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ON THE STABILITY OF METRIC SEMIGROUP HOMOMORPHISMS

HAMID REZAEI AND SOON-MO JUNG

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Abstract. In this paper, we investigate the stability of the homomorphism equation $f(x\circ_1 y) = f(x)\circ_2 f(y)$ between semigroups $(G_1,\circ_1)$ and $(G_2,\circ_2)$, where the binary operation $\circ_i$ is square-symmetric on the set G_i for $i = 1,2$. Our results generalize the classical theorem of Hyers concerning the stability of the Cauchy additive equation.

1. Introduction

The first stability problem concerning group homomorphisms was raised by Ulam [14] in 1940. *Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy),h(x)h(y)) < \delta$ for all $x,y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x),H(x)) < \varepsilon$ for all $x \in G_1$?*

If the answer is affirmative, we say that the functional equation for homomorphisms is stable.

D. H. Hyers [6] answered the question of Ulam for the case where G_1 and G_2 are Banach spaces. This result of Hyers is stated as follows:

THEOREM 1. (Hyers) *Let $f : X_1 \to X_2$ be a function between Banach spaces such that $\|f(x+y) - f(x) - f(y)\| \leqslant \delta$ for some $\delta > 0$ and for all $x,y \in X_1$. Then the limit $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for each $x \in X_1$ and $A : X_1 \to X_2$ is the unique additive function such that $\|f(x)-A(x)\| \leqslant \delta$ for every $x \in X_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X_1$, then the function A is linear.*

Hyers' result was generalized to the stability involving a sum of powers of norms by T. Aoki [1]. In 1978, Th. M. Rassias [13] addressed the Hyers's stability theorem and attempted to weaken the condition for the bound of the norm of Cauchy difference $f(x+y) - f(x) - f(y)$ and proved a considerably generalized result of Hyers by making use of a direct method:

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THEOREM 2. (Rassias) *Let E_1 and E_2 be Banach spaces and let $f : E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in t for each fixed x. Assume that there exist $\theta \geqslant 0$ and $p \in [0,1)$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \theta\big(\|x\|^p + \|y\|^p\big) \quad (x,y \in E_1).$$

Then there exists a unique linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leqslant \frac{2\theta\|x\|^p}{2-2^p} \quad (x \in E_1).$$

For the last thirty years, many results concerning the Hyers-Ulam-Rassias stability of various functional equations have been obtained, and a number of definitions of stability have been introduced (see [7, 4, 8]).

For a given nonempty set G, the operation $\circ : G \times G \to G$ is called square-symmetric provided it satisfies the following identity

$$(x \circ y) \circ (x \circ y) = (x \circ x) \circ (y \circ y) \quad (x,y \in G).$$

A semigroup is said to be square-symmetric provided the corresponding operation is square-symmetric. It is easy to see that any abelian semigroup is square-symmetric and a group is square-symmetric if and only if it is abelian. Moreover, a unital ring $(R,+,\circ)$ is commutative if and only if $\circ$ is square-symmetric. A generalization of the Hyers-Ulam-Rassias stability, using square-symmetric operations, has been obtained by Pales et al. [11]. Indeed, they investigated the stability of the following family of functional equations

$$f(x \circ y) = H(f(x), f(y)) \quad (x,y \in S),$$

where S is a nonempty set, $\circ : S \times S \to S$ is a square-symmetric binary operation, and $H : G \times G \to G$ is a G-homogeneous function of two variables, i.e., H satisfies $H(uv,uw) = uH(v,w)$ for $u,v,w \in G$ and G is a semigroup of the real or complex field.

In this paper, by making use of the direct method and fixed point method, we will prove the Hyers-Ulam stability of the following functional equation

$$f(x \circ_1 y) = f(x) \circ_2 f(y) \tag{1}$$

between arbitrary square-symmetric semigroups. As consequences of this result, we obtain the stability of several functional equations, which have been extensively investigated by a number of authors.

2. Hyers-Ulam stability of (1): direct method

In this section, we will prove the Hyers-Ulam stability of the functional equation (1). The following notation is useful for stating our main theorem: Let G be a nonempty set which is closed under the square-symmetric operation $\circ$. For any function $f : G \times G \to [0,\infty)$, we define

$$U(f,\circ) = \inf\big\{\theta \geqslant 0 : f(x \circ x, y \circ y) \leqslant \theta f(x,y) \text{ for all } x,y \in G\big\}$$

and

$$L(f,\circ) = \sup\left\{\theta \geqslant 0 : f(x\circ x, y\circ y) \geqslant \theta f(x,y) \text{ for all } x,y \in G\right\}.$$

Note that if $f(x,y)=0$ implies $f(x\circ x,y\circ y)=0$ for every $x,y\in G$ (for example, if f is a metric on G), then we have

$$U(f,\circ) = \sup\left\{\frac{f(x\circ x,y\circ y)}{f(x,y)} : x,y\in G \text{ and } f(x,y)\neq 0\right\}$$

and

$$L(f,\circ) = \inf\left\{\frac{f(x\circ x,y\circ y)}{f(x,y)} : x,y\in G \text{ and } f(x,y)\neq 0\right\}.$$

THEOREM 3. *Assume that G_i is a nonempty set which is closed under the square-symmetric operation $\circ_i$ for $i=1,2$. Let (G_2,d) be a complete metric space such that the operation $\circ_2$ is continuous and the mapping $x\mapsto x\circ_2 x$ is surjective on G_2. Moreover, assume that $\beta : G_1\times G_1 \to [0,\infty)$ is a function such that*

$$-\infty < U(\beta) := U(\beta,\circ_1) < L(d,\circ_2) =: L(d) < \infty.$$

If a mapping $f : G_1\to G_2$ is almost homomorphism, i.e., if f satisfies

$$d(f(x\circ_1 y), f(x)\circ_2 f(y)) \leqslant \beta(x,y) \quad (x,y\in G_1), \tag{2}$$

then there exists a unique homomorphism $A : G_1\to G_2$ such that

$$d(f(x),A(x)) \leqslant \frac{\beta(x,x)}{L(d)-U(\beta)} \quad (x\in G_1). \tag{3}$$

Proof. Let $\alpha_1 := U(\beta)$ and $\alpha_2 := L(d)$. Then $\alpha_1<\alpha_2$ and by the definitions of $U(\beta)$ and $L(d)$, we have

$$\beta(x\circ_1 x, y\circ_1 y) \leqslant \alpha_1\beta(x,y) \quad (x,y\in G_1)$$

and

$$d(x\circ_2 x, y\circ_2 y) \geqslant \alpha_2 d(x,y) \quad (x,y\in G_2).$$

In other words,

$$\beta(h_1(x),h_1(y)) \leqslant \alpha_1\beta(x,y) \quad (x,y\in G_1) \tag{4}$$

and

$$d(h_2(x),h_2(y)) \geqslant \alpha_2 d(x,y) \quad (x,y\in G_2), \tag{5}$$

where $h_i(x) = x\circ_i x$ for $x\in G_i$ and $i=1,2$. Putting $y=x$ in (2) yields

$$d(f(h_1(x)),h_2(f(x))) \leqslant \beta(x,x) \quad (x\in G_1). \tag{6}$$

Since the operations are square-symmetric, the function h_i is a homomorphism for $i=1,2$, i.e.,

$$h_i(x\circ_i y) = h_i(x)\circ_i h_i(y) \quad (x,y\in G_i). \tag{7}$$

In particular, $h_2 : G_2 \to G_2$ is an isomorphism. In fact, by the assumption, the mapping $x \mapsto x \circ_2 x$ is surjective and by (5), h_2 is one-to-one. Now, it follows from (4) and (5) that

$$\beta(h_1^n(x), h_1^n(y)) \leqslant \alpha_1^n \beta(x,y) \quad (x,y \in G_1) \tag{8}$$

and

$$d(h_2^n(x), h_2^n(y)) \geqslant \alpha_2^n d(x,y) \quad (x,y \in G_2). \tag{9}$$

Further, if x and y are replaced by $h_2^{-n}(x)$ and $h_2^{-n}(y)$ in the last relation, respectively, then we get

$$d(h_2^{-n}(x), h_2^{-n}(y)) \leqslant \alpha_2^{-n} d(x,y) \quad (x,y \in G_2). \tag{10}$$

Here h_i^n denotes the n-fold iteration of h_i, i.e.,

$$h_i^n = \underbrace{h_i \circ \cdots \circ h_i}_{n\text{-times}}$$

and h_2^{-n} denotes $(h_2^{-1})^n$.

Let $q_n(x) := h_2^{-n} f(h_1^n(x))$ for every $n \geqslant 1$ and $x \in G_1$. Make use of (6), (8), and (10) to see

$$\begin{aligned} d\big(q_{n+1}(x), q_n(x)\big) &= d\big(h_2^{-n-1} f(h_1^{n+1}x), h_2^{-n} f(h_1^n x)\big) \\ &= d\big(h_2^{-n-1}\big(f h_1(h_1^n x)\big), h_2^{-n-1}\big(h_2 f(h_1^n x)\big)\big) \\ &\leqslant \alpha_2^{-n-1} d\big(f h_1(h_1^n x), h_2 f(h_1^n x)\big) \\ &\leqslant \alpha_2^{-n-1} \beta(h_1^n(x), h_1^n(x)) \\ &\leqslant \alpha_2^{-1} \big(\alpha_1 \alpha_2^{-1}\big)^n \beta(x,x). \end{aligned}$$

Hence, we conclude that

$$d(q_{n+1}(x), q_n(x)) \leqslant \alpha_2^{-1} \big(\alpha_1 \alpha_2^{-1}\big)^n \beta(x,x) \quad (x \in G_1,\ n \geqslant 1) \tag{11}$$

and $\alpha_1 \alpha_2^{-1} < 1$, which implies that the sequence $\{q_n(x)\}$ is a Cauchy sequence. Since (G_2, d) is complete, there exists a limit function $A(x) := \lim_{n\to\infty} q_n(x)$.

We now apply induction on n to prove that

$$d(q_n(x), f(x)) \leqslant \sum_{i=0}^{n-1} \alpha_2^{-1} \big(\alpha_1 \alpha_2^{-1}\big)^i \beta(x,x) \quad (x \in G_1,\ n \geqslant 1). \tag{12}$$

Fix $x \in G_1$. It follows from (6) and (10) that

$$\begin{aligned} d(q_1(x), f(x)) &= d\big(h_2^{-1} f(h_1(x)), f(x)\big) \\ &\leqslant d\big(h_2^{-1} f(h_1(x)), h_2^{-1}(h_2(f(x)))\big) \\ &\leqslant \alpha_2^{-1} d(f h_1(x), h_2 f(x)) \\ &\leqslant \alpha_2^{-1} \beta(x,x). \end{aligned}$$

Now, suppose (12) holds for some integer $n \geqslant 1$. Then by using (11) and (12), we have

$$\begin{aligned} d(q_{n+1}(x), f(x)) &\leqslant d(q_{n+1}(x), q_n(x)) + d(q_n(x), f(x)) \\ &\leqslant \alpha_2^{-1}\left(\alpha_1\alpha_2^{-1}\right)^n \beta(x,x) + \sum_{i=0}^{n-1} \alpha_2^{-1}\left(\alpha_1\alpha_2^{-1}\right)^i \beta(x,x) \\ &= \sum_{i=0}^{n} \alpha_2^{-1}\left(\alpha_1\alpha_2^{-1}\right)^i \beta(x,x). \end{aligned}$$

Letting $n \to \infty$ in (12), we get

$$d(A(x), f(x)) \leqslant \frac{\beta(x,x)}{\alpha_2 - \alpha_1} = \frac{\beta(x,x)}{L(d) - U(\beta)} \quad (x \in G_1).$$

We now prove that $A : G_1 \to G_2$ is a homomorphism. By the definition of q_n and the fact that h_2 is a homomorphism, together with (2), (7), (8), and (10), we conclude that

$$\begin{aligned} d(q_n(x \circ_1 y), q_n(x) \circ_2 q_n(y)) &= d\left(h_2^{-n} f(h_1^n(x \circ_1 y)), q_n(x) \circ_2 q_n(y)\right) \\ &= d\left(h_2^{-n} f(h_1^n(x \circ_1 y)), h_2^{-n}(h_2^n(q_n(x) \circ_2 q_n(y)))\right) \\ &\leqslant \alpha_2^{-n} d\left(f(h_1^n(x \circ_1 y)), h_2^n(q_n(x) \circ_2 q_n(y))\right) \\ &= \alpha_2^{-n} d\left(f(h_1^n(x \circ_1 y)), h_2^n(q_n(x)) \circ_2 h_2^n(q_n(y))\right) \\ &= \alpha_2^{-n} d(f(h_1^n x \circ_1 h_1^n y), f(h_1^n x) \circ_2 f(h_1^n y)) \\ &\leqslant \alpha_2^{-n} \beta(h_1^n(x), h_1^n(y)) \\ &\leqslant \alpha_2^{-n} \alpha_1^n \beta(x,y) \end{aligned}$$

for every $x, y \in G_1$. Therefore, we have

$$d(q_n(x \circ_1 y), q_n(x) \circ_2 q_n(y)) \leqslant \left(\alpha_1\alpha_2^{-1}\right)^n \beta(x,x) \quad (x, y \in G_1,\ n \geqslant 1).$$

Applying the continuity of operation $\circ_2$, considering $0 < \alpha_1\alpha_2^{-1} < 1$, and letting $n \to \infty$ in the last inequality, we conclude that A is indeed a homomorphism.

It remains to prove that $A : G_1 \to G_2$ is the unique homomorphism satisfying (3). Assume that there exists another homomorphism $A' : G_1 \to G_2$ satisfying (3). Since h_i is a homomorphism for $i = 1, 2$, we see that $Ah_1x = h_2Ax$ and $A'h_1x = h_2A'x$ and more generally

$$Ah_1^n x = h_2^n Ax \text{ and } A'h_1^n x = h_2^n A'x \quad (x \in G_1,\ n \geqslant 1).$$

Hence, we have

$$A(x) = h_2^{-n} A(h_1^n x) \text{ and } A'(x) = h_2^{-n} A'(h_1^n x) \quad (x \in G_1,\ n \geqslant 1).$$

By the triangle inequality, (3), (8), and (10), we obtain

$$\begin{aligned} d(A(x),A'(x)) &= d\big(h_2^{-n}A(h_1^n x),h_2^{-n}A'(h_1^n x)\big) \\ &\leqslant \alpha_2^{-n} d(A(h_1^n x),A'(h_1^n x)) \\ &\leqslant \alpha_2^{-n}\big(d(A(h_1^n x),f(h_1^n x))+d(f(h_1^n x),A'(h_1^n x))\big) \\ &\leqslant 2\alpha_2^{-n}\frac{\beta(h_1^n x,h_1^n x)}{\alpha_2-\alpha_1} \\ &\leqslant 2\big(\alpha_1\alpha_2^{-1}\big)^n\frac{\beta(x,x)}{\alpha_2-\alpha_1} \end{aligned}$$

for all $x \in G_1$ and $n \geqslant 1$. Since $0 < \alpha_1\alpha_2^{-1} < 1$, letting $n \to \infty$, we get $A(x) = A'(x)$ for all $x \in G_1$, which ends the proof. □

Recall that if T is a bounded linear operator on a Banach space X, then

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in X, x \neq 0\right\}.$$

Moreover, if T is invertible, then

$$\|T^{-1}\| = \inf\left\{\frac{\|Tx\|}{\|x\|} : x \in X, x \neq 0\right\}.$$

COROLLARY 1. *Let $(G,\cdot)$ be an abelian semigroup, X a Banach space, $x_0 \in X$, and let T_1 and T_2 be bounded linear operators on X such that $T_1T_2 = T_2T_1$ and $T_1 + T_2$ is invertible. Assume that $\beta : G \times G \to [0,\infty)$ is a function such that $U(\beta,\cdot) < \|(T_1 + T_2)^{-1}\|$. If $f : G \to X$ is a function satisfying*

$$\|f(xy) - T_1(f(x)) - T_2(f(y)) - x_0\| \leqslant \beta(x,y) \quad (x,y \in G),$$

then there exists a unique mapping $A : G \to X$ such that

$$A(xy) = T_1A(x) + T_2A(y) + x_0 \quad (x \in G)$$

and

$$\|f(x) - A(x)\| \leqslant \frac{\beta(x,x)}{\|(T_1 + T_2)^{-1}\| - U(\beta)} \quad (x \in G).$$

Proof. Let us define the binary operations $\circ_1$ and $\circ_2$ on G and X by

$$x \circ_1 y = xy \quad (x,y \in G)$$

and

$$x \circ_2 y = T_1x + T_2y + x_0 \quad (x,y \in X).$$

We claim that $\circ_2$ is square-symmetric: To see this, we note that

$$\begin{aligned}(x\circ_2 x)\circ_2 (y\circ_2 y) &= [(T_1+T_2)(x)+x_0]\circ_2 [(T_1+T_2)(y)+x_0]\\ &= T_1[(T_1+T_2)(x)+x_0]+T_2[(T_1+T_2)(y)+x_0]+x_0\\ &= (T_1+T_2)(T_1x+T_2y+x_0)+x_0\\ &= (T_1+T_2)(x\circ_2 y)+x_0\\ &= (x\circ_2 y)\circ_2 (x\circ_2 y)\end{aligned}$$

for every $x,y\in X$. Hence, $\circ_2$ is square-symmetric. Since $(G,\cdot)$ is an abelian semigroup, $\circ_1$ is also square-symmetric.

Let $d(x,y)=\|x-y\|$. Then we have

$$\begin{aligned}L(d) &= L(d,\cdot)\\ &= \inf\left\{\frac{d(x\circ_2 x, y\circ_2 y)}{d(x,y)} : x,y\in X, x\neq y\right\}\\ &= \inf\left\{\frac{\|[(T_1+T_2)(x)+x_0]-[(T_1+T_2)(y)+x_0]\|}{\|x-y\|} : x,y\in X,\ x\neq y\right\}\\ &= \inf\left\{\frac{\|(T_1+T_2)(x-y)\|}{\|x-y\|} : x,y\in X,\ x\neq y\right\}\\ &= \inf\left\{\frac{\|(T_1+T_2)(x)\|}{\|x\|} : x\in X,\ x\neq 0\right\}\\ &= \|(T_1+T_2)^{-1}\|.\end{aligned}$$

Therefore, $U(\beta)<\|(T_1+T_2)^{-1}\|=L(d)$. Since T_1+T_2 is invertible and $x\circ_2 x=(T_1+T_2)x+x_0$, the mapping $x\mapsto x\circ_2 x$ is bounded and surjective. Finally, by Theorem 3, the assertion is true. □

If T_1 and T_2 are identity operators and $x_0=0$, then the Hyers-Ulam stability of Eq. (1) is an immediate consequence of Corollary 1. We remark that the additive functional equation $f(x+y)=f(x)+f(y)$ and the logarithmic functional equation $f(xy)=f(x)+f(y)$ are special cases of Eq. (1).

Let X be a normed space and let $H: X\times X\to X$ be a function such that there exists a nonnegative real number α with

$$H(x,x)=\alpha x \text{ and } H(\alpha x,\alpha y)=\alpha H(x,y)\quad (x,y\in X). \tag{13}$$

Then the binary operation $\circ: X\times X\to X$ defined by

$$x\circ y := H(x,y)\quad (x,y\in X)$$

is square-symmetric. To see this, we note that

$$\begin{aligned}(x\circ x)\circ(y\circ y) &= H((x\circ x),(y\circ y))\\ &= H(H(x,x),H(y,y))\\ &= H(\alpha x,\alpha y) = \alpha H(x,y)\\ &= H(H(x,y),H(x,y)) = H(x,y)\circ H(x,y)\\ &= (x\circ y)\circ(x\circ y).\end{aligned}$$

This observation leads to the following proposition.

PROPOSITION 1. *Let X_1 be a normed space, X_2 a Banach space, and let $H_i : X_i \times X_i \to X_i$ be functions satisfying (13) for all $x,y \in X_i$ and for nonnegative real numbers α_1,α_2. Assume that a function $f : X_1 \to X_2$ satisfies*

$$\|f(H_1(x,y)) - H_2(f(x),f(y))\| \leqslant \theta(\|x\|^p + \|y\|^p) \quad (x,y \in X_1),$$

where θ and p are nonnegative constants. If $\alpha_1^p < \alpha_2$, then there exists a unique mapping $A : X_1 \to X_2$ such that

$$\|f(x) - A(x)\| \leqslant \frac{2\theta\|x\|^p}{\alpha_2 - \alpha_1^p} \quad (x \in X_1)$$

and

$$A(H_1(x,y)) = H_2(A(x),A(y)) \quad (x,y \in X_1).$$

Proof. If we define $x\circ_i y := H_i(x,y)$, then the above observation implies that the binary operation $\circ_i$ is square-symmetric, $x\circ_2 x = H_2(x,x) = \alpha_2 x$ for every $x \in X_2$, and

$$\|x\circ_2 x - y\circ_2 y\| = \|\alpha_2 x - \alpha_2 y\| = \alpha_2\|x-y\|.$$

Hence the mapping $x \to x\circ_2 x$ is surjective, the operation $\circ_2$ is continuous and $L(d) = \alpha_2$. On the other hand, if we put $\beta(x,y) := \theta\left(\|x\|^p + \|y\|^p\right)$, then

$$\beta(x\circ_1 x, y\circ_1 y) = \beta(\alpha_1 x,\alpha_1 y) = \theta\alpha_1^p\left(\|x\|^p + \|y\|^p\right) = \alpha_1^p\beta(x,y).$$

Thus, we obtain $U(\beta) = \alpha_1^p$.

According to Theorem 3, there exists a unique homomorphism $A : X_1 \to X_2$ such that

$$\|f(x) - A(x)\| \leqslant \frac{2\theta\|x\|^p}{L(d) - U(\beta)} = \frac{2\theta\|x\|^p}{\alpha_2 - \alpha_1^p} \quad (x \in X_1).$$

For the verification of the last assertion, we have

$$A(H_1(x,y)) = A(x\circ_1 y) = A(x)\circ_2 A(y) = H_2(A(x),A(y)) \quad (x,y \in X_1),$$

which ends the proof. □

EXAMPLE 1. Let A be a C^*-algebra and $a \in A$ a self-adjoint element, i.e., $a = a^*$. Then a is said to be positive if it is of the form $a = bb^*$ for some $b \in A$. The set of positive elements of A is denoted by A^+. Note that A^+ is a closed convex cone (see [5]). It is well-known that for a positive element a and a positive integer n there exists a unique positive element $x \in A^+$ such that $a = x^n$. We denote such an x by $\sqrt[n]{a}$.

In view of Proposition 1, the functional equation

$$f\left(\sqrt{ax^2+by^2}\right) = \sqrt{af(x)^2+bf(y)^2}$$

where $f : A^+ \to A^+$, $a,b > 0$, and $\sqrt{a+b} > 1$, is stable in the sense of Hyers-Ulam. To see this, consider the function $H : A^+ \times A^+ \to A^+$ defined by

$$H(x,y) = \sqrt{ax^2+by^2}.$$

Then, we have $H(x,x) = (\sqrt{a+b})x$ and

$$H\left(\sqrt{a+b}x, \sqrt{a+b}y\right) = \sqrt{a+b}\sqrt{ax^2+by^2} = \sqrt{a+b}H(x,y).$$

By Proposition 1 for $\alpha_2 = \sqrt{a+b}$ and $p = 0$, we see that the above equation is stable in the sense of Hyers-Ulam.

3. Hyers-Ulam stability of (1): fixed point method

In this section, using the fixed point method, we will prove the Hyers-Ulam stability of the functional equation (1) for Banach spaces.

For a nonempty set X, we introduce the definition of the generalized metric on X. A function $d : X \times X \to [0,\infty]$ is called a generalized metric on X if and only if d satisfies

- $d(x,y) = 0$ if and only if $x = y$;
- $d(x,y) = d(y,x)$ for all $x,y \in X$;
- $d(x,z) \leqslant d(x,y) + d(y,z)$ for all $x,y,z \in X$.

We remark that the only difference between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We now introduce one of the fundamental results of the fixed point theory. For the proof, we refer to [9].

THEOREM 4. *Let $(\mathscr{X},d)$ be a generalized complete metric space. Assume that $\Lambda : \mathscr{X} \to \mathscr{X}$ is a strictly contractive operator with the Lipschitz constant $L < 1$, i.e.,*

$$d(\Lambda g, \Lambda h) \leqslant Ld(g,h) \quad (g,h \in \mathscr{X}).$$

If there exists a nonnegative integer n_0 such that $d(\Lambda^{n_0+1}f, \Lambda^{n_0}f) < \infty$ for some $f \in \mathscr{X}$, then the following statements are true:

(*i*) *The sequence* $\{\Lambda^n f\}$ *converges to a fixed point* A *of* Λ*;*

(*ii*) A *is the unique fixed point of* Λ *in* $\mathscr{X}^* = \{g \in \mathscr{X} : d(\Lambda^{n_0} f, g) < \infty\}$*;*

(*iii*) *If* $g \in \mathscr{X}^*$*, then*

$$d(g,A) \leqslant \frac{1}{1-L} d(\Lambda g, g).$$

In 2003, V. Radu proved the Hyers-Ulam-Rassias stability of the additive functional equation (1) by using the fixed point method (see [12, 2, 10]). In what follows, we give a fixed point version of the proof of Theorem 3.

Proof. Letting $y = x$ in (2), we get

$$d(fh_1(x), h_2 f(x)) \leqslant \beta(x,x) \quad (x \in G_1).$$

Consider the set $X := \{f : f : G_1 \to G_2 \text{ is a function}\}$ and define the generalized metric on X by

$$D(g,h) = \inf\{\mu \in (0,\infty) : d(g(x),h(x)) \leqslant \mu\beta(x,x) \text{ for all } x \in G_1\},$$

where we set $\inf \emptyset = \infty$ as usual. It is easy to show that (X,D) is a complete metric space (see [10]).

We now consider the mapping $\Lambda : X \to X$ defined by

$$(\Lambda g)(x) = h_2^{-1} g(h_1 x) \quad (x \in G_1).$$

For given $g,h \in X$, $d(g(x),h(x)) \leqslant D(g,h)\beta(x,x)$. Hence, it follows from (10) that

$$\begin{aligned} d((\Lambda g)(x), (\Lambda h)(x)) &= d\big(h_2^{-1} g(h_1 x), h_2^{-1} h(h_1 x)\big) \\ &\leqslant \alpha_2^{-1} d(g(h_1 x), h(h_1 x)) \\ &\leqslant \alpha_2^{-1} D(g,h)\beta(h_1 x, h_1 x) \\ &\leqslant \alpha_1 \alpha_2^{-1} D(g,h)\beta(x,x) \end{aligned}$$

for all $x \in G_1$. By the definition of D, we have

$$D(\Lambda g, \Lambda h) \leqslant \alpha_1 \alpha_2^{-1} D(g,h) \quad (g,h \in X), \tag{14}$$

which implies that Λ is strictly contractive.

By (6), we get

$$d(f(x),(\Lambda f)(x)) = d(f(x), h_2^{-1} f(h_1 x)) \leqslant \alpha_2^{-1} d(h_2 f(x), f(h_1 x)) \leqslant \alpha_2^{-1}\beta(x,x)$$

for all $x \in G_1$. Therefore, $D(\Lambda f, f) \leqslant \alpha_2^{-1} < \infty$. By Theorem 4, there exists a mapping $A : X \to X$ satisfying the following conditions:

(*i*) A is a fixed point of Λ, i.e., $\Lambda A = h_2^{-1} A h_1 = A$ and hence, $A(h_1(x)) = h_2(A(x))$ for all $x \in G_1$. Moreover, A is the unique fixed point of Λ in the set $X^* := \{g \in X : D(f,g) < \infty\}$, which implies that $d(f(x),A(x)) \leqslant D(f,A)\beta(x,x)$;

(ii) $D(\Lambda^n f, A) \to 0$ as $n \to \infty$, i.e., $A(x) = \lim_{n\to\infty} h_2^{-n} f(h_1^n x)$;

(iii) By Theorem 4 (iii) and (14), we conclude that

$$D(f,A) \leqslant \frac{1}{1-\alpha_1\alpha_2^{-1}} D(f,\Lambda f) < \frac{\alpha_2^{-1}}{1-\alpha_1\alpha_2^{-1}} = \frac{1}{\alpha_2-\alpha_1},$$

which implies that

$$d(f(x),A(x)) \leqslant D(f,A)\beta(x,x) \leqslant \frac{\beta(x,x)}{\alpha_2-\alpha_1} = \frac{\beta(x,x)}{L(d)-U(\beta)}.$$

Finally, we can prove that $A: G_1 \to G_2$ is a unique homomorphism as we did in the proof of Theorem 3. □

In 1993, J. Chmielinski and J. Tabor [3] investigated the stability of the Pexider equation

$$f(x+y) = g(x) + h(y).$$

It will be interesting to investigate the generalized Pexider equation

$$f(x \circ_1 y) = g(x) \circ_2 h(y),$$

where $f,g,h : (G_1,\circ_1) \to (G_2,\circ_2)$ are functions between square-symmetric semigroups $(G_1,\circ_1)$ and $(G_2,\circ_2)$.

REFERENCES

[1] T. AOKI, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[2] L. CADARIU AND V. RADU, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.

[3] J. CHMIELINSKI AND J. TABOR, *On approximate solutions of the Pexider equation*, Aequationes Math. **46** (1993), 143–163.

[4] S. CZERWIK, *Functional Equations and Inequalities in Several Variables*, World Scientific, Hackensacks, New Jersey, 2002.

[5] J. DIXMIER, *C^*-Algebras*, North-Holland Publ. Com., Amsterdam, New York and Oxford, 1977.

[6] D. H. HYERS, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.

[7] D. H. HYERS, G. ISAC AND TH. M. RASSIAS, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, 1998.

[8] S.-M. JUNG, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer Optimization and Its Applications Vol. **48**, Springer, New York, 2011.

[9] B. MARGOLIS AND J. B. DIAZ, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.

[10] D. MIHET AND V. RADU, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.

[11] Z. PALES, P. VOLKMANN AND R. D. LUCE, *Hyers-Ulam stability of functional equations with a square-symmetric operation*, Proc. Natl. Acad. Sci. **95** (1998), 12772–12775.

[12] V. RADU, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory. **4** (2003), 91–96.

[13] TH. M. RASSIAS, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[14] S. M. ULAM, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, No. **8**, Interscience, New York, NY, USA, 1960.

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Hamid Rezaei
Department of Mathematics
College of Sciences, Yasouj University
Yasouj-75914-74831, Iran
e-mail: rezaei@mail.yu.ac.ir

Soon-Mo Jung
Mathematics Section
College of Science and Technology
Hongik University
339–701 Sejong, Republic of Korea
e-mail: smjung@hongik.ac.kr