Journal of
Mathematical
Inequalities
Volume 10, Number 1 (2016), 159–166 doi:10.7153/jmi-10-13

MORE ON L^p–INTEGRABILITY

YI ZHAO AND SONGPING ZHOU

(*Communicated by J. Pečarić*)

Abstract. In this paper, we give a further generalization to L^p-integrability of trigonometric series connecting with derivatives of the sum-functions.

1. Introduction

The *mean value bounded variation* concept generalized monotonicity and is considered as the ultimate condition ([8]).

A nonnegative sequence $A = \{a_n\}$ is said to satisfy the *mean value bounded variation condition* if there is a $\lambda \geqslant 2$ and a positive constant M_0 depending upon the sequence A and λ only such that for all n we have

$$\sum_{k=n}^{2n} |\Delta a_k| := \sum_{k=n}^{2n} |a_k - a_{k+1}| \leqslant \frac{M_0}{n} \sum_{k=n/\lambda}^{\lambda n} a_k, \tag{1}$$

where $\sum\limits_{k=n/\lambda}^{\lambda n}$ means $\sum\limits_{n/\lambda \leqslant k \leqslant \lambda n}$, and we may assume that $M_0 > 1$ without loss of generality.

We denote the set of nonnegative sequences satisfying (1) as MVBVS (Mean Value Bounded Variation Sequences)

Let $1 \leqslant p < \infty$, denote $L_{2\pi}^p$ to be the space of p-power integrable functions $f(x)$ of period 2π equipped with the norm

$$\|f\|_{L^p} = \left(\int_{-\pi}^{\pi} |f(x)|^p \mathrm{d}x \right)^{1/p},$$

write in short $L_{2\pi} = L_{2\pi}^1$. Consider the trigonometric series

$$S(x) \equiv \sum_{n=1}^{\infty} a_n \sin nx \tag{2}$$

or

$$S(x) \equiv \sum_{n=1}^{\infty} a_n \cos nx, \tag{3}$$

and its sum function is written by $f(x)$ or $g(x)$ respectively.

In L^p-integrability, the work [6] proved the following theorem:

Mathematics subject classification (2010): 42A25, 42A50.
Keywords and phrases: Integrability, mean value bounded variation, monotonicity.

THEOREM 1.1. *Let $1<p<\infty$, $1/p-1<\gamma<1/p$. Suppose that a nonnegative sequence $\{a_n\}$ satisfies (1), consider the trigonometric series (2) or (3), and its sum function is denoted by $f(x)$. Moreover, $\{a_n\}$ is the Fourier coefficients of $f(x)\in L_{2\pi}$. Then, $x^{-\gamma}f(x)\in L^p_{2\pi}$ if and only if*

$$\sum_{n=1}^{\infty} n^{p+p\gamma-2}a_n^p<\infty.$$

The precedent references could be found in [1], [2], [5] and [3].

In this paper, we will generalize the above result (Theorem 1.1) by relaxing the restriction of γ while connecting with derivatives of the sum-function.

We remark that, in the proof of the main result in this paper, applications of several classical inequalities (Lemma 2.1–Lemma 2.3) are very useful.

We always assume that $1<p<\infty$, $1/p+1/q=1$. For $\gamma>1/p-1$, define

$$\kappa_\gamma=2k,\ \ 2k-1+1/p<\gamma<2k+1+1/p,\ \ k=0,1,2,\cdots.$$

Exactly, we prove that

THEOREM 1.2. *Suppose that a nonnegative sequence $\{a_n\}$ satisfies condition (1), and consider the trigonometric series (2), and its sum function is denoted by $f(x)$. Let $1<p<\infty$. If $2k-1+1/p<\gamma<2k+1+1/p$, $k=0,1,\cdots$, then $x^{-\gamma+\kappa_\gamma}f^{(\kappa_\gamma)}(x)\in L^p_{2\pi}$ and $\{n^{\kappa_\gamma}a_n\}$ is the Fourier coefficients of $f^{(\kappa_\gamma)}(x)$ if and only if*

$$\sum_{n=1}^{\infty} n^{p+p\gamma-2}a_n^p<\infty. \tag{4}$$

Throughout the paper, we always use M $(M(p))$ to stand for a positive constant (depending upon p only) that may not be necessarily the same at each occurrence. Sometimes, also use $O(1)$ to indicate the upper bound.

2. Proof

We first mention several inequalities.

LEMMA 2.1. *Let a nonnegative sequence $\{a_n\}$ satisfy condition (1), then*

$$a_n\leqslant\frac{2M_0}{n}\sum_{k=n/\lambda}^{\lambda n}a_k.$$

This is an already well-known inequality, one could refer to, for instance, [8] or [7].

LEMMA 2.2. *If $p \geqslant 1$ and $\alpha_n \geqslant 0$, then for any sequence $\{\mu_n\}$ of positive numbers, it holds that*

$$\sum_{n=1}^{\infty} \mu_n \left(\sum_{k=1}^{n} \alpha_k \right)^p \leqslant p^p \sum_{n=1}^{\infty} \mu_n^{1-p} \left(\sum_{k=n}^{\infty} \mu_k \right)^p \alpha_n^p, \tag{5}$$

and

$$\sum_{n=1}^{\infty} \mu_n \left(\sum_{k=n}^{\infty} \alpha_k \right)^p \leqslant p^p \sum_{n=1}^{\infty} \mu_n^{1-p} \left(\sum_{k=1}^{n} \mu_k \right)^p \alpha_n^p. \tag{6}$$

These inequalities were due to Leindler and proved in [4, Theorem 1].

LEMMA 2.3. *Let $p>1$, $s<p-1$, and $f(x)$ be a nonnegative function defined on $[0,\infty)$. Write $F(x)=\int_0^x f(t)\mathrm{d}t$. If $f^p(x)x^s$ is integrable on $[0,\infty)$, then $(x^{-1}F(x))^p x^s$ is integrable on $[0,\infty)$, and*

$$\int_0^{\infty} \left(\frac{F(x)}{x} \right)^p x^s \mathrm{d}x \leqslant \left(\frac{p}{p-s-1} \right)^p \int_0^{\infty} f^p(x) x^s \mathrm{d}x$$

holds.

It can be found in [9, page 20].

Then we establish some other inequalities or lemmas.

LEMMA 2.4. *Let a nonnegative sequence $\{a_n\}$ satisfy condition (1), then, for any natural number $\kappa \geqslant 1$, $\{n^\kappa a_n\}$ satisfies condition (1).*

Proof. Let $\{a_n\}$ satisfy condition (1), by writing $A_n = n^\kappa a_n$, we check that

$$|\Delta A_j| = \left| a_j \Delta j^\kappa + (j+1)^\kappa \Delta a_j \right| \leqslant M \left(a_j j^{\kappa-1} + j^\kappa |\Delta a_j| \right),$$

so that, by (1),

$$\sum_{j=n}^{2n} |\Delta A_j| \leqslant \frac{M}{n} \left(\sum_{j=n}^{2n} j^\kappa a_j + n^\kappa \sum_{j=n/\lambda}^{\lambda n} a_j \right) \leqslant \frac{M}{n} \sum_{j=n/\lambda}^{\lambda n} A_j. \quad \square$$

LEMMA 2.5. *Let $A_n = n^{\kappa\gamma} a_n$. If $\{a_n\}$ satisfies (1) and (4), then*

$$\lim_{n\to\infty} A_n = 0, \quad \sum_{n=1}^{\infty} |\Delta A_n| < \infty. \tag{7}$$

Proof. By condition (4), for the λ appearing in (1), we have

$$\lim_{n\to\infty} \sum_{k=n/\lambda}^{n\lambda} k^{p+p\gamma-2} a_k^p = 0. \tag{8}$$

Note for $2k-1+1/p<\gamma<2k+1+1/p$, $k=0,1,2,\cdots$ that $\kappa_\gamma=2k$, i.e.,

$$\kappa_\gamma-1+1/p<\gamma<\kappa_\gamma+1+1/p, \tag{9}$$

therefore,

$$-\varepsilon_0:=-1+1/p+\kappa_\gamma-\gamma<0. \tag{10}$$

Applying Lemma 2.1, we deduce that

$$A_n=n^{\kappa_\gamma}a_n\leqslant 2M_0n^{\kappa_\gamma-1}\sum_{k=n/\lambda}^{\lambda n}a_k\leqslant M\sum_{k=n/\lambda}^{n\lambda}k^{\kappa_\gamma-1}a_k=M\sum_{k=n/\lambda}^{n\lambda}k^{1+\gamma-2/p}a_kk^{-2+2/p+\kappa_\gamma-\gamma},$$

hence by Holder's inequlity and (10),

$$A_n\leqslant M\left(\sum_{k=n/\lambda}^{n\lambda}k^{p+p\gamma-2}a_k^p\right)^{1/p}\left(\sum_{k=n/\lambda}^{n\lambda}k^{-1-\varepsilon_0q}\right)^{1/q}\leqslant M\left(\sum_{k=n/\lambda}^{n\lambda}k^{p+p\gamma-2}a_k^p\right)^{1/p},$$

thus with (8), $\lim\limits_{n\to\infty}A_n=0$. Similarly, by Lemma 2.4,

$$\begin{aligned}\sum_{k=1}^{\infty}|\Delta A_k|&=\sum_{j=0}^{\infty}\sum_{k=2^j}^{2^{j+1}}|\Delta A_k|\leqslant\sum_{j=0}^{\infty}\frac{M}{2^j}\sum_{k=2^j/\lambda}^{2^j\lambda}A_k\leqslant M\sum_{j=0}^{\infty}\sum_{k=2^j/\lambda}^{2^j\lambda}k^{\kappa_\gamma-1}a_k\\&\leqslant M\sum_{k=1}^{\infty}k^{\kappa_\gamma-1}a_k\leqslant M\left(\sum_{k=1}^{\infty}k^{p+p\gamma-2}a_k^p\right)^{1/p}\left(\sum_{k=1}^{\infty}k^{-1-\varepsilon_0q}\right)^{1/q}<\infty,\end{aligned}$$

this proves the second inequality in (7). □

LEMMA 2.6. *Suppose that a nonnegative sequence $\{a_n\}$ satisfies condition (1), and consider the trigonometric series (2), its sum function is denoted by $f(x)$. Let $1<p<\infty$. If $2k-1+1/p<\gamma<2k+1+1/p$, $k=0,1,\cdots$, and condition (4) holds, then $x^{-\gamma+\kappa_\gamma}f^{(\kappa_\gamma)}(x)\in L^p_{2\pi}$ and $\{A_n\}$ is the Fourier coefficients of $f^{(\kappa_\gamma)}(x)$.*

Proof. Considering the series

$$\sum_{n=1}^{\infty}A_n\sin(nx+\kappa_\gamma\pi/2)=\sum_{n=1}^{\infty}(-1)^kA_n\sin nx,\ k=0,1,\cdots. \tag{11}$$

By Lemma 2.5 and the classical results (see, e.g., [9]), the series (11) converges to its sum function $h(x)$ in $(0,\pi]$. Assume that $x\in[\frac{\pi}{n+1},\frac{\pi}{n})$, by using the inequality $|\sin x|\leqslant|x|$ and Abel's transformation, we get

$$|h(x)|\leqslant\frac{\pi}{n}\sum_{j=1}^{n}jA_j+\frac{n+1}{\pi}\sum_{j=n}^{\infty}|\Delta A_j|.$$

Therefore,

$$\int_0^{\pi} x^{(\kappa_\gamma-\gamma)p}|h(x)|^p \mathrm{d}x \leqslant M\sum_{n=1}^{\infty} n^{(\gamma-\kappa_\gamma)p}\int_{\pi/(n+1)}^{\pi/n}|h(x)|^p \mathrm{d}x$$
$$\leqslant M\sum_{n=1}^{\infty} n^{(\gamma-\kappa_\gamma)p-p-2}\left(\sum_{j=1}^{n} jA_j\right)^p + M\sum_{n=1}^{\infty} n^{(\gamma-\kappa_\gamma)p+p-2}\left(\sum_{j=n}^{\infty}|\Delta A_j|\right)^p$$
$$=: I_1 + I_2.$$

By (5) and (4), we deduce that ($2+p-\gamma p+\kappa_\gamma p>1$ by (9))

$$I_1 \leqslant Mp^p\sum_{n=1}^{\infty} n^{\left((\gamma-\kappa_\gamma)p-p-2\right)(1-p)}\left(\sum_{j=n}^{\infty} j^{(\gamma-\kappa_\gamma)p-p-2}\right)^p (nA_n)^p$$
$$\leqslant M\left(\frac{p}{1+p-\gamma p+\kappa_\gamma p}\right)^p\sum_{n=1}^{\infty} n^{p+p\gamma-2}a_n^p<\infty.$$

At the same time, since $\{a_n\}$ satisfies (1), by Lemma 2.4, $\{A_n\}$ satisfies (1). Then, for any sufficiently large n, there is a $\lambda \geqslant 2$ such that

$$\sum_{j=n}^{\infty}|\Delta A_j| \leqslant \sum_{j=0}^{\infty}\sum_{l=2^jn}^{2^{j+1}n}|\Delta A_l| \leqslant M\sum_{j=0}^{\infty}\frac{1}{2^jn}\sum_{l=2^jn/\lambda}^{\lambda 2^jn} A_l \leqslant M\sum_{l=n/\lambda}^{\infty}\frac{A_l}{l}.$$

It follows that

$$I_2 \leqslant M\sum_{n=\lambda+1}^{\infty} n^{(\gamma-\kappa_\gamma)p+p-2}\left(\sum_{l=n/\lambda}^{\infty}\frac{A_l}{l}\right)^p \leqslant M(p)\sum_{n=1}^{\infty} n^{(\gamma-\kappa_\gamma)p+p-2}\left(\sum_{j=n}^{\infty}\frac{A_j}{j}\right)^p.$$

Now applying (6) and (4), we have ($\gamma p-\kappa_\gamma p+p-2>-1$ by (9))

$$I_2 \leqslant M(p)\sum_{n=1}^{\infty} n^{\left((\gamma-\kappa_\gamma)p+p-2\right)(1-p)}\left(\sum_{j=1}^{n} j^{(\gamma-\kappa_\gamma)p+p-2}\right)^p\left(\frac{A_n}{n}\right)^p$$
$$\leqslant M(p)\sum_{n=1}^{\infty} n^{\gamma p+p-2}a_n^p<\infty.$$

The estimates of I_1 and I_2 already show that $x^{-\gamma+\kappa_\gamma}h(x)\in L^p_{2\pi}$. By Hölder's inequality, it is easy to verify that $h(x)\in L_{2\pi}$. From condition (7) and that the series (11) converges to its sum function $h(x)$ in $(0,\pi]$, we know that $\{A_n\}$ is the Fourier coefficients of $h(x)$. Also it is obvious that $h(x)=f^{(\kappa_\gamma)}(x)$ almost everywhere by termwise integration. Lemma 2.6 is proved. □

Proof of Theorem 1.2. The sufficiency can be derived from Lemma 2.6. Write $h(x)=f^{(\kappa_\gamma)}(x)$, since $x^{-\gamma+\kappa_\gamma}f^{(\kappa_\gamma)}(x)\in L^p_{2\pi}$, we have $h(x)\in L_{2\pi}$. Let $A_n=n^{\kappa_\gamma}a_n$, then $\{A_n\}$ is the Fourier coefficients of $h(x)$, hence

$$H(x):=\int_0^x h(t)\mathrm{d}t=\sum_{j=1}^{\infty}\frac{A_j}{j}(1-\cos jx)=2\sum_{j=1}^{\infty}\frac{A_j}{j}\sin^2\frac{jx}{2},$$

so that

$$H\left(\frac{\pi}{2n}\right)=2\sum_{j=1}^{\infty}\frac{A_j}{j}\sin^2\frac{j\pi}{4n}\geqslant M\sum_{j=n}^{2n}\frac{A_j}{j}. \tag{12}$$

By Lemma 2.1, we have

$$\begin{aligned}J:=\sum_{n=2\lambda}^{\infty}n^{p+p\gamma-2}a_n^p&\leqslant\sum_{n=2\lambda}^{\infty}n^{p+p\gamma-2}\left(\sum_{l=n/\lambda}^{n\lambda}\frac{a_l}{l}\right)^p\\ &\leqslant M(p)\sum_{n=2}^{\infty}n^{p+p\gamma-2}\left(\sum_{l=n}^{\infty}\frac{a_l}{l}\right)^p\leqslant M(p)\sum_{j=1}^{\infty}\sum_{n=2^j}^{2^{j+1}}n^{p+p\gamma-2}\left(\sum_{l=j}^{\infty}\sum_{k=2^l}^{2^{l+1}}\frac{a_k}{k}\right)^p\\ &\leqslant M(p)\sum_{j=1}^{\infty}\sum_{n=2^j}^{2^{j+1}}n^{p+p\gamma-2}\left(\sum_{l=j}^{\infty}\sum_{k=2^l}^{2^{l+1}}\frac{a_k}{k}\right)^p\leqslant M(p)\sum_{j=1}^{\infty}2^{j(p+p\gamma-1)}\left(\sum_{l=j}^{\infty}\sum_{k=2^l}^{2^{l+1}}\frac{a_k}{k}\right)^p.\end{aligned}$$

Applying (6) and (12), we get

$$\begin{aligned}J&\leqslant M(p)\sum_{j=1}^{\infty}2^{j(p+p\gamma-1)(1-p)}\left(\sum_{k=1}^{j}2^{k(p+p\gamma-1)}\right)^p\left(\sum_{l=2^j}^{2^{j+1}}\frac{a_l}{l}\right)^p\\ &\leqslant M(p)\sum_{j=1}^{\infty}2^{j(p+p\gamma-\kappa_\gamma-1)}\left(\sum_{l=2^j}^{2^{j+1}}\frac{A_l}{l}\right)^p\\ &\leqslant M(p)\sum_{j=1}^{\infty}2^{j(p+p\gamma-p\kappa_\gamma-1)}H^p\left(\frac{\pi}{2^{j+1}}\right).\end{aligned} \tag{13}$$

Let

$$\Phi(x)=\int_0^x|h(t)|\mathrm{d}t.$$

Since $(-\gamma+\kappa_\gamma)p<p-1$, it follows from (13) with Lemma 2.3 that

$$\begin{aligned}J&\leqslant M(p)\sum_{j=1}^{\infty}\Phi^p\left(\frac{\pi}{2^{j+1}}\right)2^{j((\gamma-\kappa_\gamma)p+p-1)}\leqslant M(p)\sum_{j=1}^{\infty}\int_{\pi/2^{j+1}}^{\pi/2^j}x^{(-\gamma+\kappa_\gamma)p}\left(\frac{\Phi(x)}{x}\right)^p\mathrm{d}x\\ &\leqslant M(p)\int_0^{\pi}x^{(-\gamma+\kappa_\gamma)p}|h(x)|^p\mathrm{d}x,\end{aligned}$$

that already completes the proof of Theorem 1.2. □

For $\gamma>1/p-1$, define

$$\kappa_\gamma^*=\begin{cases}0, & -1+1/p<\gamma<1/p,\\ 2k+1, & 2k+1/p<\gamma<2k+2+1/p,\end{cases}\qquad k=0,1,2,\cdots.$$

Similarly, we have

THEOREM 2.7. *Suppose that a nonnegative sequence* $\{a_n\}$ *satisfies condition* (1), *and consider the trigonometric series* (3), *its sum function is denoted by* $g(x)$. *Let* $1<p<\infty$. *If* $-1+1/p<\gamma<1/p$ *or* $2k+1/p<\gamma<2k+2+1/p$, $k=0,1,\cdots$, *then* $x^{-\gamma+\kappa_\gamma^*}g^{(\kappa_\gamma^*)}(x)\in L_{2\pi}^p$ *and* $\{n^{\kappa_\gamma^*}a_n\}$ *is the Fourier coefficients of* $g^{(\kappa_\gamma^*)}(x)$ *if and only if*

$$\sum_{n=1}^{\infty} n^{p+p\gamma-2}a_n^p<\infty.$$

3. Approximation

The next aim of this paper is to discuss the approximation rate. Let $f(x)\in L_{2\pi}^p$, $1<p<\infty$, and $\gamma>1/p-1$. Define the weighted modulus of continuity in L^p norm as follows:

$$\omega(f,h)_{L^p,\gamma}:=\sup_{|t|\leqslant h}\left\|x^{-\gamma}(f(x+t)-f(x))\right\|_{L^p}.$$

THEOREM 3.1. *Suppose that a nonnegative sequence* $\{a_n\}$ *satisfies condition* (1) *and* (4), *consider the trigonometric series* (2), *and its sum function is denoted by* $f(x)$. *Let* $1<p<\infty$. *If* $2k-1+1/p<\gamma<2k+1+1/p$, $k=0,1,\cdots$, *then* $x^{-\gamma+\kappa_\gamma}f^{(\kappa_\gamma)}(x)\in L_{2\pi}^p$, *and*

$$\omega\left(f^{(\kappa_\gamma)},\frac{1}{n}\right)_{L^p,\gamma-\kappa_\gamma}\leqslant Mn^{-1}\left(\sum_{k=1}^{n-1}k^{2p+p\gamma-2}a_k^p\right)^{1/p}+M\left(\sum_{k=n}^{\infty}k^{p+p\gamma-2}a_k^p\right)^{1/p}.$$

THEOREM 3.2. *Suppose that a nonnegative sequence* $\{a_n\}$ *satisfies condition* (1) *and* (4), *consider the trigonometric series* (3), *and its sum function is denoted by* $g(x)$. *Let* $1<p<\infty$. *If* $-1+1/p<\gamma<1/p$ *or* $2k+1/p<\gamma<2k+2+1/p$, $k=0,1,\cdots$, *then* $x^{-\gamma+\kappa_\gamma^*}g^{(\kappa_\gamma^*)}(x)\in L_{2\pi}^p$, *and*

$$\omega\left(g^{(\kappa_\gamma^*)},\frac{1}{n}\right)_{L^p,\gamma-\kappa_\gamma^*}\leqslant Mn^{-1}\left(\sum_{k=1}^{n-1}k^{2p+p\gamma-2}a_k^p\right)^{1/p}+M\left(\sum_{k=n}^{\infty}k^{p+p\gamma-2}a_k^p\right)^{1/p}.$$

The proof of Theorem 3.1 and Theorem 3.2 is quite similar to that in [6, Theorem2], we omit it here.

REFERENCES

[1] R. A. ASKEY AND S. WAINGER, *Integrability theorems for Fourier series*, Duke Math. J., **33** (1966), 223–228.

[2] JR. P. R. BOAS, *Integrability Theorems for Trigonometric Transforms*, Springer, Berlin-Heidelberg, 1967.

[3] R. J. LE AND S. P. ZHOU, *A remark on "two-sided" monotonicity condition: an application to L^p convergence*, Acta Math. Hungar. **113** (2006), 159–169.

[4] L. LEINDLER, *Generalization of inequalities of Hardy and Littlewood*, Acta Sci. Math. (Szeged) **31** (1970), 279–285.

[5] L. LEINDLER, *Relations among Fourier series and sum-functions*, Acta Math. Hungar. **104** (2004), 171–183.
[6] D. S. YU, P. ZHOU AND S. P. ZHOU, *On L^p integrability and convergence of trigonometric series*, Studia Math. **182** (2007), 215–226.
[7] S. P. ZHOU, *Monotonicity Condition of Trigonometric Series: Development and Application*, Science Press, Beijing, 2012, in Chinese.
[8] S. P. ZHOU, P. ZHOU AND D. S. YU, *Ultimate generalization to monotonicity for uniform convergence of trigonometric series*, Science China Math. **53** (2010), 1853-1862/available: arXiv: math.CA/0611805 v1 27 Nov 2006.
[9] A. ZYGMUND, *Trigonometric Series*, Cambridge University Press, Cambridge, 1977.

(Received May 27, 2014)

Yi Zhao
School of Science, Hangzhou Dianzi University
Hangzhou 310018 China
e-mail: zhaoyi@hdu.edu.cn

Songping Zhou
Institute of Mathematics, Zhejiang Sci-Tech University
Hangzhou 310018 China
e-mail: songping.zhou@163.com