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SOME RESULTS FOR THE ZEROS OF A CLASS OF FIBONACCI–TYPE POLYNOMIALS

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Abstract. In this article, we obtain a rectangle that contains all the zeros of a class of Fibonacci-like polynomials. Then we obtain some relations and majorizations for the real and imaginary parts of the zeros of such polynomials.

1. Introduction

Let a and b be fixed complex numbers and $a \neq 0$. Consider a class of Fibonacci-type polynomials $G_n(z) = G_n(a,b;z)$ defined by the recurrence relation

$$G_n(z) = zG_{n-1}(z) + G_{n-2}(z), \quad G_0(z) = a, \quad G_1(z) = z + b. \tag{1}$$

The polynomials $G_n(1,0;z)$ are the usual Fibonacci polynomials $F_n(z)$ whose roots are known explicitly as

$$z_j = 2i\cos\frac{j\pi}{n+1}, \quad j = 1,\dots,n.$$

However, there are no general formulae for the zeros of the Fibonacci-type polynomials. In [1], the author has obtained the result

$$|z| \leqslant \max\{2, |a| + |b|\},$$

which generalizes the result in [2] for the case $a = b = 1$. It has been shown in [3] that if z is any zero of $G_n(z)$ then

$$|z| \leqslant 1 + \max\{|a|, |b|\}.$$

In this article, we establish upper and lower bounds for the real and imaginary parts of the zeros of $G_n(z)$. These bounds when combined, describe a rectangle that contains all the zeros of $G_n(z)$. Then we investigate some of the properties of the zeros of $G_n(z)$ and obtain some majorizations for the real and imaginary parts.

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2. Preliminary results

For a matrix $B \in C^{n\times n}$, an $r \times r$ matrix $\tilde{B}$ is called a principal matrix of B if it is obtained from B by deleting any $n-r$ rows and the same $n-r$ columns, let $\operatorname{Re} B = \frac{1}{2}(B+B^*)$ and $\operatorname{Im} B = \frac{1}{2i}(B-B^*)$ be the Cartesian parts of B, where B^* is the Hermitian adjoint of B, let $\det(B)$ stand for the determinant of B. If B is Hermitian, then the eigenvalues of B are arranged in such a way that $\lambda_1(B) \geqslant \lambda_2(B) \geqslant \ldots \geqslant \lambda_n(B)$.

For two sequences of real numbers arranged in decreasing order $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, we say that x is majorized by y if

$$\sum_{j=1}^{k} x_j \leqslant \sum_{j=1}^{k} y_j \quad \text{for} \quad k = 1, 2, \ldots, n-1$$

and

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j.$$

To obtain our bounds and majorizations, we need several results involving inequalities, majorization relations and the interlacing property for the eigenvalues of matrices. For the theory of the following results, the reader is referred to [4], [5] and [6].

LEMMA 1. *If λ is any eigenvalue of B, then λ belongs to the rectangle*

$$[\lambda_n(\operatorname{Re} B), \lambda_1(\operatorname{Re} B)] \times [\lambda_n(\operatorname{Im} B), \lambda_1(\operatorname{Im} B)].$$

LEMMA 2. *Let $A,\ B \in C^{n\times n}$ be Hermitian. Then*

$$\lambda_j(A) + \lambda_n(B) \leqslant \lambda_j(A+B) \leqslant \lambda_j(A) + \lambda_1(B).$$

In particular,

$$\lambda_1(A+B) \leqslant \lambda_1(A) + \lambda_1(B) \quad \textit{and} \quad \lambda_n(A+B) \geqslant \lambda_n(A) + \lambda_n(B).$$

LEMMA 3. *Let $B \in C^{n\times n}$ with eigenvalues arranged in such a way that* $\operatorname{Re}\lambda_1(B) \geqslant \operatorname{Re}\lambda_2(B) \geqslant \ldots \geqslant \operatorname{Re}\lambda_n(B)$. *Then*

$$\sum_{j=1}^{k} \operatorname{Re}\lambda_j(B) \leqslant \sum_{j=1}^{k} \lambda_j(\operatorname{Re} B) \quad \textit{for} \quad k = 1, 2, \ldots, n-1$$

and

$$\sum_{j=1}^{n} \operatorname{Re}\lambda_j(B) = \sum_{j=1}^{n} \lambda_j(\operatorname{Re} B).$$

LEMMA 4. *Let $A,\ B \in C^{n\times n}$ be Hermitian. Then*

$$\sum_{j=1}^{k} \lambda_j(A+B) \leqslant \sum_{j=1}^{k} \lambda_j(A) + \sum_{j=1}^{k} \lambda_j(B) \quad \textit{for} \quad k = 1,2,\dots,n-1$$

and

$$\sum_{j=1}^{n} \lambda_j(A+B) = \sum_{j=1}^{n} \lambda_j(A) + \sum_{j=1}^{n} \lambda_j(B).$$

LEMMA 5. (The Cauchy interlacing property) *If $B \in C^{n\times n}$ is Hermitian and $\tilde{B}$ is an $r\times r$ principal submatrix of B, then the eigenvalues of B interlace those of $\tilde{B}$, that is*

$$\lambda_i(B) \geqslant \lambda_i(\tilde{B}) \geqslant \lambda_{n-r+i}(B) \quad \textit{for} \quad i = 1,\dots,r.$$

We would like to note that the real parts in Lemma 3 can be replaced by the imaginary parts.

3. Main results

Apply the recurrence relation (1) to obtain the exact form of $G_n(z)$ for $n = 2,3$ and 4 as follows:

$$G_2(z) = z^2 + bz + a = \det\begin{bmatrix} z+b & a \\ -1 & z \end{bmatrix},$$

$$G_3(z) = z^3 + bz^2 + (a+1)z + b = \det\begin{bmatrix} z+b & a & 0 \\ -1 & z & 1 \\ 0 & -1 & z \end{bmatrix},$$

$$G_4(z) = z^4 + bz^3 + (a+2)z^2 + 2bz + a = \det\begin{bmatrix} z+b & a & 0 & 0 \\ -1 & z & 1 & 0 \\ 0 & -1 & z & 1 \\ 0 & 0 & -1 & z \end{bmatrix}.$$

Consider the $n\times n$ (companion) matrix

$$M_n = \begin{bmatrix} -b & -a & 0 & \dots & 0 \\ 1 & 0 & -1 & \ddots & \vdots \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}. \tag{2}$$

From the recurrence relation (1) and using induction, it can be easily shown that for

$n \geqslant 2$,

$$G_n(z) = \begin{vmatrix} z+b & a & 0 & \dots & 0 \\ -1 & z & 1 & \ddots & \vdots \\ 0 & -1 & z & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & -1 & z \end{vmatrix}.$$

Thus, $G_n(z)$ can be considered as the characteristic polynomial of M_n and therefore, the zeros of $G_n(z)$ are exactly the eigenvalues of M_n.

Let $R_n = \operatorname{Re}(M_n)$ then it follows from (2) that R_n has the partitioned form

$$R_n = \begin{bmatrix} -\operatorname{Re} b & x^* \\ x & 0 \end{bmatrix},$$

where $x^* = [\frac{1}{2}(1-a), 0, \dots, 0]$. It can be easily shown that the characteristic polynomial of R_n is

$$\det(R_n - \lambda I) = (-\operatorname{Re} b - \lambda)(-\lambda)^{n-1} - \frac{1}{4}|a-1|^2(-\lambda)^{n-2}.$$

Thus, the eigenvalues of R_n are

$$\lambda_1(R_n) = \frac{1}{2}\left(-\operatorname{Re} b + \sqrt{(\operatorname{Re} b)^2 + |a-1|^2}\right), \tag{3}$$

$$\lambda_n(R_n) = \frac{1}{2}\left(-\operatorname{Re} b - \sqrt{(\operatorname{Re} b)^2 + |a-1|^2}\right), \tag{4}$$

and

$$\lambda_j(R_n) = 0 \quad \text{for} \quad j = 2, \dots, n-1. \tag{5}$$

Let $J_n = \operatorname{Im}(M_n)$ then it can be easily shown that

$$J_n = S_n + T_n, \tag{6}$$

where S_n is the matrix in the partitioned form

$$S_n = \begin{bmatrix} -\operatorname{Im} b & y^* \\ y & 0 \end{bmatrix},$$

with $y^* = [\frac{1}{2}i(a-1), 0, \dots, 0]$, and T_n is the $n \times n$ tridiagonal matrix

$$T_n = \begin{bmatrix} 0 & i & 0 & \dots & 0 \\ -i & 0 & i & \ddots & \vdots \\ 0 & -i & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & i \\ 0 & \dots & 0 & -i & 0 \end{bmatrix}.$$

As for R_n, it is easy to verify that the eigenvalues of S_n are

$$\lambda_1(S_n) = \frac{1}{2}\left(-\mathrm{Im}\, b + \sqrt{(\mathrm{Im}\, b)^2 + |a-1|^2}\right), \tag{7}$$

$$\lambda_n(S_n) = \frac{1}{2}\left(-\mathrm{Im}\, b - \sqrt{(\mathrm{Im}\, b)^2 + |a-1|^2}\right) \tag{8}$$

and

$$\lambda_j(S_n) = 0 \quad \text{for} \quad j = 2, \ldots, n-1. \tag{9}$$

It is well-known that the eigenvalues of T_n are

$$\lambda_j(T_n) = 2\cos\frac{j\pi}{n+1} \quad \text{for} \quad j = 1, 2, \ldots, n. \tag{10}$$

From (6), (7), (8) and (10) and by applying Lemma 2, we have

$$\lambda_1(J_n) \leqslant \frac{1}{2}\left(-\mathrm{Im}\, b + \sqrt{(\mathrm{Im}\, b)^2 + |a-1|^2}\right) + 2\cos\frac{\pi}{n+1}$$

and

$$\lambda_n(J_n) \geqslant \frac{1}{2}\left(-\mathrm{Im}\, b - \sqrt{(\mathrm{Im}\, b)^2 + |a-1|^2}\right) - 2\cos\frac{\pi}{n+1}.$$

Now, by Lemma 1, we are in a position to give the following rectangle that contains all zeros of $G_n(z)$.

THEOREM 1. *If z is any zero of $G_n(z)$, then z belongs to the rectangle*

$$[\alpha_1, \alpha_2] \times [\beta_1, \beta_2],$$

where

$$\alpha_1 = \frac{1}{2}\left(-\mathrm{Re}\, b - \sqrt{(\mathrm{Re}\, b)^2 + |a-1|^2}\right),$$

$$\alpha_2 = \frac{1}{2}\left(-\mathrm{Re}\, b + \sqrt{(\mathrm{Re}\, b)^2 + |a-1|^2}\right),$$

$$\beta_1 = \frac{1}{2}\left(-\mathrm{Im}\, b - \sqrt{(\mathrm{Im}\, b)^2 + |a-1|^2}\right) - 2\cos\frac{\pi}{n+1}$$

and

$$\beta_2 = \frac{1}{2}\left(-\mathrm{Im}\, b + \sqrt{(\mathrm{Im}\, b)^2 + |a-1|^2}\right) + 2\cos\frac{\pi}{n+1}.$$

LEMMA 6. *The following inequalities hold.*

$$-2\cos\frac{\pi}{n} \leqslant \sum_{j=2}^{n-1} \lambda_j(J_n) \leqslant 2\cos\frac{\pi}{n} \tag{11}$$

and

$$-\operatorname{Im}b-2\cos\frac{\pi}{n}\leqslant\lambda_1(J_n)+\lambda_n(J_n)\leqslant-\operatorname{Im}b+2\cos\frac{\pi}{n}. \tag{12}$$

Proof. Since T_{n-1} is a principal submatrix of J_n, then it follows from Lemma 5 that

$$\lambda_1(J_n)\geqslant 2\cos\frac{\pi}{n}\geqslant\lambda_2(J_n)\geqslant 2\cos\frac{2\pi}{n}\geqslant\ldots\geqslant\lambda_{n-1}(J_n)\geqslant 2\cos\frac{(n-1)\pi}{n}\geqslant\lambda_n(J_n)$$

and since $\sum_{j=1}^{n-1}\cos\frac{j\pi}{n}=0$ and $\cos\frac{(n-1)\pi}{n}=-\cos\frac{\pi}{n}$, then (11) follows.

Now, (12) follows from (11) and the fact that $\sum_{j=1}^{n}\lambda_j(J_n)=\operatorname{tr}(J_n)=-\operatorname{Im}b$. □

Finally, in the following two results, we give majorizations for the real parts and the imaginary parts of the zeros of $G_n(z)$.

LEMMA 7. *If the zeros of $G_n(z)$ are arranged in such a way that* $\operatorname{Re}z_1\geqslant\operatorname{Re}z_2\geqslant\ldots\geqslant\operatorname{Re}z_n$, *then*

$$\sum_{j=1}^{k}\operatorname{Re}z_j\leqslant\frac{1}{2}\left(-\operatorname{Re}b+\sqrt{(\operatorname{Re}b)^2+|a-1|^2}\right)\quad\text{for}\quad k=1,2,\ldots,n-1 \tag{13}$$

and

$$\sum_{j=1}^{n}\operatorname{Re}z_j=-\operatorname{Re}b. \tag{14}$$

Proof. We obtain

$$\begin{aligned}\sum_{j=1}^{k}\operatorname{Re}z_j&=\sum_{j=1}^{k}\operatorname{Re}\lambda_j(M_n)\\&\leqslant\sum_{j=1}^{k}\lambda_j(R_n)\end{aligned}$$

for $k=1,2,\ldots,n-1$. Using (3), (4) and (5) yields the majorization (13). Now, (14) follows.

Now, (14) follows from the fact that $\sum_{j=1}^{n}z_j=\sum_{j=1}^{n}\lambda_j(M_n)=\operatorname{tr}(M_n)=-b$. □

THEOREM 2. *If the zeros of $G_n(z)$ are arranged in such a way that* $\operatorname{Im}z_1\geqslant\operatorname{Im}z_2\geqslant\ldots\geqslant\operatorname{Im}z_n$, *then*

$$\sum_{j=1}^{k}\operatorname{Im}z_j\leqslant\frac{1}{2}\left(-\operatorname{Im}b+\sqrt{(\operatorname{Im}b)^2+|a-1|^2}\right)+\left(\frac{\sin\left(\frac{2k+1}{2n}\right)\pi}{\sin\frac{\pi}{2n}}\right)-1 \tag{15}$$

for $k = 1, 2, \ldots, n-1$ *and*

$$\sum_{j=1}^{n} \operatorname{Im} z_j = -\operatorname{Im} b. \tag{16}$$

Proof. Replacing the real parts by the imaginary parts in Lemma 3, we obtain

$$\begin{aligned}\sum_{j=1}^{k} \operatorname{Im} z_j &= \sum_{j=1}^{k} \operatorname{Im} \lambda_j(M_n) \\ &\leqslant \sum_{j=1}^{k} \lambda_j(J_n)\end{aligned}$$

for $k = 1, 2, \ldots, n-1$. Applying Lemma 4 and using (6), (7), (8), (9) and (10), we obtain

$$\sum_{j=1}^{k} \operatorname{Im} z_j \leqslant \frac{1}{2}\left(-\operatorname{Im} b + \sqrt{(\operatorname{Im} b)^2 + |a-1|^2}\right) + 2\sum_{j=1}^{k} \cos\frac{j\pi}{n} \tag{17}$$

for $k = 1, 2, \ldots, n-1$. But, for every real number t which is not a multiple of 2π, we have

$$\sum_{j=1}^{k} \cos jt = \frac{1}{2}\left(\frac{\sin(2k+1)\frac{t}{2}}{\sin\frac{t}{2}}\right) - \frac{1}{2}.$$

Thus,

$$\sum_{j=1}^{k} \cos\frac{j\pi}{n} = \frac{1}{2}\left(\frac{\sin\left(\frac{2k+1}{2n}\right)\pi}{\sin\frac{\pi}{2n}}\right) - \frac{1}{2},$$

which, together with (17), gives the majorization (15). Now, (16) follows from the fact that $\sum_{j=1}^{n} z_j = \sum_{j=1}^{n} \lambda_j(M_n) = \operatorname{tr}(M_n) = -b$. $\square$

REMARK. We would like to refer the interested reader to [7] and [8], where the authors have investigated some fundamental properties of Fibonacci-type polynomials.

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