i, s) \right) (\beta - s)^k ds. \tag{31}$$

Moreover if $\phi^{(j)}(\beta) \leqslant 0$ for $j = 2, 4, \ldots, n-1$ and $\phi^{(j)}(\beta) \geqslant 0$ for $j = 3, 5, \ldots, n-2$, then the right hand side of (31) will be non positive, that is reverse inequality in (2) holds.

Proof. (i) Since G is convex and $\mathbf{y} \prec \mathbf{x}$ therefore by Theorem 8 we have

$$\sum_{i=1}^{m} G(x_i, s) - \sum_{i=1}^{m} G(y_i, s) \geqslant 0.$$

Also $(s-t)^{n-3} \geqslant 0$ for $s \in [t, \beta]$. Hence (21) holds for $w_i = 1$ $(i = 1, \ldots, m)$.

Using Theorem 8, the inequality (29) holds.

Similarly we can prove the other parts. □

In the following theorem we give generalization of Fuch's majorization theorem.

THEOREM 11. *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geqslant 3$ and let $\mathbf{x} = (x_1,\dots,x_m)$ and $\mathbf{y} = (y_1,\dots,y_m)$ be decreasing m-tuples and $\mathbf{w} = (w_1,\dots,w_m)$ be any m-tuple such that $x_i,\ y_i \in [\alpha,\beta]$, $w_i \in \mathbb{R}$ $(i = 1,\dots,m)$ which satisfies (3), (4) and G be the Green function as defined in (9).*

(i) If ϕ is n-convex, then

$$\sum_{i=1}^{m} w_i\,\phi(x_i) - \sum_{i=1}^{m} w_i\,\phi(y_i) \geqslant \sum_{k=0}^{n-3} \frac{\phi^{(k+2)}(\alpha)}{k!} \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G(x_i,s) - \sum_{i=1}^{m} w_i G(y_i,s) \right) (s-\alpha)^k ds. \tag{32}$$

Moreover if $\phi^{(j)}(\alpha) \geqslant 0$ for $j = 2,\dots,n-1$, then the right hand side of (32) will be non negative, that is (5) holds.

(ii) If n is even and ϕ is n-convex, then

$$\sum_{i=1}^{m} w_i\,\phi(x_i) - \sum_{i=1}^{m} w_i\,\phi(y_i) \geqslant \sum_{k=0}^{n-3} \frac{(-1)^k\phi^{(k+2)}(\beta)}{k!} \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G(x_i,s) - \sum_{i=1}^{m} w_i G(y_i,s) \right) (\beta-s)^k ds. \tag{33}$$

Moreover if $\phi^{(j)}(\beta) \geqslant 0$ for $j = 2,4,\dots,n-2$ and $\phi^{(j)}(\beta) \leqslant 0$ for $j = 3,5,\dots,n-1$, then the right hand side of (33) will be non negative, that is (5) holds.

(iii) If n is odd and ϕ is n-convex, then

$$\sum_{i=1}^{m} w_i\,\phi(x_i) - \sum_{i=1}^{m} w_i\,\phi(y_i) \leqslant \sum_{k=0}^{n-3} \frac{(-1)^k\phi^{(k+2)}(\beta)}{k!} \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G(x_i,s) - \sum_{i=1}^{m} w_i G(y_i,s) \right) (\beta-s)^k ds. \tag{34}$$

Moreover if $\phi^{(j)}(\beta) \leqslant 0$ for $j = 2,4,\dots,n-1$ and $\phi^{(j)}(\beta) \geqslant 0$ for $j = 3,5,\dots,n-2$, then the right hand side of (34) will be non positive, that is reverse inequality in (5) holds.

Proof. The proof is similar to the proof of Theorem 10 but use Theorem 2 instead of Theorem 1. □

The following generalization of integral majorization theorem holds.

THEOREM 12. *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n \geqslant 3$ and let $x, y : [a,b] \to [\alpha,\beta]$ be decreasing and $w : [a,b] \to \mathbb{R}$ be any continuous functions such that (6), (7) hold and G be the Green function as defined in (9).*

(i) *If ϕ is n-convex, then*

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau$$
$$\geqslant \sum_{k=0}^{n-3} \frac{\phi^{(k+2)}(\alpha)}{k!} \int_\alpha^\beta \left(\int_a^b w(\tau)((G(x(\tau),s) - G(y(\tau),s))d\tau\right)(s-\alpha)^k ds. \quad (35)$$

Moreover if $\phi^{(j)}(\alpha) \geqslant 0$ for $j = 2,\dots,n-1$, then the right hand side of (35) will be non negative.

(ii) *If n is even and ϕ is n-convex, then*

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau$$
$$\geqslant \sum_{k=0}^{n-3} \frac{(-1)^k\phi^{(k+2)}(\beta)}{k!} \int_\alpha^\beta \left(\int_a^b w(\tau)((G(x(\tau),s) - G(y(\tau),s))d\tau\right)(\beta-s)^k ds. \quad (36)$$

Moreover if $\phi^{(j)}(\beta) \geqslant 0$ for $j = 2,4,\dots,n-2$ and $\phi^{(j)}(\beta) \leqslant 0$ for $j = 3,5,\dots,$ $n-1$, then the right hand side of (36) will be non negative.

(iii) *If n is odd and ϕ is n-convex, then*

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau$$
$$\leqslant \sum_{k=0}^{n-3} \frac{(-1)^k\phi^{(k+2)}(\beta)}{k!} \int_\alpha^\beta \left(\int_a^b w(\tau)((G(x(\tau),s) - G(y(\tau),s))d\tau\right)(\beta-s)^k ds. \quad (37)$$

Moreover if $\phi^{(j)}(\beta) \leqslant 0$ for $j = 2,4,\dots,n-1$ and $\phi^{(j)}(\beta) \geqslant 0$ for $j = 3,5,\dots,$ $n-2$, then the right hand side of (37) will be non positive.

3. Bounds for identities related to generalizations of majorization inequality

For two Lebesgue integrable functions $f,h : [\alpha,\beta] \to \mathbb{R}$ we consider the Čebyšev functional

$$\Lambda(f,h) = \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)h(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt \cdot \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)dt.$$

In [7] the authors proved the following theorems:

THEOREM 13. *Let $f : [\alpha,\beta] \to \mathbb{R}$ be a Lebesgue integrable function and $h :$ $[\alpha,\beta] \to \mathbb{R}$ be an absolutely continuous function with $(\cdot - a)(b-\cdot)[h']^2 \in L[\alpha,\beta]$. Then we have the inequality*

$$|\Lambda(f,h)| \leqslant \frac{1}{\sqrt{2}}[\Lambda(f,f)]^{\frac{1}{2}} \frac{1}{\sqrt{\beta-\alpha}} \left(\int_\alpha^\beta (x-\alpha)(\beta-x)[h'(x)]^2 dx\right)^{\frac{1}{2}}. \quad (38)$$

The constant $\frac{1}{\sqrt{2}}$ *in (38) is the best possible.*

THEOREM 14. *Assume that* $h:[\alpha,\beta]\to\mathbb{R}$ *is monotonic nondecreasing on* $[\alpha,\beta]$ *and* $f:[\alpha,\beta]\to\mathbb{R}$ *is absolutely continuous with* $f'\in L_\infty[\alpha,\beta]$. *Then we have the inequality*

$$|\Lambda(f,h)|\leqslant\frac{1}{2(\beta-\alpha)}\|f'\|_\infty\int_\alpha^\beta(x-\alpha)(\beta-x)dh(x). \tag{39}$$

The constant $\frac{1}{2}$ *in (39) is the best possible.*

In the sequel we use the above theorems to obtain generalizations of the results proved in the previous section.

For m-tuples $\mathbf{w}=(w_1,\dots,w_m)$, $\mathbf{x}=(x_1,\dots,x_m)$ and $\mathbf{y}=(y_1,\dots,y_m)$ with x_i, y_i $\in[\alpha,\beta]$, $w_i\in\mathbb{R}$ $(i=1,\dots,m)$ and the Green function G defined by (9), denote

$$\mathfrak{R}(t)=\sum_{i=1}^m w_i\int_t^\beta\left(G(x_i,s)-G(y_i,s)\right)(s-t)^{n-3}ds,\quad t\in[\alpha,\beta], \tag{40}$$

$$\mathfrak{B}(t)=\sum_{i=1}^m w_i\int_\alpha^t\left(G(x_i,s)-G(y_i,s)\right)(s-t)^{n-3}ds,\quad t\in[\alpha,\beta], \tag{41}$$

similarly for continuous functions $x,y:[a,b]\to[\alpha,\beta]$, $w:[a,b]\to\mathbb{R}$, denote

$$\tilde{\mathfrak{R}}(t)=\int_t^\beta\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-t)^{n-3}ds,\quad t\in[\alpha,\beta], \tag{42}$$

$$\tilde{\mathfrak{B}}(t)=\int_\alpha^t\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-t)^{n-3}ds,\quad t\in[\alpha,\beta]. \tag{43}$$

Consider the Čebyšev functionals $\Lambda(\mathfrak{R},\mathfrak{R})$, $\Lambda(\mathfrak{B},\mathfrak{B})$, $\Lambda(\tilde{\mathfrak{R}},\tilde{\mathfrak{R}})$ and $\Lambda(\tilde{\mathfrak{B}},\tilde{\mathfrak{B}})$ are given by:

$$\Lambda(\mathfrak{R},\mathfrak{R})=\frac{1}{\beta-\alpha}\int_\alpha^\beta\mathfrak{R}^2(t)dt-\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta\mathfrak{R}(t)dt\right)^2 \tag{44}$$

$$\Lambda(\mathfrak{B},\mathfrak{B})=\frac{1}{\beta-\alpha}\int_\alpha^\beta\mathfrak{B}^2(t)dt-\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta\mathfrak{B}(t)dt\right)^2 \tag{45}$$

$$\Lambda(\tilde{\mathfrak{R}},\tilde{\mathfrak{R}})=\frac{1}{\beta-\alpha}\int_\alpha^\beta\tilde{\mathfrak{R}}^2(t)dt-\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta\tilde{\mathfrak{R}}(t)dt\right)^2 \tag{46}$$

$$\Lambda(\tilde{\mathfrak{B}},\tilde{\mathfrak{B}})=\frac{1}{\beta-\alpha}\int_\alpha^\beta\tilde{\mathfrak{B}}^2(t)dt-\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta\tilde{\mathfrak{B}}(t)dt\right)^2 \tag{47}$$

THEOREM 15. *Let* $\phi:[\alpha,\beta]\to\mathbb{R}$ *be such that* $\phi^{(n-1)}$ *is absolutely continuous for some* $n\geqslant 3$ *with* $(\cdot-\alpha)(\beta-\cdot)[\phi^{(n+1)}]^2\in L[\alpha,\beta]$ *and let* $\boldsymbol{w}=(w_1,\dots,w_m)$, $\boldsymbol{x}=(x_1,\dots,x_m)$ *and* $\boldsymbol{y}=(y_1,\dots,y_m)$ *be* m*-tuples such that* x_i, $y_i\in[\alpha,\beta]$, $w_i\in\mathbb{R}$ $(i=1,\dots,m)$ *and let the functions* G, $\mathfrak{R}$ *and* $\mathfrak{B}$ *be defined by (9), (40) and (41) respectively. Then*

(i)

$$\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \sum_{i=1}^{m} w_i (x_i - y_i)$$
$$+ \sum_{k=0}^{n-3} \frac{\phi^{(k+2)}(\alpha)}{k!} \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G(x_i,s) - \sum_{i=1}^{m} w_i G(y_i,s) \right) (s-\alpha)^k ds$$
$$+ \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\beta-\alpha)(n-3)!} \int_{\alpha}^{\beta} \mathfrak{R}(t)dt + \kappa_n^1(\phi;\alpha,\beta). \tag{48}$$

where the remainder $\kappa_n^1(\phi;\alpha,\beta)$ *satisfies the estimation*

$$\left|\kappa_n^1(\phi;\alpha,\beta)\right| \leqslant \frac{\sqrt{\beta-\alpha}}{\sqrt{2}(n-3)!} [\Lambda(\mathfrak{R},\mathfrak{R})]^{\frac{1}{2}} \left| \int_{\alpha}^{\beta} (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2 dt \right|^{\frac{1}{2}}. \tag{49}$$

(ii)

$$\sum_{i=1}^{m} w_i \phi(x_i) - \sum_{i=1}^{m} w_i \phi(y_i) = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \sum_{i=1}^{m} w_i (x_i - y_i)$$
$$+ \sum_{k=0}^{n-3} \frac{(-1)^k \phi^{(k+2)}(\beta)}{k!} \int_{\alpha}^{\beta} \left(\sum_{i=1}^{m} w_i G(x_i,s) - \sum_{i=1}^{m} w_i G(y_i,s) \right) (\beta-s)^k ds$$
$$+ \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\alpha-\beta)(n-3)!} \int_{\alpha}^{\beta} \mathfrak{B}(t)dt - \kappa_n^2(\phi;\alpha,\beta). \tag{50}$$

where the remainder $\kappa_n^2(\phi;\alpha,\beta)$ *satisfies the estimation*

$$\left|\kappa_n^2(\phi;\alpha,\beta)\right| \leqslant \frac{\sqrt{\beta-\alpha}}{\sqrt{2}(n-3)!} [\Lambda(\mathfrak{B},\mathfrak{B})]^{\frac{1}{2}} \left| \int_{\alpha}^{\beta} (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2 dt \right|^{\frac{1}{2}}. \tag{51}$$

Proof.

(i) If we apply Theorem 13 for $f \to \mathfrak{R}$ and $h \to \phi^{(n)}$ we obtain

$$\left| \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \mathfrak{R}(t)\phi^{(n)}(t)dt - \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \mathfrak{R}(t)dt \cdot \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \phi^{(n)}(t)dt \right|$$
$$\leqslant \frac{1}{\sqrt{2}} [\Lambda(\mathfrak{R},\mathfrak{R})]^{\frac{1}{2}} \frac{1}{\sqrt{\beta-\alpha}} \left| \int_{\alpha}^{\beta} (t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2 dt \right|^{\frac{1}{2}}.$$

Therefore we have

$$\frac{1}{(n-3)!}\int_{\alpha}^{\beta}\mathfrak{R}(t)\phi^{(n)}(t)dt = \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\beta-\alpha)(n-3)!}\int_{\alpha}^{\beta}\mathfrak{R}(t)dt + \kappa_n^1(\phi;\alpha,\beta)$$

where the remainder $\kappa_n^1(\phi;\alpha,\beta)$ satisfies the estimation (49). Now from the identity (13) we obtain (48).

(ii) Similar to the first part. □

Integral case of the above theorem can be given:

THEOREM 16. *Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous for some $n\geqslant 3$ with $(\cdot-\alpha)(\beta-\cdot)[\phi^{(n+1)}]^2\in L[\alpha,\beta]$ and let $x,y:[a,b]\to[\alpha,\beta]$, $w:[a,b]\to\mathbb{R}$ be continuous functions and let the functions G, $\tilde{\mathfrak{R}}$ and $\tilde{\mathfrak{B}}$ be defined by (9), (42) and (43) respectively. Then*

(i)

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$+\sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_{\alpha}^{\beta}\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^k ds$$
$$+\frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\beta-\alpha)(n-3)!}\int_{\alpha}^{\beta}\tilde{\mathfrak{R}}(t)dt + \tilde{\kappa}_n^1(\phi;\alpha,\beta). \tag{52}$$

where the remainder $\tilde{\kappa}_n^1(\phi;\alpha,\beta)$ satisfies the estimation

$$\left|\tilde{\kappa}_n^1(\phi;\alpha,\beta)\right| \leqslant \frac{\sqrt{\beta-\alpha}}{\sqrt{2}(n-3)!}\left[\Lambda(\tilde{\mathfrak{R}},\tilde{\mathfrak{R}})\right]^{\frac{1}{2}}\left|\int_{\alpha}^{\beta}(t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}. \tag{53}$$

(ii)

$$\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau$$
$$+\sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_{\alpha}^{\beta}\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\beta)^k ds$$
$$+\frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{(\alpha-\beta)(n-3)!}\int_{\alpha}^{\beta}\tilde{\mathfrak{B}}(t)dt - \tilde{\kappa}_n^2(\phi;\alpha,\beta). \tag{54}$$

where the remainder $\tilde{\kappa}_n^2(\phi;\alpha,\beta)$ satisfies the estimation

$$\left|\tilde{\kappa}_n^2(\phi;\alpha,\beta)\right| \leqslant \frac{\sqrt{\beta-\alpha}}{\sqrt{2}(n-3)!}\left[\Lambda(\tilde{\mathfrak{B}},\tilde{\mathfrak{B}})\right]^{\frac{1}{2}}\left|\int_{\alpha}^{\beta}(t-\alpha)(\beta-t)[\phi^{(n+1)}(t)]^2 dt\right|^{\frac{1}{2}}. \tag{55}$$

Using Theorem 14 we obtain the following Grüss type inequalities.

THEOREM 17. *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n)}$ $(n \geqslant 3)$ is absolutely continuous function and $\phi^{(n+1)} \geqslant 0$ on $[\alpha,\beta]$ and let the functions $\mathfrak{R}$ and $\mathfrak{B}$ be defined by (40) and (41) respectively. Then we have*

(i) the representation (48) and the remainder $\kappa_n^1(\phi;\alpha,\beta)$ satisfies the bound

$$\left|\kappa_n^1(\phi;\alpha,\beta)\right| \leqslant \frac{1}{(n-3)!}\|\mathfrak{R}'\|_\infty \left\{\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right\}. \tag{56}$$

(ii) the representation (50) and the remainder $\kappa_n^2(\phi;\alpha,\beta)$ satisfies the bound

$$\left|\kappa_n^2(\phi;\alpha,\beta)\right| \leqslant \frac{1}{(n-3)!}\|\mathfrak{B}'\|_\infty \left\{\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right\}.$$

Proof.

(i) Applying Theorem 14 for $f \to \mathfrak{R}$ and $h \to \phi^{(n)}$ we obtain

$$\begin{aligned}&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}(t)\phi^{(n)}(t)dt - \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{R}(t)dt \cdot \frac{1}{\beta-\alpha}\int_\alpha^\beta \phi^{(n)}(t)dt\right| \\ &\leqslant \frac{1}{2(\beta-\alpha)}\|\mathfrak{R}'\|_\infty \int_\alpha^\beta (t-\alpha)(\beta-t)\phi^{(n+1)}(t)dt.\end{aligned} \tag{57}$$

Since

$$\begin{aligned}\int_\alpha^\beta (t-\alpha)(\beta-t)\phi^{(n+1)}(t)dt &= \int_\alpha^\beta [2t-(\alpha+\beta)]\phi^{(n)}(t)dt \\ &= (\beta-\alpha)\left[\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)\right] - 2\left(\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)\right),\end{aligned}$$

using the identity (13) and the inequality (57) we deduce (56).

Similarly, we can prove part (ii). □

Integral case of the above theorem can be given:

THEOREM 18. *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n)}$ $(n \geqslant 3)$ is absolutely continuous function and $\phi^{(n+1)} \geqslant 0$ on $[\alpha,\beta]$ and let $x,y : [a,b] \to [\alpha,\beta]$, $w : [a,b] \to \mathbb{R}$ be continuous functions and the functions G, $\tilde{\mathfrak{R}}$ and $\tilde{\mathfrak{B}}$ be defined by (9), (42) and (43) respectively. Then we have* *(i) the representation* (52) *and the remainder* $\tilde{\kappa}_n^1(\phi;\alpha,\beta)$ *satisfies the bound*

$$\left|\tilde{\kappa}_n^1(\phi;\alpha,\beta)\right| \leqslant \frac{1}{(n-3)!}\|\tilde{\mathfrak{R}}'\|_\infty \left\{\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right\}. \tag{58}$$

(ii) the representation (54) *and the remainder* $\tilde{\kappa}_n^2(\phi;\alpha,\beta)$ *satisfies the bound*

$$\left|\tilde{\kappa}_n^2(\phi;\alpha,\beta)\right| \leqslant \frac{1}{(n-3)!}\|\tilde{\mathfrak{B}}'\|_\infty \left\{\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right\}.$$

Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be a function then the p-norm of ϕ is defined by

$$\|\phi\|_p = \begin{cases} \left(\int_\alpha^\beta |\phi(t)|^p dt\right)^{\frac{1}{p}}, & \text{for } 1\leqslant p<\infty, \text{ if } |\phi|^p \text{ is } R\text{-integrable function,} \\ \text{essential supremum of } \phi, & \text{for } p=\infty, \text{ if } \phi \text{ is essentially bounded.} \end{cases}$$

We present the Ostrowski-type inequalities related to generalizations of majorization's inequality.

THEOREM 19. *Suppose that all assumptions of Theorem* 6 *hold. Assume* (p,q) *is a pair of conjugate exponents, that is* $1\leqslant p,q\leqslant\infty$, $1/p+1/q=1$. *Let* $\left|\phi^{(n)}\right|^p : [\alpha,\beta]\to\mathbb{R}$ *be an R-integrable function for some* $n\geqslant 3$. *Then we have:*

(i)

$$\begin{aligned}&\left|\sum_{i=1}^m w_i\phi(x_i) - \sum_{i=1}^m w_i\phi(y_i) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^m w_i(x_i-y_i)\right. \\ &\left.\quad - \sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_\alpha^\beta\left(\sum_{i=1}^m w_iG(x_i,s)-\sum_{i=1}^m w_iG(y_i,s)\right)(s-\alpha)^k ds\right| \\ &\leqslant \frac{1}{(n-3)!}\left\|\phi^{(n)}\right\|_p \|f\|_q,\end{aligned} \tag{59}$$

where

$$f(t)=\int_t^\beta\left(\sum_{i=1}^m w_iG(x_i,s)-\sum_{i=1}^m w_iG(y_i,s)\right)(s-t)^{n-3}ds.$$

The constant on the right-hand side of (59) *is sharp for* $1<p\leqslant\infty$ *and the best possible for* $p=1$.

(ii)

$$\left|\sum_{i=1}^{m} w_i\,\phi(x_i) - \sum_{i=1}^{m} w_i\,\phi(y_i) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^{m} w_i\,(x_i-y_i)\right.$$
$$\left. - \sum_{k=0}^{n-3}\frac{(-1)^k\phi^{(k+2)}(\beta)}{k!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m} w_i G(x_i,s) - \sum_{i=1}^{m} w_i G(y_i,s)\right)(\beta-s)^k ds\right|$$
$$\leqslant \frac{1}{(n-3)!}\left\|\phi^{(n)}\right\|_p \left\|\bar{f}\right\|_q, \tag{60}$$

where

$$\bar{f}(t) = \int_{\alpha}^{t}\left(\sum_{i=1}^{m} w_i G(x_i,s) - \sum_{i=1}^{m} w_i G(y_i,s)\right)(s-t)^{n-3} ds.$$

The constant on the right-hand side of (60) is sharp for $1 < p \leqslant \infty$ and the best possible for $p = 1$.

Proof.

(i) Let us denote

$$\Im(t) = \frac{1}{(n-3)!}\int_{t}^{\beta}\left(\sum_{i=1}^{m} w_i G(x_i,s) - \sum_{i=1}^{m} w_i G(y_i,s)\right)(s-t)^{n-3} ds.$$

Using the identity (13) and applying Hölder's inequality we obtain

$$\left|\sum_{i=1}^{m} w_i\,\phi(x_i) - \sum_{i=1}^{m} w_i\,\phi(y_i) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^{m} w_i\,(x_i-y_i)\right.$$
$$\left. - \sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m} w_i G(x_i,s) - \sum_{i=1}^{m} w_i G(y_i,s)\right)(s-\alpha)^k ds\right|$$
$$= \left|\int_{\alpha}^{\beta}\Im(t)\phi^{(n)}(t)dt\right| \leqslant \left\|\phi^{(n)}\right\|_p \|\Im\|_q.$$

For the proof of the sharpness of the constant $\|\Im\|_q$ let us find a function ϕ for which the equality in (59) is obtained.

For $1 < p < \infty$ take ϕ to be such that

$$\phi^{(n)}(t) = \operatorname{sgn}\Im(t)\,|\Im(t)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $\phi^{(n)}(t) = \operatorname{sgn}\Im(t)$.

For $p = 1$ we prove that

$$\left|\int_{\alpha}^{\beta}\Im(t)\phi^{(n)}(t)dt\right| \leqslant \max_{t\in[\alpha,\beta]}|\Im(t)|\left(\int_{\alpha}^{\beta}\left|\phi^{(n)}(t)\right| dt\right) \tag{61}$$

is the best possible inequality. As $\Im(t)$ is continuous on $[\alpha,\beta]$ so assume that $|\Im(t)|$ attains its maximum at $t_0 \in [\alpha,\beta]$. First we assume that $\Im(t_0) > 0$. For ε small enough we define $\phi_\varepsilon(t)$ by

$$\phi_\varepsilon(t) = \begin{cases} 0, & \alpha \leqslant t \leqslant t_0, \\ \frac{1}{\varepsilon n!}(t-t_0)^n, & t_0 \leqslant t \leqslant t_0+\varepsilon, \\ \frac{1}{n!}(t-t_0)^{n-1}, & t_0+\varepsilon \leqslant t \leqslant \beta. \end{cases}$$

Then for ε small enough

$$\left|\int_\alpha^\beta \Im(t)\phi^{(n)}(t)dt\right| = \left|\int_{t_0}^{t_0+\varepsilon} \Im(t)\frac{1}{\varepsilon}dt\right| = \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} \Im(t)dt.$$

Now from the inequality (61) we have

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} \Im(t)dt \leqslant \Im(t_0)\int_{t_0}^{t_0+\varepsilon} \frac{1}{\varepsilon}dt = \Im(t_0).$$

Since,

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} \Im(t)dt = \Im(t_0)$$

the statement follows. In the case $\Im(t_0) < 0$, we define $f_\varepsilon(t)$ by

$$\phi_\varepsilon(t) = \begin{cases} \frac{1}{n!}(t-t_0-\varepsilon)^{n-1}, & \alpha \leqslant t \leqslant t_0, \\ -\frac{1}{\varepsilon n!}(t-t_0-\varepsilon)^n, & t_0 \leqslant t \leqslant t_0+\varepsilon, \\ 0, & t_0+\varepsilon \leqslant t \leqslant \beta, \end{cases}$$

and the rest of the proof is the same as above.

(ii) Similar to the first part. □

Integral case can be given as:

THEOREM 20. *Suppose that all assumptions of Theorem 7 hold. Assume (p,q) is a pair of conjugate exponents, that is $1 \leqslant p,q \leqslant \infty$, $1/p+1/q = 1$. Let $\left|\phi^{(n)}\right|^p$: $[\alpha,\beta] \to \mathbb{R}$ be an R-integrable function for some $n \geqslant 3$. Then we have:*

(i)

$$\begin{aligned} &\left|\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau \right. \\ &\left. \quad - \sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_\alpha^\beta\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^k ds\right| \\ &\leqslant \frac{1}{(n-3)!}\left\|\phi^{(n)}\right\|_p \|g\|_q, \end{aligned} \tag{62}$$

where

$$g(t)=\int_t^\beta\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-t)^{n-3}ds.$$

The constant on the right-hand side of (62) is sharp for $1<p\leqslant\infty$ *and the best possible for* $p=1$.

(ii)

$$\left|\int_a^b w(\tau)\phi(x(\tau))d\tau-\int_a^b w(\tau)\phi(y(\tau))d\tau-\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau\right.$$
$$\left.-\sum_{k=0}^{n-3}\frac{(-1)^k\phi^{(k+2)}(\beta)}{k!}\int_\alpha^\beta\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(\beta-s)^k ds\right|$$
$$\leqslant\frac{1}{(n-3)!}\left\|\phi^{(n)}\right\|_p\|\bar{g}\|_q, \tag{63}$$

where

$$\bar{g}(t)=\int_\alpha^t\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-t)^{n-3}ds.$$

The constant on the right-hand side of (63) is sharp for $1<p\leqslant\infty$ *and the best possible for* $p=1$.

4. n-exponential convexity and exponential convexity

We begin this section by giving some definitions and notions which are used frequently in the results. For more details see e.g. [6], [9] and [16].

DEFINITION 4. A function $\phi:I\to\mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^n\xi_i\xi_j\,\phi\left(\frac{x_i+x_j}{2}\right)\geqslant 0,$$

hold for all choices $\xi_1,\ldots,\xi_n\in\mathbb{R}$ and all choices $x_1,\ldots,x_n\in I$. A function $\phi:I\to\mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

DEFINITION 5. A function $\phi:I\to\mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n\in\mathbb{N}$.

A function $\phi:I\to\mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

PROPOSITION 1. *If $\phi : I \to \mathbb{R}$ is an n-exponentially convex in the Jensen sense, then the matrix $\left[\phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{m}$ is a positive semi-definite matrix for all $m \in \mathbb{N}$, $m \leqslant n$. Particularly,*

$$\det\left[\phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{m} \geqslant 0$$

for all $m \in \mathbb{N}$, $m = 1,2,\ldots,n$.

REMARK 1. It is known that $\phi : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2\phi(x) + 2\alpha\beta\phi\left(\frac{x+y}{2}\right) + \beta^2\phi(y) \geqslant 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex.

Motivated by inequalities (22), (24), (26) and (28), under the assumptions of Theorems 8 and 9 we define the following linear functionals:

$$\begin{aligned}
\digamma_1(\phi) = &\sum_{i=1}^{m} w_i\,\phi(x_i) - \sum_{i=1}^{m} w_i\,\phi(y_i) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^{m} w_i(x_i - y_i)\\
&-\sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m} w_iG(x_i,s) - \sum_{i=1}^{m} w_iG(y_i,s)\right)(s-\alpha)^k ds \qquad (64)
\end{aligned}$$

$$\begin{aligned}
\digamma_2(\phi) = &\sum_{i=1}^{m} w_i\,\phi(x_i) - \sum_{i=1}^{m} w_i\,\phi(y_i) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\sum_{i=1}^{m} w_i(x_i - y_i)\\
&-\sum_{k=0}^{n-3}\frac{(-1)^k\phi^{(k+2)}(\beta)}{k!}\int_{\alpha}^{\beta}\left(\sum_{i=1}^{m} w_iG(x_i,s) - \sum_{i=1}^{m} w_iG(y_i,s)\right)(\beta-s)^k ds \qquad (65)
\end{aligned}$$

$$\begin{aligned}
\digamma_3(\phi) = &\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau\\
&-\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau\\
&-\sum_{k=0}^{n-3}\frac{\phi^{(k+2)}(\alpha)}{k!}\int_{\alpha}^{\beta}\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(s-\alpha)^k ds \qquad (66)
\end{aligned}$$

$$\begin{aligned}
\digamma_4(\phi) = &\int_a^b w(\tau)\phi(x(\tau))d\tau - \int_a^b w(\tau)\phi(y(\tau))d\tau\\
&-\frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}\int_a^b w(\tau)(x(\tau)-y(\tau))d\tau\\
&-\sum_{k=0}^{n-3}\frac{(-1)^k\phi^{(k+2)}(\beta)}{k!}\int_{\alpha}^{\beta}\left(\int_a^b w(\tau)((G(x(\tau),s)-G(y(\tau),s))d\tau\right)(\beta-s)^k ds
\end{aligned}$$

(67)

REMARK 2. Under the assumptions of Theorems 8 and 9, it holds $\digamma_i(\phi) \geqslant 0$, $i = 1,\dots,4$ for all n-convex functions ϕ.

Lagrange and Cauchy type mean value theorems related to defined functionals are given in the following theorems.

THEOREM 21. *Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi \in C^n[\alpha,\beta]$. If the inequalities in (21) $(i = 1)$, (23) $(i = 2)$, (25) $(i = 3)$ and (27) $(i = 4)$ hold, then there exist $\xi_i \in [\alpha,\beta]$ such that*

$$\digamma_i(\phi) = \phi^{(n)}(\xi_i)\digamma_i(\varphi), \quad i = 1,\dots,4 \tag{68}$$

where $\varphi(x) = \frac{x^n}{n!}$ and $\digamma_i$, $i = 1,\dots,4$ are defined by (64)–(67).

Proof. Similar to the proof of Theorem 4.1 in [10]. □

THEOREM 22. *Let $\phi,\psi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi,\psi \in C^n[\alpha,\beta]$. If the inequalities in (21) $(i = 1)$, (23) $(i = 2)$, (25) $(i = 3)$ and (27) $(i = 4)$ hold, then there exist $\xi_i \in [\alpha,\beta]$ such that*

$$\frac{\digamma_i(\phi)}{\digamma_i(\psi)} = \frac{\phi^{(n)}(\xi_i)}{\psi^{(n)}(\xi_i)}, \quad i = 1,\dots,4 \tag{69}$$

provided that the denominators are non-zero and $\digamma_i$, $i = 1,\dots,4$ are defined by (64)–(67).

Proof. Similar to the proof of Corollary 4.2 in [10]. □

Now we will produce n-exponentially and exponentially convex functions applying defined functionals. We use an idea from [16]. In the sequel I and J will be intervals in $\mathbb{R}$.

THEOREM 23. *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$ such that the function $t \mapsto [x_0,\dots,x_n;\phi_t]$ is n-exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\dots,x_n \in I$. Then for the linear functionals $\digamma_i(\phi_t)$ $(i = 1,2,..,4)$ as defined by (64)–(67), the following statements hold:*

(i) The function $t \to \digamma_i(\phi_t)$ is n-exponentially convex in the Jensen sense on J and the matrix $[\digamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m$ is a positive semi-definite for all $m \in \mathbb{N}$, $m \leqslant n$, $t_1,..,t_m \in J$. Particularly,

$$\det[\digamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^m \geqslant 0 \quad \textit{for all} \quad m \in \mathbb{N},\ m = 1,2,\dots,n.$$

(ii) If the function $t \to \digamma_i(\phi_t)$ is continuous on J, then it is n-exponentially convex on J.

Proof. (i) For $\xi_j \in \mathbb{R}$ and $t_j \in J$, $j = 1,\ldots,n$, we define the function

$$h(x) = \sum_{j,l=1}^{n} \xi_j \xi_l \phi_{\frac{t_j+t_l}{2}}(x).$$

Using the assumption that the function $t \mapsto [x_0,\ldots,x_n;\phi_t]$ is n-exponentially convex in the Jensen sense, we have

$$[x_0,\ldots,x_n,h] = \sum_{j,l=1}^{n} \xi_j \xi_l [x_0,\ldots,x_n;\phi_{\frac{t_j+t_l}{2}}] \geqslant 0,$$

which in turn implies that h is a n-convex function on J, so $\digamma_i(h) \geqslant 0$, $i = 1,\ldots,4$. Hence

$$\sum_{j,l=1}^{n} \xi_j \xi_l \digamma_i\left(\phi_{\frac{t_j+t_l}{2}}\right) \geqslant 0.$$

We conclude that the function $t \mapsto \digamma_i(\phi_t)$ is n-exponentially convex on J in the Jensen sense.

The remaining part follows from Proposition 1.

(ii) If the function $t \to \digamma_i(\phi_t)$ is continuous on J, then it is n-exponentially convex on J by definition. □

The following corollary is an immediate consequence of the above theorem

COROLLARY 1. *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0,\ldots,x_n;\phi_t]$ is exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\ldots,x_n \in I$. Then for the linear functionals $\digamma_i(\phi_t)$ $(i = 1,..,4)$ as defined by (64)–(67), the following statements hold:*

(i) The function $t \to \digamma_i(\phi_t)$ is exponentially convex in the Jensen sense on J and the matrix $[\digamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^{m}$ is a positive semi-definite for all $m \in \mathbb{N}$, $m \leqslant n$, $t_1,..,t_m \in J$. Particularly,

$$\det[\digamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^{m} \geqslant 0 \quad \textit{for all} \quad m \in \mathbb{N},\ m = 1,2,\ldots,n.$$

(ii) If the function $t \to \digamma_i(\phi_t)$ is continuous on J, then it is exponentially convex on J.

COROLLARY 2. *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $t \mapsto [x_0,\ldots,x_n;\phi_t]$ is 2-exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0,\ldots,x_n \in I$. Let $\digamma_i$, $i = 1,\ldots,4$ be linear functionals defined by (64)–(67). Then the following statements hold:*

(i) If the function $t \mapsto \mathcal{F}_i(\phi_t)$ *is continuous on* J*, then it is* 2*-exponentially convex function on* J*. If* $t \mapsto \mathcal{F}_i(\phi_t)$ *is additionally strictly positive, then it is also log-convex on* J*. Furthermore, the following inequality holds true:*

$$[\mathcal{F}_i(\phi_s)]^{t-r} \leqslant [\mathcal{F}_i(\phi_r)]^{t-s} [\mathcal{F}_i(\phi_t)]^{s-r}, \quad i = 1, \ldots, 4$$

for every choice $r, s, t \in J$*, such that* $r < s < t$*.*

(ii) If the function $t \mapsto \mathcal{F}_i(\phi_t)$ *is strictly positive and differentiable on J, then for every* $p, q, u, v \in J$*, such that* $p \leqslant u$ *and* $q \leqslant v$*, we have*

$$\mu_{p,q}(\mathcal{F}_i, \Omega) \leqslant \mu_{u,v}(\mathcal{F}_i, \Omega), \tag{70}$$

where

$$\mu_{p,q}(\mathcal{F}_i, \Omega) = \begin{cases} \left(\frac{\mathcal{F}_i(\phi_p)}{\mathcal{F}_i(\phi_q)} \right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}\mathcal{F}_i(\phi_p)}{\mathcal{F}_i(\phi_p)} \right), & p = q, \end{cases} \tag{71}$$

for $\phi_p, \phi_q \in \Omega$*.*

Proof.

(i) This is an immediate consequence of Theorem 23 and Remark 1.

(ii) Since $p \mapsto \mathcal{F}_i(\phi_t)$ is positive and continuous, by (i) we have that $t \mapsto \mathcal{F}_i(\phi_t)$ is log-convex on J, that is, the function $t \mapsto \log \mathcal{F}_i(\phi_t)$ is convex on J. Hence we get

$$\frac{\log \mathcal{F}_i(\phi_p) - \log \mathcal{F}_i(\phi_q)}{p-q} \leqslant \frac{\log \mathcal{F}_i(\phi_u) - \log \mathcal{F}_i(\phi_v)}{u-v}, \tag{72}$$

for $p \leqslant u, q \leqslant v, p \neq q, u \neq v$. So, we conclude that

$$\mu_{p,q}(\mathcal{F}_i, \Omega) \leqslant \mu_{u,v}(\mathcal{F}_i, \Omega).$$

Cases $p = q$ and $u = v$ follow from (72) as limit cases. □

5. Examples

In this section, we present some families of functions which fulfil the conditions of Theorem 23, Corollary 1 and Corollary 2. This enables us to construct a large families of functions which are exponentially convex. Explicit form of this functions is obtained after we calculate explicit action of functionals on a given family.

EXAMPLE 1. Let us consider a family of functions

$$\Omega_1 = \{\phi_t : \mathbb{R} \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$\phi_t(x) = \begin{cases} \frac{e^{tx}}{t^n}, \ t \neq 0, \\ \frac{x^n}{n!}, \ t = 0. \end{cases}$$

Since $\frac{d^n\phi_t}{dx^n}(x) = e^{tx} > 0$, the function ϕ_t is n-convex on $\mathbb{R}$ for every $t \in \mathbb{R}$ and $t \mapsto \frac{d^n\phi_t}{dx^n}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 23 we also have that $t \mapsto [x_0,\ldots,x_n;\phi_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 1 we conclude that $t \mapsto \digamma_i(\phi_t)$, $i = 1,\ldots,4$, are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although the mapping $t \mapsto \phi_t$ is not continuous for $t = 0$), so it is exponentially convex. For this family of functions, $\mu_{p,q}(\digamma_i,\Omega_1)$, $i = 1,\ldots,4$, from (71), becomes

$$\mu_{p,q}(\digamma_i,\Omega_1) = \begin{cases} \left(\frac{\digamma_i(\phi_p)}{\digamma_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\digamma_i(id\cdot\phi_p)}{\digamma_i(\phi_p)} - \frac{n}{p}\right), & p = q \neq 0, \\ \exp\left(\frac{1}{n+1}\frac{\digamma_i(id\cdot\phi_0)}{\digamma_i(\phi_0)}\right), & p = q = 0, \end{cases}$$

where id is the identity function. By Corollary 2 $\mu_{p,q}(\digamma_i,\Omega_1)$ is a monotonic function in parameters p and q.

Since

$$\left(\frac{\frac{d^n\phi_p}{dx^n}}{\frac{d^n\phi_q}{dx^n}}\right)^{\frac{1}{p-q}} (\log x) = x,$$

using Theorem 22 it follows that:

$$M_{p,q}(\digamma_i,\Omega_1) = \log\mu_{p,q}(\digamma_i,\Omega_1), \quad i = 1,\ldots,4$$

satisfies

$$a \leqslant M_{p,q}(\digamma_i,\Omega_1) \leqslant b, \quad i = 1,\ldots,4.$$

So, $M_{p,q}(\digamma_i,\Omega_1)$ is a monotonic mean.

EXAMPLE 2. Let us consider a family of functions

$$\Omega_2 = \{g_t : (0,\infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$g_t(x) = \begin{cases} \frac{x^t}{t(t-1)\cdots(t-n+1)}, & t \notin \{0,1,\ldots,n-1\}, \\ \frac{x^j \log x}{(-1)^{n-1-j}j!(n-1-j)!}, & t = j \in \{0,1,\ldots,n-1\}. \end{cases}$$

Since $\frac{d^n g_t}{dx^n}(x) = x^{t-n} > 0$, the function g_t is n-convex for $x > 0$ and $t \mapsto \frac{d^n g_t}{dx^n}(x)$ is exponentially convex by definition. Arguing as in Example 1 we get that the mappings

$t \mapsto \mathcal{F}_i(g_t)$, $i = 1,\dots,4$ are exponentially convex. Hence, for this family of functions $\mu_{p,q}(\mathcal{F}_i,\Omega_2)$, $i = 1,\dots,4$, from (71), is equal to

$$\mu_{p,q}(\mathcal{F}_i,\Omega_2) = \begin{cases} \left(\frac{\mathcal{F}_i(g_p)}{\mathcal{F}_i(g_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left((-1)^{n-1}(n-1)!\frac{\mathcal{F}_i(g_0 g_p)}{\mathcal{F}_i(g_p)} + \sum_{k=0}^{n-1}\frac{1}{k-p}\right), & p = q \notin \{0,1,\dots,n-1\}, \\ \exp\left((-1)^{n-1}(n-1)!\frac{\mathcal{F}_i(g_0 g_p)}{2\mathcal{F}_i(g_p)} + \sum_{\substack{k=0 \\ k\neq p}}^{n-1}\frac{1}{k-p}\right), & p = q \in \{0,1,\dots,n-1\}. \end{cases}$$

Again, using Theorem 22 we conclude that

$$a \leqslant \left(\frac{\mathcal{F}_i(g_p)}{\mathcal{F}_i(g_q)}\right)^{\frac{1}{p-q}} \leqslant b, \quad i = 1,\dots,4. \tag{73}$$

So, $\mu_{p,q}(\mathcal{F}_i,\Omega_2)$, $i = 1,\dots,4$ is a mean and by (70) it is monotonic.

EXAMPLE 3. Let

$$\Omega_3 = \{\zeta_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$$

be a family of functions defined by

$$\zeta_t(x) = \begin{cases} \frac{t^{-x}}{(\ln t)^n}, & t \neq 1; \\ \frac{x^n}{n!}, & t = 1. \end{cases}$$

Since $\frac{d^n \zeta_t}{dx^n}(x) = t^{-x}$ is the Laplace transform of a non-negative function (see [18]) it is exponentially convex. Obviously ζ_t are n-convex functions for every $t > 0$.

For this family of functions, $\mu_{t,q}(\mathcal{F}_i,\Omega_3)$, $i = 1,\dots,4$, in this case for $[a,b] \in \mathbb{R}^+$, from (71) becomes

$$\mu_{t,q}(\mathcal{F}_i,\Omega_3) = \begin{cases} \left(\frac{\mathcal{F}_i(\zeta_t)}{\mathcal{F}_i(\zeta_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ \exp\left(-\frac{\mathcal{F}_i(id.\zeta_t)}{t\mathcal{F}_i(\zeta_t)} - \frac{n}{t\ln t}\right), & t = q \neq 1; \\ \exp\left(-\frac{1}{n+1}\frac{\mathcal{F}_i(id.\zeta_1)}{\mathcal{F}_i(\zeta_1)}\right), & t = q = 1. \end{cases}$$

This is monotonous function in parameters t and q by (70).

Using Theorem 22 it follows that

$$M_{t,q}(\mathcal{F}_i,\Omega_3) = -L(t,q)\ln\mu_{t,q}(\mathcal{F}_i,\Omega_3), \quad i = 1,..,4.$$

satisfy

$$a \leqslant M_{t,q}(\mathcal{F}_i,\Omega_3) \leqslant b, \quad i = 1,..,4.$$

This shows that $M_{t,q}(\digamma_i,\Omega_3)$ is mean for $i=1,\ldots,4$. Because of the above inequality (70), this mean is also monotonic. $L(t,q)$ is logarithmic mean defined by

$$L(t,q) = \begin{cases} \frac{t-q}{\log t-\log q}, & t \neq q; \\ t, & t=q. \end{cases}$$

EXAMPLE 4. Let

$$\Omega_4 = \{\gamma_t : (0,\infty) \to (0,\infty) : t \in (0,\infty)\}$$

be a family of functions defined by

$$\gamma_t(x) = \frac{e^{-x\sqrt{t}}}{t^n}.$$

Since $\frac{d^n\gamma_t}{dx^n}(x) = e^{-x\sqrt{t}}$ is the Laplace transform of a non-negative function (see [18]) it is exponentially convex. Obviously γ_t are n-convex function for every $t>0$.

For this family of functions, $\mu_{t,q}(\digamma_i,\Omega_4)$, $i=1,\ldots,4$, in this case for $[a,b]\in\mathbb{R}^+$, from (71) becomes

$$\mu_{t,q}(\digamma_i,\Omega_4) = \begin{cases} \left(\frac{\digamma_i(\gamma_t)}{\digamma_i(\gamma_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ \exp\left(-\frac{\digamma_i(id.\gamma_t)}{2\sqrt{t}\digamma_i(\gamma_t)} - \frac{n}{t}\right), & t=q. \end{cases}$$

This is monotonous function in parameters t and q by (70).

Using Theorem 22 it follows that

$$M_{t,q}(\digamma_i,\Omega_4) = -\left(\sqrt{t}+\sqrt{q}\right) ln\mu_{t,q}(\digamma_i,\Omega_4), \quad i=1,..,4$$

satisfy

$$a \leqslant M_{t,q}(\digamma_i,\Omega_4) \leqslant b, \quad i=1,..,4.$$

This shows that $M_{t,q}(\digamma_i,\Omega_4)$ is mean for $i=1,\ldots,4$. Because of the above inequality (70), this mean is also monotonic.

REFERENCES

[1] M. ADIL KHAN, NAVEED LATIF, I. PERIĆ AND J. PEČARIĆ, *On Sapogov's extension of Čebyšev's inequality*, Thai J. Math., **10** (2) (2012), 617–633.

[2] M. ADIL KHAN, NAVEED LATIF, I. PERIĆ AND J. PEČARIĆ, *On majorization for matrices*, Math. Balkanica, to appear.

[3] M. ADIL KHAN, M. NIEZGODA AND J. PEČARIĆ, *On a refinement of the majorization type inequality*, Demonstratio Math., **44** (1) (2011), 49–57.

[4] M. ADIL KHAN, SADIA KHALID AND J. PEČARIĆ, *Refinements of some majorization type inequalities*, J. Math. Inequal., **7** (1) (2013), 73–92.

[5] N. S. BARNETT, P. CERONE, S. S. DRAGOMIR, *Majorisation inequalities for Stieltjes integrals*, Appl. Math. Lett., **22** (2009), 416–421.

[6] S. N. BERNSTEIN, *Sur les fonctions absolument monotones*, Acta Math. **52** (1929), 1–66.
[7] P. CERONE, S. S. DRAGOMIR, *Some new Ostrowski-type bounds for the Čebyšev functional and applications*, J. Math. Inequal. **8** (1) (2014), 159–170.
[8] L. FUCHS, *A new proof of an inequality of Hardy-Littlewood-Polya*, Mat. Tidsskr, **B** (1947), 53–54.
[9] J. JAKŠETIĆ, J. PEČARIĆ, *Exponential convexity method*, J. Convex Anal. (2013), no. 1, 181–197.
[10] J. JAKŠETIĆ, J. PEČARIĆ, A. PERUŠIĆ, *Steffensen inequality, higher order convexity and exponential convexity*, Rend. Circ. Mat. Palermo **63** (1) (2014), 109–127.
[11] N. LATIF, J. PEČARIĆ AND I. PERIĆ, *On Majorization, Favard and Berwald's Inequalities*, Annals of Functional Analysis, **2** (2011), no. 1, 31–50, ISSN: 2008-8752.
[12] L. MALIGRANDA, J. PEČARIĆ, L. E. PERSSON, *Weighted Favard's and Berwald's Inequalities*, J. Math. Anal. Appl. **190** (1995), 248–262.
[13] A. W. MARSHALL, I. OLKIN AND BARRY C. ARNOLD, *Inequalities: Theory of Majorization and Its Applications* (Second Edition), Springer Series in Statistics, New York 2011.
[14] M. NIEZGODA, *Remarks on convex functions and separable sequences*, Discrete Math. **308** (2008), 1765–1773.
[15] J. PEČARIĆ, F. PROSCHAN AND Y. L. TONG, *Convex functions, Partial Orderings and Statistical Applications*, Academic Press, New York, 1992.
[16] J. PEČARIĆ, J. PERIĆ, *Improvements of the Giaccardi and the Petrović inequality and related results*, An. Univ. Craiova Ser. Mat. Inform., **39** (1) (2012), 65–75.
[17] J. PEČARIĆ, *On some inequalities for functions with nondecreasing increments*, J. Math. Anal. Appl., **98** (1984), 188–197.
[18] D. V. WIDDER, *The Laplace Transform*, Princeton Univ. Press, New Jersey, 1941.
[19] D. V. WIDDER, *Completely convex function and Lidstone series*, Trans. Am. Math. Soc. **51** (1942), 387–398.

(Received September 8, 2014)

M. Adil Khan
Department of Mathematics
University of Peshawar
Peshawar, Pakistan
e-mail: adilswati@gmail.com

N. Latif
Department of Mathematics
Govt. College University
Faisalabad 38000, Pakistan
e-mail: naveed707@gmail.com

J. Pečarić
Faculty of Textile Technology
University of Zagreb
Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia
e-mail: pecaric@element.hr