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THE HORNICH–HLAWKA INEQUALITY AND BERNSTEIN FUNCTIONS

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Abstract. The Hornich-Hlawka inequality for three real numbers is extended from the identity function to all Bernstein functions on the half-line. For vectors in a Euclidean space it is shown to hold for the square-root function.

The inequality

$$|x+y|+|y+z|+|z+x| \leqslant |x|+|y|+|z|+|x+y+z| \tag{1}$$

goes back to Hornich and Hlawka, see [2], [3] and [5], p. 171 ff. It is also called the quadrilateral inequality, and is valid for real numbers x,y,z but also for vectors in a Euclidean space (so in particular for complex numbers as well). We will first consider the case of real numbers where we show the following extension of (1): for every Bernstein function f on $\mathbb{R}_+ = [0,\infty[$ we have

$$f(|x+y|)+f(|y+z|)+f(|z+x|) \leqslant f(|x|)+f(|y|)+f(|z|)+f(|x+y+z|)$$

for $x,y,z \in \mathbb{R}$, where f being a *Bernstein function* means $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is continuous, C^∞ on $]0,\infty[$, with a completely monotone derivative f', i.e.

$$(-1)^{j+1} f^{(j)}(x) \geqslant 0 \qquad \forall\, j \in \mathbb{N}, \quad \forall\, x \in]0,\infty[.$$

Famous examples of such functions are $\log(1+x), x/(1+x)$ and x^α for $0 < \alpha \leqslant 1$. One might think that at least the inequality corresponding to $\alpha = 1/2$, i.e.

$$\sqrt{|x+y|}+\sqrt{|y+z|}+\sqrt{|z+x|} \leqslant \sqrt{|x|}+\sqrt{|y|}+\sqrt{|z|}+\sqrt{|x+y+z|}$$

should allow for a short, direct proof, but this does not seem to be the case. A few basic facts on Bernstein functions can be found in [1]. A far more thorough treatment is of course the recent monograph [8], at the end of which a list of 138 (classes of) Bernstein functions is given.

For any function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ we introduce the shift $(E_a f)(s) := f(a+s)$, $a \in \mathbb{R}_+$, and $\nabla_a := E_0 - E_a$, i.e. $(\nabla_a f)(s) := f(s) - f(a+s)$. (These notions clearly make sense on any abelian semigroup.) Since the operators $\{E_a \mid a \in \mathbb{R}_+\}$ commute, so do also $\{\nabla_a \mid a \in \mathbb{R}_+\}$. A function f on $\mathbb{R}_+$ is by definition n*-alternating* iff

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$\nabla_{a_1}\nabla_{a_2}\dots\nabla_{a_k}f \leqslant 0$ for all $a_1,\dots,a_k \in \mathbb{R}_+$ and for all $k = 1,\dots,n$. And f is *completely alternating* if this holds for every $n \in \mathbb{N}$. For a continuous non-negative function f on $\mathbb{R}_+$ this property is equivalent with being a Bernstein function; cf. [1], Corollary 4.6.8 and the Remark on page 114.

If f is n times differentiable on $]0,\infty[$, and fulfills $f' \geqslant 0,\ f'' \leqslant 0,\ f''' \geqslant 0,\dots,$ $(-1)^{n+1}f^{(n)} \geqslant 0$, then f is n-alternating, by the mean value theorem.

Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be 2-alternating; then for $0 \leqslant a \leqslant b$

$$0 \geqslant \left(\nabla^2_{(b-a)/2}f\right)(a) = f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)$$

and

$$0 \geqslant (\nabla_a\nabla_b f)(0) = f(0) - f(a) - f(b) + f(a+b)$$

implying f to be concave and subadditive.

THEOREM 1. *Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be 3-alternating. Then we have for all $x,y,z \in \mathbb{R}$*

$$f(|x+y|) + f(|y+z|) + f(|z+x|) \leqslant f(|x|) + f(|y|) + f(|z|) + f(|x+y+z|). \qquad (2)$$

This holds in particular for every Bernstein function f.

Proof. We may and do assume $f(0) = 0$. Five cases will be considered.

1. $x,y,z \geqslant 0$: we have

$$0 \geqslant (\nabla_x\nabla_y\nabla_z f)(0) = -f(x) - f(y) - f(z) + f(x+y) + f(y+z) + f(z+x) - f(x+y+z).$$

In all the remaining cases it is obviously sufficient to have just one of the three numbers x,y,z negative, which we choose to be z; so $x,y \geqslant 0$ in what follows.

2. $x+y \leqslant |z|$: Put $c := |z| - x - y = |x+y+z|$, then

$$0 \geqslant (\nabla_x\nabla_y\nabla_c f)(0) = -f(x) - f(y) - f(|x+y+z|) + f(x+y) + f(|z|-y) + f(|z|-x) - f(|z|)$$

where $|z| - y = |y+z|$ and $|z| - x = |z+x|$.

3. $x,y \leqslant |z| \leqslant x+y$: Since f is 2-alternating, we get

$$0 \geqslant \left(\nabla_y\nabla_{x+y-|z|}f\right)(|z|-y) = f(|z|-y) - f(|z|) - f(x) + f(x+y)$$

The function f is also 1-alternating, i.e. increasing, so $f(|z|-x) \leqslant f(y)$, and thus

$$\begin{aligned}
&f(|x+y|) + f(|x+z|) + f(|z+y|)\\
&= f(x+y) + f(|z|-x) + f(|z|-y)\\
&\leqslant f(x) + f(y) + f(|z|)\\
&\leqslant f(x) + f(y) + f(|z|) + f(|x+y+z|).
\end{aligned}$$

4. $x \leqslant |z| \leqslant y$: We have $f(x+y) \leqslant f(x)+f(y)$, hence

$$f(x+y)+f(|z|-x)+f(y-|z|) \leqslant f(x)+f(y)+f(|z|)+f(x+y-|z|).$$

5. $|z| \leqslant x,y$: Again by subadditivity

$$f(x+y) \leqslant f(x+y-|z|)+f(|z|)$$

and then the desired inequality is immediate. □

A function $g:\mathbb{R}_+ \longrightarrow \mathbb{R}$ is called *n-monotone* iff $-g$ is n-alternating, in which case $f := g(0)-g$ is a non-negative n-alternating function. If g is n-monotone for each $n \in \mathbb{N}$, and non-negative, it is called *completely monotone*.

COROLLARY 1. *For every 3-monotone function g on $\mathbb{R}_+$ and $x,y,z \in \mathbb{R}$ we have*

$$g(|x|)+g(|y|)+g(|z|)+g(|x+y+z|) \leqslant g(0)+g(|x+y|)+g(|y+z|)+g(|z+x|). \quad (3)$$

This holds in particular if g is completely monotone.

So for example the following inequality holds for $x,y,z \in \mathbb{R}$:

$$e^{-|x|}+e^{-|y|}+e^{-|z|}+e^{-|x+y+z|} \leqslant 1+e^{-|x+y|}+e^{-|y+z|}+e^{-|z+x|}.$$

REMARK 1. The sufficient condition in the above theorem is of course not necessary: for $f(x) := x^2$ we even have equality in (2) – called *Hlawka's identity*. For $f(x) := x^{2+\varepsilon}$ with $\varepsilon > 0$, however, the inequality (2) does no longer hold: take $x=y=-z=1$.

As already mentioned, the Hornich-Hlawka inequality is also true in Euclidean spaces, i.e. replacing in (1) x,y,z by vectors in $\mathbb{R}^n$, and the absolute value by the Euclidean norm. See [2] for a nice direct proof, by simple computation. If an analogue of Theorem 1 holds likewise in this generality, perhaps "only" for Bernstein functions, remains open for the time being. However, the interesting special case where $f(t)=\sqrt{t}$ can be answered positively. We consider a slightly more general "frame" first.

THEOREM 2. *Let X be an abelian group, $x \longmapsto |x|$ a non-negative symmetric and subadditive function on X (i.e. $|-x|=|x|$ and $|x+y| \leqslant |x|+|y|$ $\forall\, x,y \in X$), and let $f:\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be concave. Then, if f^2 fulfills* (2)*, i.e. if*

$$f^2(|x+y|)+f^2(|y+z|)+f^2(|z+x|) \leqslant f^2(|x|)+f^2(|y|)+f^2(|z|)+f^2(|x+y+z|)$$

for all $x,y,z \in X$, so does also f.

Proof. We develop an idea from the paper [9], used there for a prove of the original inequality (1).

Let $x,y,z \in X$ be a given. We then put

$$\begin{aligned}
p(x,y) &:= f(|x|)+f(|y|)+f(|x+y|)\\
q(x,y) &:= f(|x|)+f(|y|)-f(|x+y|)\\
U &:= p(x,y)\,q(x,y)+p(x,z)\,q(x,z)+p(y,z)q(y,z)\\
&\quad = 2[f^2(|x|)+f^2(|y|)+f^2(|z|)+f(|x|)f(|y|)+f(|x|)f(|z|)\\
&\qquad +f(|y|)f(|z|)]-f^2(|x+y|)-f^2(|x+z|)-f^2(|y+z|)\\
P &:= p(x,y)\vee p(x,z)\vee p(y,z)\\
Q &:= q(x,y)+q(x,z)+q(y,z)\\
S_1 &:= f(|x|)+f(|y|)+f(|z|)\\
S_2 &:= f(|x+y|)+f(|x+z|)+f(|y+z|)\\
S_3 &:= f(|x+y+z|)
\end{aligned}$$

Then $Q = 2S_1 - S_2$, $U \leqslant P\cdot Q$, and $S_3 \leqslant S_1$ because f is subadditive. Since f is furthermore increasing, we get

$$S_1+S_3 = f(|x|)+f(|y|)+f(|z|)+f(|x+y+z|) \geqslant p(x,y)$$

and then of course

$$S_1+S_3 \geqslant P.$$

We obtain, using now our assumption on f^2,

$$\begin{aligned}
S_1^2-S_3^2 &= f^2(|x|)+f^2(|y|)+f^2(|z|)\\
&\quad +2\cdot[f(|x|)f(|y|)+f(|x|)f(|z|)+f(|y|)f(|z|)]-f^2(|x+y+z|)\\
&\leqslant 2\cdot[f^2(|x|)+f^2(|y|)+f^2(|z|)+f(|x|)f(|y|)\\
&\quad +f(|x|)f(|z|)+f(|y|)f(|z|)]\\
&\quad -f^2(|x+y|)-f^2(|x+z|)-f^2(|y+z|)\\
&= U \leqslant P\cdot Q,
\end{aligned}$$

and since $S_1+S_3 \geqslant P$, it follows

$$S_1 - S_3 \leqslant Q = 2S_1 - S_2$$

or equivalently

$$S_2 \leqslant S_1 + S_3$$

which had to be shown. □

COROLLARY 2. *Let* $\|.\|$ *denote the Euclidean norm on* $\mathbb{R}^n$, *or the* 1*-norm* $\|x\| = |x_1| + \ldots + |x_n|$. *Then for arbitrary* $x, y, z \in \mathbb{R}^n$

$$\sqrt{\|x+y\|} + \sqrt{\|x+z\|} + \sqrt{\|y+z\|} \leqslant \sqrt{\|x\|} + \sqrt{\|y\|} + \sqrt{\|z\|} + \sqrt{\|x+y+z\|}, \quad (4)$$

and also the corresponding result for the 4th root, the 8th root, etc.

Proof. Since the square of the Euclidean norm fulfills even Hlawka's identity

$$\|x+y\|^2 + \|x+z\|^2 + \|y+z\|^2 = \|x\|^2 + \|y\|^2 + \|z\|^2 + \|x+y+z\|^2,$$

Theorem 2 implies first the original Hornich-Hlawka inequality, and by reapplication what is claimed.

Since inequality (1) (for real numbers) extends trivially to the 1-norm, we get also in this case the square-root inequality (4) from Theorem 2. □

REMARK 2. It is certainly of interest for which norms on $\mathbb{R}^n$ $(n \leqslant 3$ suffices) the inequality (4) still holds. It remains true if the norm is a negative definite function on the group $\mathbb{R}^n$, since in this case there exists an isometric (linear) embedding into some L^1-space, see for ex. [10], Theorem 5.10 (with $p = 1$). It is well known (and easy to see) that L^p-norms for $p \in [1,2]$ are negative definite. Since every norm on $\mathbb{R}^2$ is negative definite, inequality (4) holds for every norm in the plane. However, this result can also be shown in a very elementary way, cf. [4].

REMARK 3. A "natural" analogue of inequality (1) for more than 3 numbers does not hold: take $n = 4$ and consider

$$\sum_{i=1}^{4} |x_i| - \sum_{i<j} |x_i + x_j| + \sum_{i<j<k} |x_i + x_j + x_k| - \left| \sum_{i=1}^{4} x_i \right|.$$

This expression has for $x_1 = x_2 = x_3 = -x_4 = 1$ the value 2, and changing x_4 to -2 gives -2 as a result.

However, on $\mathbb{R}^n_+$ we see immediately (as in step 1. of the proof of Theorem 1) that any n-alternating function f fulfills

$$\begin{aligned} 0 &\geqslant (\nabla_{x_1} \ldots \nabla_{x_n} f)(0) \\ &= f(0) - \sum_i f(x_i) + \sum_{i<j} f(x_i + x_j) - \sum_{i<j<k} f(x_i + x_j + x_k) \pm \ldots \end{aligned} \quad (5)$$

With $\sigma(x) := \sum_{i=1}^n x_i$, assuming $f(0) = 0$, this inequality is of some importance in probability theory: it means (essentially) that $\exp(-f \circ \sigma)$ is a multivariate survival function. This remains true if σ is replaced by a so-called *n-max-decreasing* function $\varphi : \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+$, defined by

$$\begin{aligned} 0 \geqslant \varphi(0) &- \sum_i \varphi(x_i e_i) + \sum_{i<j} \varphi(x_i e_i + x_j e_j) \\ &- \sum_{i<j<k} \varphi(x_i e_i + x_j e_j + x_k e_k) \pm \ldots \end{aligned} \quad (6)$$

(where $e_1,\ldots,e_n$ are the usual unit vectors) and the same inequality for every translate $x \longmapsto \varphi(a+x), a \in \mathbb{R}_+^n$. Note that for $\varphi = \sigma$ we have equality in (6). Details can be found in [7]. The following interesting sufficient condition for (5) to hold is given in [6], Theorem 6.5: the function $-f(x)/x$ should be convex of order $n-1$ on $]0,\infty[$.

REFERENCES

[1] C. BERG, J. P. R. CHRISTENSEN AND P. RESSEL, *Harmonic Analysis on Semigroups*, Springer-Verlag, New York, 1984.

[2] E. HLAWKA, *Ungleichungen*, Manz-Verlag, Wien, 1990.

[3] H. HORNICH, *Eine Ungleichung für Vektorlängen*, Math. Zeitschrift, **48** (1942), 268–274.

[4] L. M. KELLY, D. M. SMILEY AND M. F. SMILEY, *Two dimensional spaces are quadrilateral spaces*, Amer. Math. Monthly, **72** (1965), 753–754.

[5] D. S. MITRINOVIC, *Analytic Inequalities*, Springer-Verlag, New York, 1965.

[6] J. E. PECARIĆ, F. PROSCHAN AND Y. L. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, Acad. Press, Boston, 1992.

[7] P. RESSEL, *Functions operating on multivariate distribution and survival functions – With applications to classical mean-values and to copulas*, J. Multivar. Analysis, **105** (2012), 55–67.

[8] R. SCHILLING, R. SONG AND Z. VONDRAČEK, *Bernstein Functions*, de Gruyter-Verlag, Berlin, 2010.

[9] D. M. SMILEY AND M. F. SMILEY, *The polygonal inequalities*, Amer. Math. Monthly, **71** (1964), 755–760.

[10] J. H. WELLS AND L. R. WILLIAMS, *Embeddings and Extensions in Analysis*, Springer-Verlag, New York, 1975.

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