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IMPROVEMENTS OF THE HERMITE–HADAMARD INEQUALITY ON TIME SCALES

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Abstract. In this paper we give refinements of converse Jensen's inequality as well as of the Hermite-Hadamard inequality on time scales. We give mean value theorems and investigate logarithmic and exponential convexity of the linear functionals related to the obtained refinements. We also give several examples which illustrate possible applications for our results.

1. Introduction

The Hermite-Hadamard inequality is known to be the first inequality for convex functions. It is stated as:

$$(b-a)\Phi\left(\frac{a+b}{2}\right) \leqslant \int_a^b \Phi(s)\mathrm{d}s \leqslant (b-a)\frac{\Phi(a)+\Phi(b)}{2}, \tag{1.1}$$

where $a,b \in \mathbb{R}$ with $a<b$ and $\Phi\colon [a,b] \to \mathbb{R}$ is a convex function. It was first established by Hermite in 1881. Also, Beckenbach, a leading expert on the history and theory of complex functions, wrote that the first inequality in (1.1) was proved in 1893 by Hadamard who apparently was not aware of Hermite's result (see [11]). In general, (1.1) is now known as the Hermite-Hadamard inequality. Note that first inequality in (1.1) is Jensen's inequality,

$$\Phi\left(\frac{\int_a^b f(s)\mathrm{d}s}{b-a}\right) \leqslant \frac{\int_a^b \Phi(f(s))\mathrm{d}s}{b-a},$$

when $f(s)=s$ and the second one gives a converse of Jensen's inequality. Various generalizations of the Hermite-Hadamard inequality are given in the literature. Let us recall some generalizations from time scales theory given in [1].

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First we give an introduction to the time scale theory. Time scale $\mathbb{T}$ is an arbitrary closed subset of $\mathbb{R}$ and time scale calculus provides unification and extension of classical results. For example, when $\mathbb{T} = \mathbb{R}$ the time scale integral is Lebesgue integral and when $\mathbb{T} = \mathbb{Z}$ the time scale integral becomes sum. For detailed introduction to the time scale theory we refer to [2, 3, 4, 5].

In [3], multiple Lebesgue integral is defined in the following way.

Let $\mathbb{T}_i$, $i = 1, \ldots, n$, be time scales and

$$\Lambda^n = \mathbb{T}_1 \times \ldots \times \mathbb{T}_n = \{t = (t_1, \ldots, t_n) \colon \ t_i \in \mathbb{T}_i, \ 1 \leqslant i \leqslant n\}$$

an n-dimensional time scale. Let $f \colon E \to \mathbb{R}$ be a Δ-measurable function, where $E \subset \Lambda^n$ is Δ-measurable. Then the corresponding Δ-integral, called Lebesgue Δ-integral, is denoted by

$$\int_E f(t_1, \ldots, t_n) \Delta_1 t_1 \ldots \Delta_n t_n, \quad \int_E f(t) \Delta t, \quad \int_E f \mathrm{d}\mu_\Delta, \quad \text{or} \quad \int_E f(t) \mathrm{d}\mu_\Delta(t),$$

where μ_Δ is a σ-additive Lebesgue Δ-measure on Λ^n. By [3, Section 3] all theorems of the general Lebesgue integration theory hold also for Lebesgue Δ-integral on Λ^n. In what follows, we consider E to be Δ-measurable subset of Λ^n.

In the following, Theorem 1.1 recalls that the multiple Lebesgue Δ-integral is an isotonic linear functional. Theorem 1.2 recalls Jensen's inequality for multiple Lebesgue Δ-integral, Theorem 1.4 is a generalization of the Hermite-Hadamard inequality, while other theorems recall some converses of Jensen's inequality for multiple Lebesgue Δ-integral.

THEOREM 1.1. ([1, Theorem 3.7]) *If f and g are Δ-integrable functions on E then*

$$\int_E (\alpha f + \beta g) \mathrm{d}\mu_\Delta = \alpha \int_E f \mathrm{d}\mu_\Delta + \beta \int_E g \mathrm{d}\mu_\Delta \quad \textit{for all} \quad \alpha, \beta \in \mathbb{R}$$

and

$$f(s) \geqslant 0 \quad \textit{for all} \quad s \in E \quad \textit{implies} \quad \int_E f \mathrm{d}\mu_\Delta \geqslant 0.$$

THEOREM 1.2. ([1, Theorem 4.2]) *Let $\Phi \in \mathrm{C}(I, \mathbb{R})$ be a convex function, where $I \subset \mathbb{R}$ is an interval. Suppose that f is a Δ-integrable function on E such that $f(E) = I$ and that $h \colon E \to \mathbb{R}$ is a nonnegative Δ-integrable function such that $\int_E h \mathrm{d}\mu_\Delta > 0$. Then*

$$\Phi\left(\frac{\int_E h f \mathrm{d}\mu_\Delta}{\int_E h \mathrm{d}\mu_\Delta}\right) \leqslant \frac{\int_E h \Phi(f) \mathrm{d}\mu_\Delta}{\int_E h \mathrm{d}\mu_\Delta}.$$

THEOREM 1.3. ([1, Theorem 5.2]) *Let $\Phi \in C(I, \mathbb{R})$ be a convex function, where $I = [m, M] \subset \mathbb{R}$ with $m < M$. Suppose that f is a Δ-integrable function on E such that $f(E) = I$ and that $h \colon E \to \mathbb{R}$ is a nonnegative Δ-integrable function such that $\int_E h \mathrm{d}\mu_\Delta > 0$. Then*

$$\frac{\int_E h \Phi(f) \mathrm{d}\mu_\Delta}{\int_E h \mathrm{d}\mu_\Delta} \leqslant \frac{M - \int_E h f \mathrm{d}\mu_\Delta / \int_E h \mathrm{d}\mu_\Delta}{M - m} \Phi(m) + \frac{\int_E h f \mathrm{d}\mu_\Delta / \int_E h \mathrm{d}\mu_\Delta - m}{M - m} \Phi(M).$$

THEOREM 1.4. ([1, Theorem 5.5]) *Let $\Phi \in C(I,\mathbb{R})$ be a convex function, where $[m,M] \subset I$ with $m < M$ and $I \subset \mathbb{R}$ is an interval. Suppose that f is Δ-integrable on E such that $f(E) \subset [m,M]$ and that $h\colon E \to \mathbb{R}$ is a nonnegative Δ-integrable function such that $\int_E h\mathrm{d}\mu_\Delta > 0$. Let $p,q \geqslant 0$ be such that $p+q>0$ and*

$$\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} = \frac{pm+qM}{p+q}$$

holds. Then

$$\Phi\left(\frac{pm+qM}{p+q}\right) \leqslant \frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} \leqslant \frac{p\Phi(m)+q\Phi(M)}{p+q}.$$

THEOREM 1.5. ([1, Theorem 12.2]) *Let $\Phi \in C^1(I,\mathbb{R})$ be such that Φ' is strictly increasing on I, where $I = [m,M]$ with $m < M$. Suppose that f is a Δ-integrable function on E such that $f(E) = I$ and that $h\colon E \to \mathbb{R}$ is a nonnegative Δ-integrable function such that $\int_E h\mathrm{d}\mu_\Delta > 0$. Then*

$$\frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} \leqslant \lambda + \Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right) \tag{1.2}$$

holds for some λ satisfying $0 < \lambda < (M-m)(\nu - \Phi'(m))$, where $\nu = (\Phi(M) - \Phi(m))/(M-m)$. More precisely λ may be determined as follows: Let $\tilde{x}$ be the unique solution of the equation $\Phi'(x) = \nu$. Then

$$\lambda = \Phi(m) - \Phi(\tilde{x}) + \nu(\tilde{x} - m)$$

satisfies (1.2).

THEOREM 1.6. ([1, Theorem 12.3]) *In addition to the assumptions of Theorem 1.3 let $J \subset \mathbb{R}$ be an interval such that $J \supset \Phi(I)$ and suppose that $F\colon J \times J \to \mathbb{R}$ is increasing in the first variable. Then*

$$\begin{aligned}
&F\left(\frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}, \Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)\right) \\
&\leqslant \max_{x\in[m,M]} F\left(\frac{M-x}{M-m}\Phi(m) + \frac{x-m}{M-m}\Phi(M), \Phi(x)\right) \\
&= \max_{\sigma\in[0,1]} F\left(\sigma\Phi(m) + (1-\sigma)\Phi(M), \Phi(\sigma m + (1-\sigma)M)\right),
\end{aligned}$$

and the right-hand side of the inequality is an increasing function of M and a decreasing function of m.

REMARK 1.7. If we choose $F(x,y) = x - y$, as a simple consequence of Theorem 1.6 it follows

$$\begin{aligned}
&\frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} - \Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right) \qquad (1.3)\\
&\leqslant \max_{x\in[m,M]}\left(\frac{M-x}{M-m}\Phi(m) + \frac{x-m}{M-m}\Phi(M) - \Phi(x)\right)\\
&= \max_{\sigma\in[0,1]}\left(\sigma\Phi(m) + (1-\sigma)\Phi(M) - \Phi(\sigma m + (1-\sigma)M)\right).
\end{aligned}$$

REMARK 1.8. As a time scale is an arbitrary closed subset of $\mathbb{R}$, we can obtain both discrete and continuous versions of the above results. Namely, let $E = \{a, a+1, \ldots, b\} \subset \mathbb{N}$. Then $\int_E f(s)\mathrm{d}\mu_\Delta(s)$ becomes $\sum_{s=a}^{b-1} f(s)$. On the other hand, if we take $\Lambda = \mathbb{R}$ and $E = [a,b)$ an interval in $\mathbb{R}$, then $\int_E f\mathrm{d}\mu_\Delta$ becomes the Lebesgue integral $\int_a^b f(s)\mathrm{d}\mu(s)$. Similarly, if $\Lambda = h\mathbb{Z}$, $h > 0$ and $E = [a, b-h] \cap h\mathbb{Z}$ then $\int_E f(s)\mathrm{d}\mu_\Delta(s)$ becomes $h\sum_{s=a/h}^{b/h-1} f(sh)$.

REMARK 1.9. Throughout this paper we give the results for multiple Lebesgue Δ-integral but all the results can be given for many other time scales integrals in a similar way, such as Cauchy, Riemann, Lebesgue and multiple Riemann, delta, nabla and diamond-α time scales integrals and also for multiple Lebesgue nabla and diamond-α time scales integrals.

Now, we quote some definitions and results from [9] about log-convexity and exponential convexity which will be used in Section 3.

DEFINITION 1.10. A function $\psi\colon I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i\xi_j\psi\left(\frac{x_i+x_j}{2}\right) \geqslant 0,$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, \ldots, n$.

A function $\psi\colon I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

DEFINITION 1.11. A function $\psi\colon I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\psi\colon I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

REMARK 1.12. It is known (and easy to show) that $\psi\colon I \to (0,\infty)$ is log-convex in the Jensen sense if and only if

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geqslant 0$$

holds for every α, $\beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

Also, using basic convexity theory, it follows that a positive function is log-convex if and only if it is 2-exponentially convex.

PROPOSITION 1.13. *If ψ is a convex function on an interval I and if $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$ then the following inequality is valid*

$$\frac{\psi(x_2)-\psi(x_1)}{x_2-x_1} \leqslant \frac{\psi(y_2)-\psi(y_1)}{y_2-y_1}.$$

When dealing with functions with different degree of smoothness divided differences are found to be very useful.

DEFINITION 1.14. The second order divided difference of a function $f\colon [a,b] \to \mathbb{R}$ at mutually different points $x_0, x_1, x_2 \in [a,b]$ is defined recursively by

$$[x_i; f] = f(x_i), \quad i = 0,1,2,$$

$$[x_i, x_{i+1}; f] = \frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}, \quad i = 0,1,$$

$$[x_0, x_1, x_2; f] = \frac{[x_1, x_2; f]-[x_0, x_1; f]}{x_2-x_0}.$$

The value $[x_0, x_1, x_2; f]$ is independent of the order of the points x_0, x_1 and x_2. This definition may be extended to include the case in which some or all the points coincide (see [11, page 14]). Namely, taking the limits $x_1 \to x_0$, we obtain

$$\lim_{x_1 \to x_0} [x_0, x_1, x_2; f] = [x_0, x_0, x_2; f] = \frac{f(x_2)-f(x_0)-f'(x_0)(x_2-x_0)}{(x_2-x_0)^2}, \quad x_2 \neq x_0$$

provided that f' exists, and furthermore, taking the limits $x_i \to x_0$, $i = 1,2$, we obtain

$$\lim_{x_2 \to x_0} \lim_{x_1 \to x_0} [x_0, x_1, x_2; f] = [x_0, x_0, x_0; f] = \frac{f''(x_0)}{2}$$

provided that f'' exists.

In next section we give improvements of converses of Jensen's inequality as stated above and as a consequence improvements of generalization of the Hermite-Hadamard inequality. In Section 3 we discuss log-convexity, n-exponential convexity and exponential convexity of the differences obtained from new results of Section 2.

2. Main results

In what follows, we assume I to be an interval in $\mathbb{R}$ and $[m, M]$ an interval in $\mathbb{R}$ with $m < M$. To prove our main results we need the following lemma (see [10, Lemma 1]).

LEMMA 2.1. *Let Φ be a convex function on I, $x,y\in I$ and $p,q\in[0,1]$ such that $p+q=1$. Then*

$$\begin{aligned}&\min\{p,q\}\left[\Phi(x)+\Phi(y)-2\Phi\left(\frac{x+y}{2}\right)\right]\\ &\leqslant p\Phi(x)+q\Phi(y)-\Phi(px+qy)\\ &\leqslant \max\{p,q\}\left[\Phi(x)+\Phi(y)-2\Phi\left(\frac{x+y}{2}\right)\right].\end{aligned}\tag{2.1}$$

THEOREM 2.2. *Let $\Phi\in \mathrm{C}(I,\mathbb{R})$ be a convex function and let $f\colon E\to[m,M]$ be a Δ-integrable function, where $[m,M]\subseteq I$. Suppose that $h\colon E\to\mathbb{R}$ is a nonnegative Δ-integrable function such that $\int_E h\mathrm{d}\mu_\Delta>0$. Then*

$$\begin{aligned}\frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\leqslant{}&\frac{M-\int_E hf\mathrm{d}\mu_\Delta/\int_E h\mathrm{d}\mu_\Delta}{M-m}\Phi(m)\\ &+\frac{\int_E hf\mathrm{d}\mu_\Delta/\int_E h\mathrm{d}\mu_\Delta-m}{M-m}\Phi(M)-\frac{\int_E h\tilde{f}\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\delta_\Phi,\end{aligned}\tag{2.2}$$

where

$$\tilde{f}=\frac{1}{2}-\frac{|f-(m+M)/2|}{M-m},\quad \delta_\Phi=\Phi(m)+\Phi(M)-2\Phi\left(\frac{m+M}{2}\right).\tag{2.3}$$

Proof. Let the functions $p,q\colon[m,M]\to\mathbb{R}$ be defined by

$$p(x)=\frac{M-x}{M-m},\quad q(x)=\frac{x-m}{M-m}.\tag{2.4}$$

For any $x\in[m,M]$ we can write

$$\Phi(x)=\Phi\left(\frac{M-x}{M-m}m+\frac{x-m}{M-m}M\right)=\Phi(p(x)m+q(x)M).$$

By using Lemma 2.1, we obtain

$$\Phi(x)\leqslant p(x)\Phi(m)+q(x)\Phi(M)-\min\{p(x),q(x)\}\left(\Phi(m)+\Phi(M)-2\Phi\left(\frac{m+M}{2}\right)\right).$$

Now by replacing x with $f(s)$, where $s\in E$, we obtain

$$\Phi(f(s))\leqslant p(f(s))\Phi(m)+q(f(s))\Phi(M)-\tilde{f}(s)\delta_\Phi,\tag{2.5}$$

where the function $\tilde{f}$ is defined on E by

$$\tilde{f}(s)=\frac{1}{2}-\frac{|f(s)-(m+M)/2|}{M-m}.$$

Since h is nonnegative Δ-integrable and $\int_E h \mathrm{d}\mu_\Delta > 0$, multiplying (2.5) with h, applying integral and then dividing by $\int_E h \mathrm{d}\mu_\Delta$, we have

$$\frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} \leqslant \frac{\int_E hp(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\Phi(m) + \frac{\int_E hq(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\Phi(M) - \frac{\int_E h\tilde{f}\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\delta_\Phi$$

from which (2.2) follows. $\square$

REMARK 2.3. Theorem 2.2 gives a refinement of Theorem 1.3 as under the required assumptions we have

$$\frac{\int_E h\tilde{f}\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\delta_\Phi = \frac{\int_E h\left(\frac{1}{2} - \frac{|f-(m+M)/2|}{M-m}\right)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\delta_\Phi \geqslant 0. \tag{2.6}$$

REMARK 2.4. Since Δ-integral is an isotonic linear functional by Theorem 1.1, Theorem 2.2 can also be obtained from [8, Theorem 12]. If we take $E \subset \mathbb{N}$, we obtain a discrete version of Theorem 2.2 given in [8, Corrolary 1].

THEOREM 2.5. *Let $\Phi \in \mathrm{C}(I,\mathbb{R})$ be a convex function and let $f\colon E \to [m,M]$ be a Δ-integrable function, where $[m,M] \subseteq I$. Suppose that $h\colon E \to \mathbb{R}$ is a nonnegative Δ-integrable function such that $\int_E h\mathrm{d}\mu_\Delta > 0$. Then*

$$\begin{aligned} &\frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} - \Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right) \\ &\leqslant \max_{x\in[m,M]}\left\{\frac{M-x}{M-m}\Phi(m) + \frac{x-m}{M-m}\Phi(M) - \Phi(x)\right\} - \frac{\int_E h\tilde{f}\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\delta_\Phi \\ &= \max_{\sigma\in[0,1]}\{\sigma\Phi(m) + (1-\sigma)\Phi(M) - \Phi(\sigma m + (1-\sigma)M)\} - \frac{\int_E h\tilde{f}\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\delta_\Phi, \end{aligned} \tag{2.7}$$

where $\tilde{f}$ and δ_Φ are defined as in (2.3).

Proof. This is an immediate consequence of Theorem 2.2. The identity follows from the change of variables $\sigma = (M-x)/(M-m)$, so that for $x \in [m,M]$ we have $\sigma \in [0,1]$ and $x = \sigma m + (1-\sigma)M$. $\square$

REMARK 2.6. Arguing as in Remark 2.3, (2.7) is a refinement of (1.3).

REMARK 2.7. Arguing as in Remark 2.4, Theorem 2.5 can also be obtained from [8, Theorem 13].

THEOREM 2.8. *Let $\Phi \in \mathrm{C}(I,\mathbb{R})$ be a convex function and let $f\colon E \to [m,M]$ be a Δ-integrable function, where $[m,M] \subseteq I$. Suppose that $h\colon E \to \mathbb{R}$ is a nonnegative Δ-integrable function such that $\int_E h\mathrm{d}\mu_\Delta > 0$. Then*

$$\frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} - \Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right) \leqslant \frac{1}{M-m}\left\{\left|\frac{m+M}{2} - \frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right| + \frac{\int_E h|(m+M)/2 - f|\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right\}\delta_\Phi, \quad (2.8)$$

where δ_Φ *is defined as in* (2.3).

Proof. Let the functions $p,q\colon [m,M] \to \mathbb{R}$ be defined as in (2.4). Then for any $x \in [m,M]$ we can write

$$\Phi(x) = \Phi(p(x)m + q(x)M).$$

Since $\int_E hf\mathrm{d}\mu_\Delta / \int_E h\mathrm{d}\mu_\Delta \in [m,M]$, the above equation implies that

$$\Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right) = \Phi\left(p\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)m + q\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)M\right).$$

By Lemma 2.1, we get

$$\begin{aligned}\Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right) &\geqslant p\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)\Phi(m) + q\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)\Phi(M) \\ &\quad - \max\left\{p\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right), q\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)\right\}\delta_\Phi \\ &= p\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)\Phi(m) + q\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)\Phi(M) \\ &\quad - \left\{\frac{1}{2} + \frac{|(m+M)/2 - \int_E hf\mathrm{d}\mu_\Delta / \int_E h\mathrm{d}\mu_\Delta)|}{M-m}\right\}\delta_\Phi.\end{aligned} \quad (2.9)$$

Again by Lemma 2.1, we get

$$\Phi(f) \leqslant p(f)\Phi(m) + q(f)\Phi(M) - \min\{p(f), q(f)\}\delta_\Phi,$$

which implies that

$$\begin{aligned}\frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} &\leqslant \frac{\int_E hp(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\Phi(m) + \frac{\int_E hq(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\Phi(M) \\ &\quad - \frac{\int_E h\min\{p(f), q(f)\}\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\delta_\Phi \\ &= p\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)\Phi(m) + q\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)\Phi(M) \\ &\quad - \left\{\frac{1}{2} - \frac{\int_E h|f-(m+M)/2|\mathrm{d}\mu_\Delta / \int_E h\mathrm{d}\mu_\Delta}{M-m}\right\}\delta_\Phi.\end{aligned} \quad (2.10)$$

Now, from inequalities (2.9) and (2.10) we get desired inequality (2.8). □

REMARK 2.9. Arguing as in Remark 2.4, Theorem 2.8 can also be obtained from [10, Theorem 8].

COROLLARY 2.10. *Under the assumptions of Theorem 2.8 the following inequality holds:*

$$\frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} - \Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right) \leqslant \left\{\frac{1}{2} + \frac{1}{M-m}\left|\frac{m+M}{2} - \frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right|\right\}\delta_\Phi. \tag{2.11}$$

Proof. Since

$$\frac{1}{M-m}\left|\frac{m+M}{2} - f\right| \leqslant \frac{1}{2},$$

we have

$$\frac{1}{M-m}\frac{\int_E h\,|(m+M)/2 - f|\,\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} \leqslant \frac{1}{2}.$$

Now inequality (2.11) directly follows from Theorem 2.8. □

REMARK 2.11. Arguing as in Remark 2.4, Corollary 2.10 can also be obtained from [10, Theorem 6].

The following two theorems give improvements of Theorem 1.4.

THEOREM 2.12. *Let $\Phi \in \mathrm{C}(I,\mathbb{R})$ be a convex function and let $f\colon E \to [m,M]$ be a Δ-integrable function, where $[m,M] \subseteq I$. Suppose that $h\colon E \to \mathbb{R}$ is a nonnegative Δ-integrable function such that $\int_E h\mathrm{d}\mu_\Delta > 0$ and that p,q are nonnegative numbers such that $p+q > 0$ and*

$$\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} = \frac{pm+qM}{p+q}.$$

Then

$$\Phi\left(\frac{pm+qM}{p+q}\right) \leqslant \frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} \leqslant \frac{p\Phi(m)+q\Phi(M)}{p+q} - \frac{\int_E h\tilde{f}\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\delta_\Phi, \tag{2.12}$$

where $\tilde{f}$ and δ_Φ are defined as in (2.3).

Proof. The first inequality in (2.12) follows from Theorem 1.2 and the second one follows from Theorem 2.2. □

REMARK 2.13. Arguing as in Remark 2.4, Theorem 2.12 can also be obtained from [7, Theorem 5].

REMARK 2.14. Theorem 2.12 gives an improvement of Theorem 1.4 as under the required assumptions we have

$$\frac{\int_E h\tilde{f}\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\delta_\Phi \geqslant 0.$$

THEOREM 2.15. *Let $\Phi \in C(I,\mathbb{R})$ be a convex function and let $f\colon E \to [m,M]$ be a Δ-integrable function where $[m,M] \subseteq I$. Suppose that $h\colon E \to \mathbb{R}$ is a nonnegative Δ-integrable function such that $\int_E h\mathrm{d}\mu_\Delta > 0$ and that p,q are nonnegative numbers such that $p+q>0$ and*

$$\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} = \frac{pm+qM}{p+q}, \quad 0<y\leqslant \frac{M-m}{p+q}\min\{p,q\}. \tag{2.13}$$

Then

$$\begin{aligned}\Phi\left(\frac{pm+qM}{p+q}\right) &\leqslant \frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\\ &\leqslant \frac{p\Phi(m)+q\Phi(M)}{p+q} - 2\frac{\int_E h\tilde{f}_1\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\left(\frac{p\Phi(m)+q\Phi(M)}{p+q} - \Phi\left(\frac{pm+qM}{p+q}\right)\right),\end{aligned} \tag{2.14}$$

where

$$\tilde{f}_1 = \frac{1}{2} - \frac{|f-(pm+qM)/(p+q)|}{2y}. \tag{2.15}$$

Proof. The first inequality in (2.14) follows from Theorem 1.2. By using (2.13), we have

$$m \leqslant \frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} - y < \frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} + y \leqslant M.$$

Suppose $m_1 = \int_E hf\mathrm{d}\mu_\Delta / \int_E h\mathrm{d}\mu_\Delta - y$ and $M_1 = \int_E hf\mathrm{d}\mu_\Delta / \int_E h\mathrm{d}\mu_\Delta + y$, then

$$\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} = \frac{\int_E hf\mathrm{d}\mu_\Delta / \int_E h\mathrm{d}\mu_\Delta - y + \int_E hf\mathrm{d}\mu_\Delta / \int_E h\mathrm{d}\mu_\Delta + y}{2} = \frac{m_1+M_1}{2}.$$

By applying Theorem 2.12 with $p=q=1$, we obtain

$$\begin{aligned}\frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} \leqslant\ & \frac{\Phi(\int_E hf\mathrm{d}\mu_\Delta / \int_E h\mathrm{d}\mu_\Delta - y) + \Phi(\int_E hf\mathrm{d}\mu_\Delta / \int_E h\mathrm{d}\mu_\Delta + y)}{2}\\ & - \frac{\int_E h\tilde{f}_1\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\left(\Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} - y\right) + \Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} + y\right)\right.\\ & \left. -2\Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)\right)\\ =\ & \left(1-2\frac{\int_E h\tilde{f}_1\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)\frac{\Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} - y\right) + \Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} + y\right)}{2}\\ & + 2\frac{\int_E h\tilde{f}_1\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right).\end{aligned}$$

Now by using Theorem 1.3, we obtain

$$\begin{aligned}\Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} - y\right) \leqslant\ & \frac{M - (\int_E hf\mathrm{d}\mu_\Delta / \int_E h\mathrm{d}\mu_\Delta - y)}{M-m}\Phi(m)\\ & + \frac{\int_E hf\mathrm{d}\mu_\Delta / \int_E h\mathrm{d}\mu_\Delta - y - m}{M-m}\Phi(M),\end{aligned}$$

$$\Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}+y\right)\leqslant\frac{M-(\int_E hf\mathrm{d}\mu_\Delta/\int_E h\mathrm{d}\mu_\Delta+y)}{M-m}\Phi(m)$$
$$+\frac{\int_E hf\mathrm{d}\mu_\Delta/\int_E h\mathrm{d}\mu_\Delta+y-m}{M-m}\Phi(M).$$

hence

$$\frac{\Phi(\int_E hf\mathrm{d}\mu_\Delta/\int_E h\mathrm{d}\mu_\Delta-y)+\Phi(\int_E hf\mathrm{d}\mu_\Delta/\int_E h\mathrm{d}\mu_\Delta+y)}{2}$$
$$\leqslant\frac{M-\int_E hf\mathrm{d}\mu_\Delta/\int_E h\mathrm{d}\mu_\Delta}{M-m}\Phi(m)+\frac{\int_E hf\mathrm{d}\mu_\Delta/\int_E h\mathrm{d}\mu_\Delta-m}{M-m}\Phi(M).$$

If p and q are any nonnegative numbers such that (2.13) holds (observe that they are different from those we started with), we obtain

$$\frac{\Phi(\int_E hf\mathrm{d}\mu_\Delta/\int_E h\mathrm{d}\mu_\Delta-y)+\Phi(\int_E hf\mathrm{d}\mu_\Delta/\int_E h\mathrm{d}\mu_\Delta+y)}{2}\leqslant\frac{p\Phi(m)+q\Phi(M)}{p+q}.$$

Considering all this and the fact that $1-2\int_E h\tilde{f}_1\mathrm{d}\mu_\Delta/\int_E h\mathrm{d}\mu_\Delta\geqslant 0$, we deduce

$$\frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\leqslant\left(1-2\frac{\int_E h\tilde{f}_1\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)\frac{p\Phi(m)+q\Phi(M)}{p+q}+2\frac{\int_E h\tilde{f}_1\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)$$
$$=\frac{p\Phi(m)+q\Phi(M)}{p+q}-2\frac{\int_E h\tilde{f}_1\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\left[\frac{p\Phi(m)+q\Phi(M)}{p+q}-\Phi\left(\frac{pm+qM}{p+q}\right)\right],$$

hence the proof is complete. □

REMARK 2.16. Arguing as in Remark 2.4, Theorem 2.15 can also be obtained from [7, Theorem 6].

From (2.14) easily follows a Hammer-Bullen type inequality for multiple Lebesgue Δ-integral.

COROLLARY 2.17. *Under the assumptions of Theorem 2.15 the following inequality holds:*

$$\left(1-2\frac{\int_E h\tilde{f}_1\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)\left[\frac{p\Phi(m)+q\Phi(M)}{p+q}-\frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right]$$
$$\geqslant 2\frac{\int_E h\tilde{f}_1\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\left[\frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}-\Phi\left(\frac{pm+qM}{p+q}\right)\right].\qquad(2.16)$$

Proof. It follows directly from Theorem 2.15. □

REMARK 2.18. Arguing as in Remark 2.4, Corollary 2.17 can also be obtained from [7, Corollary 1].

THEOREM 2.19. *Let* $\Phi \in C^1(I,\mathbb{R})$ *be such that* Φ' *is strictly increasing on* I*, where* $I=[m,M]$ *with* $m<M$*. Suppose that* $f\colon E \to I$ *is a* Δ*-integrable function and that* $h\colon E\to\mathbb{R}$ *is a nonnegative* Δ*-integrable function such that* $\int_E h\mathrm{d}\mu_\Delta > 0$*. Let* $\tilde f$ *and* δ_Φ *be defined as in* (2.3)*. Then*

$$\frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} \leqslant \lambda + \Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right) - \frac{\int_E h\tilde f\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\delta_\Phi \tag{2.17}$$

holds for some λ *satisfying* $0<\lambda<(M-m)(\nu-\Phi'(m))$*, where* $\nu=(\Phi(M)-\Phi(m))/(M-m)$*. More precisely,* λ *may be determined in the following way: Let* $\tilde x$ *be the unique solution of the equation* $\Phi'(x)=\nu$*. Then*

$$\lambda = \Phi(m)-\Phi(\tilde x)+\nu(\tilde x - m)$$

satisfies (2.17).

Proof. By Theorem 2.5, we have

$$\frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} - \Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right) \leqslant \max_{x\in I} g(x) - \frac{\int_E h\tilde f\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\delta_\Phi$$

where

$$g(x)=\frac{M-x}{M-m}\Phi(m)+\frac{x-m}{M-m}\Phi(M)-\Phi(x).$$

Then

$$g'(x)=\nu-\Phi'(x),$$

which is strictly decreasing on I with $g'(\tilde x)=0$ for a unique $\tilde x\in I$. Consequently $g(x)$ achieves its maximum value at $x=\tilde x$. Hence the result follows. □

REMARK 2.20. Arguing as in Remark 2.4, Theorem 2.19 can also be obtained from [8, Theorem 14]. Furthermore, it gives a refinement of Theorem 1.5.

COROLLARY 2.21. *Let* f *be a* Δ*-integrable function on* E *such that* $f(E)=[m,M]\subset(0,\infty)$ *and let* $h\colon E\to\mathbb{R}$ *be a nonnegative* Δ*-integrable function such that* $\int_E h\mathrm{d}\mu_\Delta>0$*. Then*

$$\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} \leqslant \exp\left(\frac{\int_E h\log f\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)\frac{\exp\left(S\left(M/m\right)\right)}{[(m+M)^2/4mM]^{\left(\int_E h\tilde f\mathrm{d}\mu_\Delta/\int_E h\mathrm{d}\mu_\Delta\right)}}, \tag{2.18}$$

where $S(\cdot)$ *is Specht ratio and* $\tilde f$ *is defined as in Theorem* 2.2.

Proof. This is a special case of Theorem 2.19 for $\Phi=-\log$. In this case (2.17) becomes

$$-\frac{\int_E h\log f\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} \leqslant \lambda - \log\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right) - \frac{\int_E h\tilde f\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\delta_{-\log},$$

that is,

$$\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} \leqslant \exp\left(\frac{\int_E h\log f\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} + \lambda - \frac{\int_E h\tilde{f}\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\delta_{-\log}\right)$$
$$= \exp\left(\frac{\int_E h\log f\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right)\frac{\exp\lambda}{\exp\left(\left(\int_E h\tilde{f}\mathrm{d}\mu_\Delta / \int_E h\mathrm{d}\mu_\Delta\right)\delta_{-\log}\right)},$$

where

$$\delta_{-\log} = -\log m - \log M + 2\log\frac{m+M}{2} = \log\frac{(m+M)^2}{4mM},$$
$$\nu = \frac{\log m - \log M}{M-m}, \quad \tilde{x} = -\frac{1}{\nu} = \frac{M-m}{\log M - \log m},$$

hence

$$\lambda = -\log m + \nu(\tilde{x} - m) + \log\tilde{x} = \log\frac{(M/m)^{m/(M-m)}}{e\log(M/m)^{m/(M-m)}} = S\left(\frac{M}{m}\right),$$

where $S(\cdot)$ is Specht ratio defined by

$$S(a) = \frac{a^{1/(a-1)}}{e\log a^{1/(a-1)}}, \quad a \in (0,\infty)\setminus\{1\}.$$

Considering all this we obtain (2.18). □

REMARK 2.22. Arguing as in Remark 2.4, Corollary 2.21 can also be obtained from [8, Corollary 2].

COROLLARY 2.23. *Let f be a Δ-integrable function on E such that $f(E) = [m,M] \subset (0,\infty)$ and let $h\colon E \to \mathbb{R}$ be a nonnegative Δ-integrable function such that $\int_E h\mathrm{d}\mu_\Delta > 0$. Then*

$$\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} \leqslant \exp\left(\frac{\int_E h\log f\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right) + \frac{M-m}{\log(M/m)}S\left(\frac{M}{m}\right)$$
$$-\frac{\int_E h\tilde{f}_2\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\left(m+M-2\sqrt{mM}\right), \tag{2.19}$$

where $S(\cdot)$ is Specht ratio and $\tilde{f}_2$ is defined by

$$\tilde{f}_2 = \frac{1}{2} - \frac{\left|\log f - \log\sqrt{mM}\right|}{\log M - \log m}. \tag{2.20}$$

Proof. This is a special case of Theorem 2.19 for $\Phi = \exp$ and $f = \log$. In this case (2.17) becomes

$$\frac{\int_E h\exp\log f\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} \leqslant \lambda + \exp\left(\frac{\int_E h\log f\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right) - \frac{\int_E h\tilde{f}_2\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\delta_{\exp},$$

where

$$\delta_{\exp} = \exp\log m + \exp\log M - 2\exp\frac{\log m + \log M}{2} = m + M - 2\sqrt{mM},$$
$$\nu = \frac{M-m}{\log M - \log m}, \quad \tilde{x} = \log\nu = \log\frac{M-m}{\log M - \log m},$$

hence

$$\begin{aligned}\lambda &= \exp\log m + \nu(\tilde{x} - \log m) - \exp\tilde{x} \\ &= m + \frac{M-m}{\log M - \log m}\left(\log\frac{M-m}{\log M - \log m} - \log m - 1\right) \\ &= \frac{M-m}{\log(M/m)} S\left(\frac{M}{m}\right).\end{aligned}$$

Considering all this we obtain (2.19). □

REMARK 2.24. Arguing as in Remark 2.4, Corollary 2.23 can also be obtained from [8, Corollary 3].

3. Log-convexity and exponential convexity

Motivated by results from previous Section, we define linear functionals $\mathscr{H}_i\colon L_f \to \mathbb{R}$, $i = 1,2,3$, by

$$\begin{aligned}\mathscr{H}_1(\Phi) &= \frac{M - \int_E hf\,\mathrm{d}\mu_\Delta / \int_E h\mathrm{d}\mu_\Delta}{M-m}\Phi(m) + \frac{\int_E hf\,\mathrm{d}\mu_\Delta / \int_E h\mathrm{d}\mu_\Delta - m}{M-m}\Phi(M) \\ &\quad - \frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} - \frac{\int_E h\tilde{f}\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\delta_\Phi, \end{aligned} \tag{3.1}$$

$$\begin{aligned}\mathscr{H}_2(\Phi) &= \Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right) - \frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} \\ &\quad + \frac{1}{M-m}\left\{\left|\frac{m+M}{2} - \frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right| + \frac{\int_E h|(m+M)/2 - f|\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right\}\delta_\Phi, \end{aligned} \tag{3.2}$$

$$\begin{aligned}\mathscr{H}_3(\Phi) &= \Phi\left(\frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right) - \frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} \\ &\quad + \left\{\frac{1}{2} + \frac{1}{M-m}\left|\frac{m+M}{2} - \frac{\int_E hf\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\right|\right\}\delta_\Phi, \end{aligned} \tag{3.3}$$

where $f, h, \tilde{f}, \delta_\Phi$ are as in Theorem 2.2, $L_f = \{\Phi\colon I \to \mathbb{R} : \Phi(f)$ is a Δ-integrable function$\}$, $[m,M] \subseteq I$.

Also, if p, q and $\tilde{f}_1$ are as in Theorems 2.12 and 2.15, we define linear functionals $\mathscr{H}_4$ and $\mathscr{H}_5$ by

$$\mathscr{H}_4(\Phi) = \frac{p\Phi(m) + q\Phi(M)}{p+q} - \frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} - \frac{\int_E h\tilde{f}\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\delta_\Phi, \tag{3.4}$$

$$\begin{aligned}\mathscr{H}_5(\Phi) = &\frac{p\Phi(m)+q\Phi(M)}{p+q} - \frac{\int_E h\Phi(f)\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta} \\ &-2\frac{\int_E h\tilde{f}_1\mathrm{d}\mu_\Delta}{\int_E h\mathrm{d}\mu_\Delta}\left(\frac{p\Phi(m)+q\Phi(M)}{p+q} - \Phi\left(\frac{pm+qM}{p+q}\right)\right). \end{aligned} \tag{3.5}$$

If Φ is additionally continuous and convex on I, then using Theorems 2.2 and 2.8, Corollary 2.10 and Theorems 2.12 and 2.15, respectively, we have

$$\mathscr{H}_i(\Phi) \geqslant 0, \quad i = 1, \dots, 5.$$

THEOREM 3.1. *Let $\Phi\colon I \to \mathbb{R}$, where $[m,M] \subseteq I$, be such that $\Phi \in C^2(I)$. If $\mathscr{H}_i$, $i = 1, \dots, 5$, are defined as in* (3.1)*,...,*(3.5)*, then there exist $\xi_i \in [m,M]$, $i = 1, \dots, 5$, such that*

$$\mathscr{H}_i(\Phi) = \frac{\Phi''(\xi_i)}{2}\mathscr{H}_i(\Phi_0), \quad i = 1, \dots, 5, \tag{3.6}$$

where $\Phi_0(x) = x^2$.

Proof. We give a proof for the functional $\mathscr{H}_1$. Since $\Phi \in C^2(I)$ there exists η, $\zeta \in \mathbb{R}$ such that

$$\eta = \min_{x\in[m,M]} \Phi''(x) \quad \text{and} \quad \zeta = \max_{x\in[m,M]} \Phi''(x).$$

Let

$$\phi_1(x) = \frac{\zeta}{2}x^2 - \Phi(x) \quad \text{and} \quad \phi_2(x) = \Phi(x) - \frac{\eta}{2}x^2.$$

Then ϕ_1 and ϕ_2 are continuous and convex on $[m,M]$, and we have

$$\mathscr{H}_1(\phi_1) \geqslant 0, \quad \mathscr{H}_1(\phi_2) \geqslant 0,$$

which implies

$$\frac{\eta}{2}\mathscr{H}_1(\Phi_0) \leqslant \mathscr{H}_1(\Phi) \leqslant \frac{\zeta}{2}\mathscr{H}_1(\Phi_0).$$

If $\mathscr{H}_1(\Phi_0) = 0$, there is nothing to prove. Suppose $\mathscr{H}_1(\Phi_0) > 0$. Then we have

$$\eta \leqslant \frac{2\mathscr{H}_1(\Phi)}{\mathscr{H}_1(\Phi_0)} \leqslant \zeta.$$

Hence, there exists $\xi_1 \in [m,M]$ such that

$$\frac{2\mathscr{H}_1(\Phi)}{\mathscr{H}_1(\Phi_0)} = \Phi''(\xi_1),$$

and the result follows. $\square$

THEOREM 3.2. *Let $\Phi,\psi\colon I\to\mathbb{R}$, where $[m,M]\subseteq I$, be such that $\Phi,\psi\in C^2(I)$. If $\mathscr{H}_i$, $i=1,\dots,5$, are defined as in* (3.1)$,\dots,$(3.5)*, then there exist $\xi_i\in[m,M]$, $i=1,\dots,5$, such that*

$$\frac{\mathscr{H}_i(\Phi)}{\mathscr{H}_i(\psi)}=\frac{\Phi''(\xi_i)}{\psi''(\xi_i)},\quad i=1,\dots,5, \tag{3.7}$$

provided that the denominators in (3.7) *are nonzero.*

Proof. We give a proof for the functional $\mathscr{H}_1$. Consider the function χ defined by

$$\chi(t)=\mathscr{H}_1(\psi)\Phi(t)-\mathscr{H}_1(\Phi)\psi(t).$$

As the function χ is linear combination of functions Φ and ψ, so $\chi\in C^2(I)$. Now by applying Theorem 3.1 on χ, there exists some $\xi_1\in[m,M]$, such that

$$\mathscr{H}_1(\chi)=\frac{\chi''(\xi_1)}{2}\mathscr{H}_1(\Phi_0).$$

But $\mathscr{H}_1(\chi)=0$ and $\mathscr{H}_1(\Phi_0)\neq 0$ (otherwise we have a contradiction with $\mathscr{H}_1(\psi)\neq 0$, by Theorem 3.1), therefore

$$\chi''(\xi_1)=0.$$

From here the result follows. □

REMARK 3.3. If the inverse of the function $\dfrac{\Phi''}{\psi''}$ exists, then (3.7) gives

$$\xi_i=\left(\frac{\Phi''}{\psi''}\right)^{-1}\left(\frac{\mathscr{H}_i(\Phi)}{\mathscr{H}_i(\psi)}\right),\quad i=1,\dots,5.$$

Now we study log-convexity, n-exponential convexity and exponential-convexity of the functionals $\mathscr{H}_i$, $i=1,\dots,5$, using the idea from [6].

THEOREM 3.4. *Let $\mathscr{H}_i$, $i=1,\dots,5$, be defined as in* (3.1)$,\dots,$(3.5)*. Let J be an interval in $\mathbb{R}$ and let $\Omega=\{\Phi_t:\ t\in J\}$ be a family of functions defined on an open interval I such that $[m,M]\subset I$. If the function $t\mapsto[x_0,x_1,x_2;\Phi_t]$ is n-exponentially convex in the Jensen sense on J for every choice of mutually different numbers $x_0,x_1,x_2\in I$ then*

(*i*) *$t\mapsto\mathscr{H}_i(\Phi_t)$, $i=1,\dots,5$, is an n-exponentially convex function in the Jensen sense on J.*

(*ii*) *if $t\mapsto\mathscr{H}_i(\Phi_t)$, $i=1,\dots,5$, is continuous on J, then it is n-exponentially convex on J.*

Proof.

(i) Let the function $\nu : I \to \mathbb{R}$ be defined by

$$\nu(x) = \sum_{j,k=1}^{n} \xi_j \xi_k \Phi_{r_{jk}}(x)$$

where $\xi_j \in \mathbb{R}$, $r_{jk} = \frac{r_j + r_k}{2}$, $r_j \in J$, $1 \leqslant j,k \leqslant n$ and $\Phi_{r_{jk}} \in \Omega$. Using the assumption that $t \mapsto [x_0, x_1, x_2; \Phi_t]$ is n-exponentially convex in the Jensen sense on J, we obtain

$$[x_0, x_1, x_2; \nu] = \sum_{j,k=1}^{n} \xi_j \xi_k [x_0, x_1, x_2; \Phi_{r_{jk}}] \geqslant 0.$$

Therefore ν is a convex (and continuous) function. Hence $\mathscr{H}_i(\nu) \geqslant 0$, $i = 1, \ldots, 5$, which implies that

$$\sum_{j,k=1}^{n} \xi_j \xi_k \mathscr{H}_i(\Phi_{r_{jk}}) \geqslant 0, \quad i = 1, \ldots, 5.$$

We conclude that the function $t \mapsto \mathscr{H}_i(\Phi_t)$, $i = 1, \ldots, 5$, is n-exponentially convex on J in the Jensen sense.

(ii) If $t \mapsto \mathscr{H}_i(\Phi_t)$, $i = 1, \ldots, 5$, is continuous on J, then $t \mapsto \mathscr{H}_i(\Phi_t)$, $i = 1, \ldots, 5$, is n-exponentially convex by definition. $\square$

The following corollary is an immediate consequence of the above theorem.

COROLLARY 3.5. *Let $\mathscr{H}_i$, $i = 1, \ldots, 5$, be defined as in* (3.1), ..., (3.5). *Let J be an interval in $\mathbb{R}$ and let $\Omega = \{\Phi_t : t \in J\}$ be a family of functions defined on an open interval I such that $[m, M] \subset I$. If the function $t \mapsto [x_0, x_1, x_2; \Phi_t]$ is exponentially convex in the Jensen sense on J for every choice of mutually different numbers $x_0, x_1, x_2 \in I$ then*

(i) $t \mapsto \mathscr{H}_i(\Phi_t)$, $i = 1, \ldots, 5$, is an exponentially convex function in the Jensen sense on J.

(ii) if $t \mapsto \mathscr{H}_i(\Phi_t)$, $i = 1, \ldots, 5$, is continuous on J, then it is exponentially convex on J.

COROLLARY 3.6. *Let $\mathscr{H}_i$, $i = 1, \ldots, 5$, be defined as in* (3.1), ..., (3.5). *Let J be an interval in $\mathbb{R}$ and let $\Omega = \{\Phi_t : t \in J\}$ be a family of functions defined on an open interval I such that $[m, M] \subset I$. If the function $t \mapsto [x_0, x_1, x_2; \Phi_t]$ is 2-exponentially convex in the Jensen sense on J for every choice of mutually different numbers $x_0, x_1, x_2 \in I$ then* (*i*) $t \mapsto \mathscr{H}_i(\Phi_t)$, $i=1,\dots,5$, *is a* 2*-exponentially convex function in the Jensen sense on* J.

(*ii*) *if* $t \mapsto \mathscr{H}_i(\Phi_t)$, $i=1,\dots,5$, *is continuous on* J, *then it is also* 2*-exponentially convex on* J. *If* $t \mapsto \mathscr{H}_i(\Phi_t)$, $i=1,\dots,5$, *is additionally strictly positive then it is also log-convex on* J.

(*iii*) *if* $t \mapsto \mathscr{H}_i(\Phi_t)$, $i=1,\dots,5$, *is strictly positive differentiable function on* J, *then for any* $p \leqslant u$, $q \leqslant v$, $p,q,u,v \in J$, *we have*

$$\mathscr{M}_{p,q}(\mathscr{H}_i,\Omega) \leqslant \mathscr{M}_{u,v}(\mathscr{H}_i,\Omega), \quad i=1,\dots,5, \tag{3.8}$$

where

$$\mathscr{M}_{p,q}(\mathscr{H}_i,\Omega) = \begin{cases} \left(\dfrac{\mathscr{H}_i(\Phi_p)}{\mathscr{H}_i(\Phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\[2ex] \exp\left(\dfrac{\frac{d}{dp}\mathscr{H}_i(\Phi_p)}{\mathscr{H}_i(\Phi_p)}\right), & p = q. \end{cases} \tag{3.9}$$

Proof. (i) and (ii) are immediate consequences of Theorem 3.4. To prove (*iii*), let $t \mapsto \mathscr{H}_i(\Phi_t)$, $i=1,\dots,5$, be positive and differentiable and therefore continuous too. By (ii), the function $t \mapsto \mathscr{H}_i(\Phi_t)$, $i=1,\dots,5$, is log-convex and by Proposition 1.13, we obtain

$$\frac{\log \mathscr{H}_i(\Phi_p) - \log \mathscr{H}_i(\Phi_q)}{p-q} \leqslant \frac{\log \mathscr{H}_i(\Phi_u) - \log \mathscr{H}_i(\Phi_v)}{u-v}, \quad i=1,\dots,5,$$

for $p \leqslant u$, $q \leqslant v$, $p \neq q$, $u \neq v$, concluding

$$\mathscr{M}_{p,q}(\mathscr{H}_i,\Omega) \leqslant \mathscr{M}_{u,v}(\mathscr{H}_i,\Omega), \quad i=1,\dots,5.$$

If $p=q \leqslant v$ we apply the limit $q \to p$ to the above equation, concluding

$$\mathscr{M}_{p,p}(\mathscr{H}_i,\Omega) \leqslant \mathscr{M}_{u,v}(\mathscr{H}_i,\Omega), \quad i=1,\dots,5.$$

Other possible cases are treated similarly. □

REMARK 3.7. Note that by Definition 1.14 the results from Theorem 3.4, Corollary 3.5 and Corollary 3.6 still hold when two of the points $x_0,x_1,x_2 \in I$ or all three points coincide.

Now we present several families of convex functions which fulfil the conditions of Theorem 3.4 and Remark 3.7. In what follows, *id* denotes identity function.

EXAMPLE 3.8. Consider the family of functions

$$F_1 = \{\alpha_t :\ \mathbb{R} \to [0,\infty);\ t \in \mathbb{R}\}$$

defined by

$$\alpha_t(x) = \begin{cases} \dfrac{1}{t^2} e^{tx}, \ t \neq 0; \\ \\ \dfrac{1}{2} x^2, \quad t = 0. \end{cases}$$

We have $\alpha_t''(x) = e^{tx} > 0$ which shows that α_t is convex on $\mathbb{R}$ for every $t \in \mathbb{R}$ and $t \mapsto \alpha_t''(x)$ is exponentially convex by definition. By using analogous arguing as in the proof of Theorem 3.4 we also have that $t \mapsto [x_0, x_1, x_2; \alpha_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now using Corollary 3.5 we conclude that $t \mapsto \mathscr{H}_i(\alpha_t)$, $i = 1, \ldots, 5$, are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous, so they are exponentially convex.

For this family of functions, $\mathscr{M}_{p,q}(\mathscr{H}_i, \Omega)$, $i = 1, \ldots, 5$, from (3.9) becomes

$$\mathscr{M}_{p,q}(\mathscr{H}_i, F_1) = \begin{cases} \left(\dfrac{\mathscr{H}_i(\alpha_p)}{\mathscr{H}_i(\alpha_q)} \right)^{\frac{1}{p-q}}, & p \neq q; \\ \\ \exp\left(\dfrac{\mathscr{H}_i(id \cdot \alpha_p)}{\mathscr{H}_i(\alpha_p)} - \dfrac{2}{p} \right), & p = q \neq 0; \\ \\ \exp\left(\dfrac{\mathscr{H}_i(id \cdot \alpha_0)}{3\mathscr{H}_i(\alpha_0)} \right), & p = q = 0, \end{cases}$$

and by (3.8) it is monotonous function in parameters p and q. Using Theorem 3.2 it follows that for $i = 1, \ldots, 5$

$$\aleph_{p,q}(\mathscr{H}_i, F_1) = \log \mathscr{M}_{p,q}(\mathscr{H}_i, F_1)$$

satisfy $\aleph_{p,q}(\mathscr{H}_i, F_1) \in [m, M]$ which shows that $\aleph_{p,q}(\mathscr{H}_i, F_1)$ are means. Note that by (3.8) they are monotonous means.

EXAMPLE 3.9. Consider the family of functions

$$F_2 = \{\beta_t :\ (0,\infty) \to \mathbb{R};\ t \in \mathbb{R}\}$$

defined by

$$\beta_t(x) = \begin{cases} \dfrac{x^t}{t(t-1)}, & t \neq 0, 1; \\ \\ -\log x, & t = 0; \\ \\ x \log x, & t = 1. \end{cases}$$

Here $\beta_t''(x) = x^{t-2} = e^{(t-2)\ln x} > 0$, which shows that β_t is convex for $x > 0$ and $t \mapsto \beta_t''(x)$ is exponentially convex by definition. Arguing as in Example 3.8, we have that $t \mapsto \mathscr{H}_i(\beta_t)$, $i = 1,\dots,5$, are exponentially convex. In this case $\mathscr{M}_{p,q}(\mathscr{H}_i,\Omega)$, $i = 1,\dots,5$, from (3.9) becomes

$$\mathscr{M}_{p,q}(\mathscr{H}_i,F_2) = \begin{cases} \left(\dfrac{\mathscr{H}_i(\beta_p)}{\mathscr{H}_i(\beta_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\dfrac{1-2p}{p(p-1)} - \dfrac{\mathscr{H}_i(\beta_p\beta_0)}{\mathscr{H}_i(\beta_p)}\right), & p = q \neq 0,1; \\ \exp\left(1 - \dfrac{\mathscr{H}_i(\beta_0^2)}{2\mathscr{H}_i(\beta_0)}\right), & p = q = 0; \\ \exp\left(-1 - \dfrac{\mathscr{H}_i(\beta_0\beta_1)}{2\mathscr{H}_i(\beta_1)}\right), & p = q = 1. \end{cases}$$

As $\mathscr{H}_i$, $i = 1,\dots,5$, is positive for $\Phi = \beta_p \in F_2$ and $\Psi = \beta_q \in F_2$, by Theorem 3.2 there exists $\xi_i \in [m,M]$, $i = 1,\dots,5$, such that for $p \neq q$ we have

$$\xi_i = \left(\frac{\mathscr{H}_i(\beta_p)}{\mathscr{H}_i(\beta_q)}\right)^{\frac{1}{p-q}}, \quad i = 1,\dots,5.$$

Also $\mathscr{M}_{p,q}(\mathscr{H}_i,F_2)$, $i = 1,\dots,5$, is continuous, symmetric and monotonous (by (3.8)), which shows that $\mathscr{M}_{p,q}(\mathscr{H}_i,F_2)$, $i = 1,\dots,5$, is a mean.

EXAMPLE 3.10. Consider the family of functions

$$F_3 = \{\gamma_t : \ (0,\infty) \to (0,\infty) : \ t \in (0,\infty)\}$$

defined by

$$\gamma_t(x) = \begin{cases} \dfrac{t^{-x}}{(\ln^2 t)}, & t \neq 1; \\ \dfrac{x^2}{2}, & t = 1. \end{cases}$$

Here $t \mapsto \gamma_t''(x) = t^{-x} > 0$, which shows that γ_t is convex and exponential convexity of $t \mapsto \gamma_t''(x)$ is given by Example 2 in [6].

In this case $\mathscr{M}_{p,q}(\mathscr{H}_i,\Omega)$, $i = 1,\dots,5$, from (3.9) becomes

$$\mathscr{M}_{p,q}(\mathscr{H}_i,F_3) = \begin{cases} \left(\dfrac{\mathscr{H}_i(\gamma_p)}{\mathscr{H}_i(\gamma_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\dfrac{\mathscr{H}_i(id\cdot\gamma_p)}{p\mathscr{H}_i(\gamma_p)} - \dfrac{2}{p\ln p}\right), & p = q \neq 0,1; \\ \exp\left(\dfrac{-2\mathscr{H}_i(id\cdot\gamma_1)}{3\mathscr{H}_i(\gamma_1)}\right), & p = q = 1, \end{cases}$$

and by (3.8) it is monotonous function in parameters s and q. Using Theorem 3.2, it follows that for $i = 1,\dots,5$,

$$\aleph_{p,q}(\mathscr{H}_i, F_3) = -L(p,q)\log \mathscr{M}_{p,q}(\mathscr{H}_i, F_3)$$

satisfy $\aleph_{p,q}(\mathscr{H}_i, F_3) \in [m,M]$, which shows that $\aleph_{p,q}(\mathscr{H}_i, F_3)$ is a mean. Here $L(p,q)$ is the logarithmic mean defined by $L(p,q) = \dfrac{p-q}{\log p - \log q}$, $p \neq q$, $L(p,p) = p$.

EXAMPLE 3.11. Consider the family of functions

$$F_4 = \{\delta_t : \ (0,\infty) \to (0,\infty) : \ t \in (0,\infty)\}$$

defined by

$$\delta_t(x) = \frac{e^{-x\sqrt{t}}}{t}.$$

Here $t \mapsto \delta_t''(x) = e^{-x\sqrt{t}} > 0$, which shows that δ_t is convex and $t \mapsto \delta_t''(x)$ is exponentially convex by Example 3 in [6]. In this case $\mathscr{M}_{p,q}(\mathscr{H}_i, \Omega)$, $i = 1,\dots,5$, from (3.9) becomes

$$\mathscr{M}_{p,q}(\mathscr{H}_i, F_4) = \begin{cases} \left(\dfrac{\mathscr{H}_i(\delta_p)}{\mathscr{H}_i(\delta_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\[2ex] \exp\left(-\dfrac{\mathscr{H}_i(id \cdot \delta_p)}{2\sqrt{p}\,\mathscr{H}_i(\delta_p)} - \dfrac{1}{p}\right), & p = q, \end{cases}$$

and it is monotonous function in parameters p and q by (3.8). Using Theorem 3.2, it follows that for $i = 1,\dots,5$,

$$\aleph_{p,q}(\mathscr{H}_i, F_4) = -(\sqrt{s} + \sqrt{q})\log \mathscr{M}_{p,q}(\mathscr{H}_i, F_4)$$

satisfy $\aleph_{p,q}(\mathscr{H}_i, F_4) \in [m,M]$, which shows that $\aleph_{p,q}(\mathscr{H}_i, F_4)$ is a mean.

REFERENCES

[1] M. ANWAR, R. BIBI, M. BOHNER AND J. PEČARIĆ, *Integral inequalities on time scales via the theory of isotonic linear functionals*, Abstr. Appl. Anal., **2011**, (2011), Article ID 483595, 1–16.

[2] M. BOHNER AND G. SH. GUSEINOV, *Multiple integration on time scales*, Dynam. Systems Appl., **2006**, 14 (2005), no. 3–4, 579–606.

[3] M. BOHNER AND G. SH. GUSEINOV, *Multiple Lebesgue integration on time scales*, Adv. Difference Equ., **2006**, (2006), Article ID 26391, 1–12.

[4] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[5] M. BOHNER AND A. PETERSON, *Advances in Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Massachusetts, 2003.

[6] J. JAKŠETIĆ AND J. PEČARIĆ, *Exponential Convexity Method*, J. Convex Anal., **20**, 1 (2013), 181–197.

[7] M. KLARIČIĆ BAKULA, J. PEČARIĆ AND J. PERIĆ, *Extensions of the Hermite-Hadamard inequality with applications*, Math. Inequal. Appl., **15**, 4 (2012), 899–921.

[8] M. KLARIČIĆ BAKULA, J. PEČARIĆ AND J. PERIĆ, *On the converse Jensen Inequality*, Appl. Math. Comput., **218**, 11 (2012), 6566–6575.

[9] J. PEČARIĆ AND J. PERIĆ, *Improvements of the Giaccardi and the Petrović inequality and related stolarsky type means*, An. Univ. Craiova Ser. Mat. Inform., **39**, 1 (2012), 65–75.

[10] J. PEČARIĆ, J. PERIĆ, *Remarks on the paper "Jensen's inequality and new entropy bounds" of S. Simić*, J. Math. Inequal., **6**, 4 (2012), 631–636.

[11] J. E. PEČARIĆ, F. PROSCHAN, AND Y. L. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, volume 187 of Mathematics in Science and Engineering, Academic Press Inc., Boston, MA, 1992.

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