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THE GENERALIZED TSALLIS RELATIVE OPERATOR ENTROPY VIA SOLIDARITY

ROZARIJA MIKIĆ, JOSIP PEČARIĆ, IVAN PERIĆ AND YUKI SEO

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Abstract. In this paper, some operator inequalities related to the solidarity and the generalized Tsallis relative operator entropy are shown. As an application, we show the Shannon type operator inequalities and its reverse in terms of the generalized Tsallis relative operator entropy.

1. Introduction and preliminaries

The theory for connections and means of pairs of positive operators has been developed by Kubo and Ando in [18]. In [4], J. I. Fujii, M. Fujii and Seo introduced the solidarity as a generalization of operator means: A binary operation $(A,B) \mapsto A\ s\ B$ from positive invertible operators to selfadjoint operators is called a solidarity if s satisfies the following axiomatic properties:

(S1) right monotonicity: $B \leqslant C$ implies $A\ s\ B \leqslant A\ s\ C$,

($S2_r$) right lower continuity: $B_n \downarrow B$ implies $A\ s\ B_n \downarrow A\ s\ B$.

($S2_l$) $\text{s-}\lim_{n\to\infty} A_n = A$ implies $\text{s-}\lim_{n\to\infty} A_n\ s\ I = A\ s\ I$.

(S3) transformer inequality: $T^*(A\ s\ B)T \leqslant (T^*AT)\ s\ (T^*BT)$ for every operator T.

Then the function $f_s(x) = 1\ s\ x$ is operator monotone by (S1). Moreover a map $s \mapsto f_s$ is a bijection from the solidarities onto the operator monotone functions on $(0,\infty)$, where the inverse map is constructed by

$$A\ s\ B = A^{1/2} f_s\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

Here f_s is called the representing function by s. Let F be the transpose function of f in the sense of Kubo-Ando:

$$F(t) = tf\left(\frac{1}{t}\right) \quad \text{or} \quad f(r) = rF\left(\frac{1}{r}\right)$$

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and $F(x) = x\ s\ 1$ and $f(x) = 1\ s\ x$. If f_s is nonnegative, then s is called connection and besides if $f_s(1) = 1$, then s is called an operator mean in the Kubo-Ando theory. A connection s is called symmetric if $A\ s\ B = B\ s\ A$.

Though the solidarity $A\ s\ B$ does not always exist for non-invertible operators A and B, it is known in [4] the following existence conditions for the solidarity: Let s be the solidarity for the representing function f and $F(t) = t\ s\ 1$. For positive operators A and B, the solidarity $A\ s\ B$ exists if and only if the function $E(\alpha) = F'(\alpha)A + f'(1/\alpha)B$ is bounded below for $\alpha > 0$. Also, if $\alpha A \leqslant B$ for some positive scalar α, the solidarity $A\ s\ B$ exists and $f(\alpha)A \leqslant A\ s\ B \leqslant \max\{F(t) : t \leqslant 1/\alpha\}B$. If $f(t) = \log t$, the solidarity $A\ s\ B$ is the relative operator entropy $S(A|B) := A^{1/2}\log(A^{-1/2}BA^{-1/2})A^{1/2}$ defined by Fujii and Kamei [6]. In addition, the solidarity has entropy-like properties:

1. subadditivity: $(A+B)\ s\ (C+D) \geqslant A\ s\ C + B\ s\ D$.
2. joint concavity: $A\ s\ B \geqslant tA_1\ s\ B_1 + (1-t)A_2\ s\ B_2$ if $A = tA_1 + (1-t)A_2$ and $B = tB_1 + (1-t)B_2$ for $t \in [0,1]$.

Next, we recall an operator version of Uhlmann's interpolation [5, 7]. For a symmetric operator mean σ, a parameterized operator mean σ_t $(t \in [0,1])$ is called an interpolational path for σ if it satisfies

1. $A\ \sigma_0\ B = A, A\ \sigma_{1/2}\ B = A\ \sigma\ B$ and $A\ \sigma_1\ B = B$;
2. $(A\ \sigma_p\ B)\ \sigma\ (A\ \sigma_q\ B) = A\ \sigma_{\frac{p+q}{2}}\ B$ for all $p,q \in [0,1]$;
3. the map $t \in [0,1] \mapsto A\ \sigma_t\ B$ is norm continuous for each A and B.

Typical interpolational means are so-called power means;

$$A\ m_r\ B = A^{1/2}\left[\frac{1}{2}(I + (A^{-1/2}BA^{-1/2})^r\right]^{1/r} A^{1/2} \qquad \text{for } r \in [-1,1]$$

and their interpolational paths are

$$A\ m_{r,t}\ B = A^{1/2}\left[(1-t)I + t(A^{-1/2}BA^{-1/2})^r\right]^{1/r} A^{1/2} \qquad \text{for } t \in [0,1].$$

For each $r \in [-1,1]$, $A\ m_{r,t}\ B$ $(t \in [0,1])$ is an operator mean and a path from A to B via $A\ m_{r,1/2}\ B = A\ m_r\ B$. In particular, we have

$$A\ m_{1,t}\ B = A\ \nabla_t\ B = (1-t)A + tB;$$
$$A\ m_{0,t}\ B = A\ \sharp_t\ B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2};$$
$$A\ m_{-1,t}\ B = A\ !_t\ B = ((1-t)A^{-1} + tB^{-1})^{-1}.$$

They are called the arithmetic, geometric and harmonic interpolations respectively. We denote the representing function $g_{r,t}$ for $m_{r,t}$ by

$$g_{r,t}(x) = 1\ m_{r,t}\ x = (1-t+tx^r)^{1/r} \quad \text{for all } x > 0,$$

and we have

$$A\, m_{r,t}\, B = A^{\frac{1}{2}} g_{r,t}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

A binary operation s_m on positive operators is called a derivative solidarity of an interpolational path m_t if

$$A\, s_m\, B = \text{u-}\lim_{t \downarrow 0} \frac{A\, m_t\, B - A}{t}$$

for invertible A and B in the uniform operator topology. A map $m \mapsto s_m$ is called Uhlmann's transformation. The class of derivative solidarities is a proper subclass of solidarities. In fact, if s is a derivative solidarity, then $A\, s\, A = 0$ for all $A \geqslant 0$, or equivalently, $f(1) = 0$ for the representing function f of s.

The Tsallis relative operator entropy in Yanagi, Kuriyama and Furuichi [9, 10, 20] is defined by

$$\begin{aligned} T_t(A|B) &= \frac{A \natural_t B - A}{t} \\ &= \lim_{r \to 0} \frac{A\, m_{r,t}\, B - A}{t} \end{aligned}$$

and hence $T_t(A|B)$ is a derivative solidarity of an interpolational path $m_{r,t}$. In [15, 16], Isa, Ito, Kamei, Tohyama and Watanabe defined the generalized Tsallis relative entropy by

$$T_{r,t}(A|B) = \frac{A\, m_{r,t}\, B - A}{t} \quad \text{for } t \in [0,1],\ r \in [-1,1]$$

and the solidarity $T_{r,t}(A|B)$ has the representing function $f_{r,t}$ defined by

$$f_{r,t}(x) = \frac{(1-t+tx^r)^{1/r} - 1}{t}.$$

In this paper, some operator inequalities related to the solidarity and the generalized Tsallis relative operator entropy are shown. As an application, we show the Shannon type operator inequalities and its reverse in terms of the generalized Tsallis relative operator entropy.

2. Information monotonicity for solidarity

Let $B(H)$ be the C*-algebra of all bounded linear operators on a Hilbert space H. An operator $A \in B(H)$ is said to be positive (denoted by $A \geqslant 0$) if $\langle Ax, x\rangle \geqslant 0$ for all $x \in H$ and we denote by $A > 0$ if A is positive and invertible. A linear map $\Phi : B(H) \mapsto B(K)$ is called positive if $\Phi(A) \geqslant 0$ whenever $A \geqslant 0$. It is said to be unital if $\Phi(I) = I$ where I is an identity operator. We will say Φ is strictly positive if $\Phi(A) > 0$ whenever $A > 0$. It is known that a positive linear map Φ is strictly positive if and only if $\Phi(I) > 0$. Note that if Φ is a positive linear map, then $\Phi(A)$ is not always invertible for positive invertible A.

The following lemma is well-known, see [1, 3]. For the readers convenience we give a proof:

LEMMA 2.1. *If a continuous function f on $(0,\infty)$ is operator monotone, then the transpose function F of f is operator concave on $[0,\infty)$ with $F(0) \geqslant 0$.*

Proof. For each $\varepsilon > 0$, put $f_\varepsilon(t) = f(t+\varepsilon) - f(\varepsilon)$. Then f_ε is nonnegative operator monotone on $[0,\infty)$ and hence so is the transpose F_ε of f_ε by [18, Corollary 4.2] and $F_\varepsilon(0) \geqslant 0$. It follows from [13, Theorem 1.15] that F_ε is operator concave on $[0,\infty)$. Since $f(\varepsilon)t$ is linear, the function

$$G_\varepsilon(t) = F_\varepsilon(t) + f(\varepsilon)t = tf\left(\varepsilon + \frac{1}{t}\right)$$

is operator concave with $G_\varepsilon(0) \geqslant 0$ and so is $F = \lim_{\varepsilon\to 0} G_\varepsilon$, that is, F is operator concave on $[0,\infty)$ and $F(0) \geqslant 0$. $\square$

We shall present that the solidarity has the following information monotonicity:

THEOREM 2.2. *Let Φ be a positive linear map from $B(H)$ to $B(K)$ and s be a solidarity. Then*

$$\Phi(A\ s\ B) \leqslant \Phi(A)\ s\ \Phi(B) \tag{2.1}$$

for all positive operators A and B such that $A\ s\ B$ exists.

Proof. Let A and B be positive operators such that the solidarity $A\ s\ B$ exists. Firstly, suppose that B is invertible. For $n \in \mathbb{N}$, put $\Phi_n(B) = \Phi(B) + \frac{1}{n}\phi(B)I$ where ϕ is a state. Then the linear map Φ_n is strictly positive for all $n \in \mathbb{N}$. Moreover, put

$$\Psi_n(X) = \Phi_n(B)^{-\frac{1}{2}}\Phi_n(B^{\frac{1}{2}}XB^{\frac{1}{2}})\Phi_n(B)^{-\frac{1}{2}}.$$

Then Ψ_n is a unital positive linear map. Let f be the representing function of s. If F is the transpose function of f, then F is operator concave with $F(0) \geqslant 0$ by Lemma 2.1 and hence it follows from Davis-Choi-Jensen inequality [13, Theorem 1.20] that

$$\begin{aligned}
\Phi_n(A\ s\ B) &= \Phi_n(B^{\frac{1}{2}}F(B^{-\frac{1}{2}}XB^{-\frac{1}{2}})B^{\frac{1}{2}}) \\
&= \Phi_n(B)^{\frac{1}{2}}\Psi_n(F(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}))\Phi_n(B)^{\frac{1}{2}} \\
&\leqslant \Phi_n(B)^{\frac{1}{2}}F(\Psi_n(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}))\Phi_n(B)^{\frac{1}{2}} \\
&= \Phi_n(A)\ s\ \Phi_n(B).
\end{aligned}$$

Since $\Phi(A\ s\ B) \leqslant \Phi_n(A\ s\ B)$ for all $n \in \mathbb{N}$, we have $\Phi(A\ s\ B) \leqslant \Phi_n(A)\ s\ \Phi_n(B)$ and hence $\Phi(A\ s\ B) \leqslant \Phi(A)\ s\ \Phi(B)$ as $n \to \infty$.

Next, suppose that B is non invertible. Since $A\ s\ B = \text{s-}\lim_{\varepsilon\downarrow 0} A\ s\ (B+\varepsilon I)$, it follows that

$$\Phi(A\ s\ B) \leqslant \Phi(A\ s\ (B+\varepsilon I)) \leqslant \Phi(A)\ s\ \Phi(B+\varepsilon I)$$

and hence we have the desired inequality (2.1) as $\varepsilon \to 0$. $\square$

REMARK 2.3. Theorem 2.2 is a generalization of Ando's type inequality:

$$\Phi(A\ m\ B) \leqslant \Phi(A)\ m\ \Phi(B)$$

where Φ is a positive linear map and m is an operator mean, also see [2].

In [19], we showed reverses of Ando type inequality for operator means. We show the solidarity version:

THEOREM 2.4. *Let Φ be a positive linear map and s a solidarity with the representing function f which is not affine. If A and B are positive invertible operators such that $mA \leqslant B \leqslant MA$ for some scalars $0 < m < M$, then*

$$\Phi(A\ s\ B) \geqslant \Phi(A)\ s\ \Phi(B) + \beta(m,M,f)\Phi(A)$$

holds for

$$\beta(m,M,f) = \min\{a_f t + b_f - f(t) : m \leqslant t \leqslant M\} = a_f t_0 + b_f - f(t_0) \tag{2.2}$$

where $t_0 \in [m,M]$ is a unique solution of $f'(t) = a_f$ and

$$a_f = \frac{f(M)-f(m)}{M-m} \quad \text{and} \quad b_f = \frac{Mf(m)-mf(M)}{M-m}.$$

Proof. We may assume that Φ is strictly positive. We consider the map Ψ by $\Psi(X) = \Phi(A)^{-\frac{1}{2}}\Phi(A^{\frac{1}{2}}XA^{\frac{1}{2}})\Phi(A)^{-\frac{1}{2}}$. Since Ψ is a unital positive linear map and the representing function f is operator concave, it follows from $m \leqslant \Psi(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \leqslant M$ that

$$\begin{aligned}\Psi(f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})) - f(\Psi(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})) &\geqslant \Psi(a_f A^{-1/2}BA^{-1/2} + b_f I) - f(\Psi(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}))\\ &= a_f\Psi(A^{-1/2}BA^{-1/2}) + b_f I - f(\Psi(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}))\\ &\geqslant \beta(m,M,f)I\end{aligned}$$

holds for $\beta(m,M,f) = a_f t_0 + b_f - f(t_0)$ and hence

$$\begin{aligned}\Phi(A\ s\ B) &= \Phi(A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2})\\ &= \Phi(A)^{1/2}\Psi(f(A^{-1/2}BA^{-1/2}))\Phi(A)^{1/2}\\ &\geqslant \Phi(A)^{1/2}\left[f(\Psi(A^{-1/2}BA^{-1/2})) + \beta(m,M,f)I\right]\Phi(A)^{1/2}\\ &= \Phi(A)\ s\ \Phi(B) + \beta(m,M,f)\Phi(A). \quad \Box\end{aligned}$$

As applications, we show a reverse Hölder type inequality for the solidarity, and a lower bound of Tsallis relative operator entropy by means of the operator geometric mean.

By subadditivity of the solidarity, we have the following Hölder type inequality: Let s be a solidarity, A_j and B_j positive operators such that $A_j \; s \; B_j$ exist for $j = 1,\dots,n$. Then

$$\sum_{j=1}^{n} A_j \; s \; B_j \leqslant \left(\sum_{j=1}^{n} A_j\right) \; s \; \left(\sum_{j=1}^{n} B_j\right).$$

By Theorem 2.4, we have the following difference type reverse inequality:

COROLLARY 2.5. *Let s be a solidarity with the representing function f which is not affine. If A_j and B_j are positive invertible operators such that $mA_j \leqslant B_j \leqslant MA_j$ for some scalars $0 < m < M$ and $j = 1,\dots,n$, then*

$$\sum_{j=1}^{n} A_j \; s \; B_j - \left(\sum_{j=1}^{n} A_j\right) \; s \; \left(\sum_{j=1}^{n} B_j\right) \geqslant \beta(m,M,f) \sum_{j=1}^{n} A_j,$$

where $\beta(m,M,f)$ is defined by (2.2).

Proof. For $\mathbb{A} = A_1 \oplus A_2 \oplus \cdots \oplus A_n \in B(H \oplus \cdots \oplus H)$, put $\Phi(\mathbb{A}) = \sum_{j=1}^{n} A_j$. Then it follows that Φ is a positive linear map from $B(H \oplus \cdots \oplus H)$ to $B(H)$. Since

$$\Phi(\mathbb{A} \; s \; \mathbb{B}) = \Phi((A_1 \; s \; B_1) \oplus \cdots \oplus (A_n \; s \; B_n)) = \sum_{j=1}^{n} A_j \; s \; B_j,$$

we have this corollary by Theorem 2.4. □

Next, we show the following lower bound of the Tsallis relative operator entropy by means of the operator geometric mean:

COROLLARY 2.6. *Let A and B be positive invertible operators such that $mA \leqslant B \leqslant MB$ for some scalars $0 < m \leqslant M$, and Φ a positive linear map. Then for each $t \in [0,1]$*

$$\Phi(T_t(A|B)) \geqslant \alpha(m,M,t)\Phi(A)\sharp_t\Phi(B)$$

where the constant $\alpha(m,M,t)$ is defined by

$$\alpha(m,M,t) = \begin{cases} \frac{1}{t}\frac{m^t-1}{m^t} & \text{if} \quad x_0 \leqslant m \\ \frac{1}{(1-t)^{1-t}t^{1+t}}\left(\frac{M^t-m^t}{M-m}\right)^t \left(\frac{Mm^t-mM^t-M+m}{M-m}\right)^{1-t} & \text{if} \quad m \leqslant x_0 \leqslant M \\ \frac{1}{t}\frac{M^t-1}{M^t} & \text{if} \quad M \leqslant x_0 \end{cases}$$

and $x_0 = \frac{t}{1-t}\frac{Mm^t-mM^t-M+m}{M^t-m^t}$.

Proof. It follows from [17, Remark 2.3] that for each $t \in [0,1]$

$$\Phi(T_t(A|B)) \geqslant \min_{m \leqslant x \leqslant M} \left\{ \frac{1}{x^t}\left(\frac{M^t-m^t}{t(M-m)}(x-m) + \frac{m^t-1}{t}\right)\right\} \Phi(A) \; \sharp_t \; \Phi(B).$$

Put $F(x) = \frac{1}{x^t}\left(\frac{M^t-m^t}{t(M-m)}(x-m) + \frac{m^t-1}{t}\right)$. Then the unique solution of $F'(x) = 0$ is $x = x_0 = \frac{t}{1-t}\frac{Mm^t-mM^t-M+m}{M^t-m^t}$. Hence we have this corollary. □

3. The generalized Tsallis relative operator entropy

In this section, we discuss the generalized Tsallis relative operator entropy. By Theorem 2.2, we have the following information monotonicity: Let Φ be a positive linear map. Then for each $r \in [-1,1]$ and $t \in [0,1]$

$$\Phi(T_{r,t}(A|B)) \leqslant T_{r,t}(\Phi(A)|\Phi(B))$$

for positive operators A and B. We state two reverse inequalities involving the generalized Tsallis relative operator entropy. For this, we need the following theorem by Jakšić, Pečarić and Perić [17, Theorem 2.4, Theorem 2.6]:

THEOREM A. Let Φ be a positive linear map, A and B positive invertible operators such that $mA \leqslant B \leqslant MA$ for some scalars $0 < m < M$. Then for each $r \in [-1,1]$ and $t \in [0,1]$

$$\Phi(A)\ m_{r,t}\ \Phi(B) \leqslant K(m,M,r,t)\Phi(A\ m_{r,t}\ B); \tag{3.1}$$

$$\Phi(A)\ m_{r,t}\ \Phi(B) - \Phi(A\ m_{r,t}\ B) \leqslant C(m,M,r,t)\Phi(A), \tag{3.2}$$

where

$$\mu_f = \mu_f(r) = \frac{(1-t+tM^r)^{1/r} - (1-t+tm^r)^{1/r}}{M-m}$$

$$\nu_f = \nu_f(r) = \frac{M(1-t+tm^r)^{1/r} - m(1-t+tM^r)^{1/r}}{M-m}.$$

and

$$K(m,M,r,t) = \frac{1}{\nu_f}\left[(1-t)^{\frac{1}{1-r}} + t^{\frac{1}{1-r}}\left(\frac{\nu_f}{\mu_f}\right)^{\frac{r}{1-r}}\right]^{\frac{1-r}{r}} \tag{3.3}$$

$$C(m,M,r,t) = \left(\frac{1-t}{t}\right)^{\frac{1}{r}} \mu_f\left(t^{\frac{1}{r-1}}\mu_f^{\frac{r}{1-r}} - 1\right)^{\frac{r-1}{r}} - \nu_f. \tag{3.4}$$

LEMMA 3.1. *Let* $0 < m < M$, $r \in [-1,1]$ *and* $t \in [0,1]$. *Then*

(i) $K(m,M,r,t) \to K(m,M,t)^{-1}$ *as* $r \to 0$.

(ii) $C(m,M,r,t) \to C(m,M,t)$ *as* $r \to 0$,

where the generalized Kantorovich constant $K(m,M,t)$ *and the Mond-Shisha difference* $C(m,M,t)$ *are defined by*

$$K(m,M,t) = \frac{Mm^t - mM^t}{(1-t)(M-m)}\left(\frac{1-t}{t}\frac{M^t - m^t}{Mm^t - mM^t}\right)^t$$

and

$$C(m,M,t) = (1-t)\left(\frac{M^t - m^t}{t(M-m)}\right)^{\frac{t}{t-1}} - \frac{Mm^t - mM^t}{M-m}, \tag{3.5}$$

also see [13, 8].

Proof. First we show (i). Since it is easy to see that

$$\mu_f(0)=\lim_{r\to 0}\mu_f(r)=\frac{M^t-m^t}{M-m} \quad \text{and} \quad \nu_f(0)=\lim_{r\to 0}\nu_f(r)=\frac{Mm^t-mM^t}{M-m},$$

in this way we can write

$$K(m,M,t)=\frac{1}{t^t(1-t)^{1-t}}\nu_f(0)\left(\frac{\mu_f(0)}{\nu_f(0)}\right)^t.$$

It is straightforward to see that

$$\lim_{r\to 0}\frac{d}{dr}(1-t+ta^r)^{1/r}=\frac{1}{2}t(1-t)a^t\log^2 a,$$

so

$$\lim_{r\to 0}\frac{d\nu_f}{dr}=\frac{1}{2}t(1-t)\frac{Mm^t\log^2 m-mM^t\log^2 M}{M-m}$$

and

$$\lim_{r\to 0}\frac{d\mu_f}{dr}=\frac{1}{2}t(1-t)\frac{M^t\log^2 M-m^t\log^2 m}{M-m}.$$

Using the L'Hospital rule we have

$$\begin{aligned}
&\lim_{r\to 0}\left[(1-t)^{\frac{1}{1-r}}+t^{\frac{1}{1-r}}\left(\frac{\nu_f}{\mu_f}\right)^{\frac{r}{1-r}}\right]^{\frac{1}{r}}\\
&=\exp\left(\lim_{r\to 0}\frac{d}{dr}\log\left((1-t)^{\frac{1}{1-r}}+t^{\frac{1}{1-r}}\left(\frac{\nu_f}{\mu_f}\right)^{\frac{r}{1-r}}\right)\right)\\
&=\exp\left(\lim_{r\to 0}\left((1-t)^{\frac{1}{1-r}}\log(1-t)\frac{1}{(1-r)^2}+t^{\frac{1}{1-r}}\log t\frac{1}{(1-r)^2}\left(\frac{\nu_f}{\mu_f}\right)^{\frac{r}{1-r}}\right.\right.\\
&\qquad\left.\left.+\,t^{\frac{1}{1-r}}\left(\frac{\nu_f}{\mu_f}\right)^{\frac{r}{1-r}}\left(\frac{1}{(1-r)^2}\log\frac{\nu_f}{\mu_f}+\frac{r}{r-1}\frac{\mu_f}{\nu_f}\frac{d}{dr}\left(\frac{\nu_f}{\mu_f}\right)\right)\right)\right)\\
&=\exp\left((1-t)\log(1-t)+t\log t+t\left(\log\frac{\nu_f(0)}{\mu_f(0)}-\frac{\mu_f(0)}{\nu_f(0)}\lim_{r\to 0}r\frac{d}{dr}\left(\frac{\nu_f}{\mu_f}\right)\right)\right)\\
&=\exp\left((1-t)\log(1-t)+t\log t+t\log\frac{\nu_f(0)}{\mu_f(0)}\right)\\
&=(1-t)^{1-t}t^t\left(\frac{\nu_f(0)}{\mu_f(0)}\right)^t.
\end{aligned}$$

Therefore, it follows that

$$\lim_{r\to 0}K(m,M,r,t)=\frac{1}{\nu_f(0)}(1-t)^{(1-t)}t^t\left(\frac{\nu_f(0)}{\mu_f(0)}\right)^t=\frac{1}{K(m,M,t)}.$$

Next we show (ii). It follows that

$$\begin{aligned} C(m,M,t) &= (1-t)\left(\frac{M^t - m^t}{t(M-m)}\right)^{\frac{t}{t-1}} - \frac{Mm^t - mM^t}{M-m} \\ &= (1-t)t^{\frac{t}{1-t}}\mu_f(0)^{\frac{t}{t-1}} - \nu_f(0). \end{aligned}$$

Since

$$\left(\frac{1-t}{t}\right)^{1/r}\left[t^{\frac{1}{r-1}}\mu_f^{\frac{r}{1-r}} - 1\right]^{1-\frac{1}{r}} = \left(t^{\frac{1}{r-1}}\mu_f^{\frac{r}{1-r}} - 1\right)\left(\frac{t}{1-t}\right)^{-\frac{1}{r}}\left(t^{\frac{1}{r-1}}\mu_f^{\frac{r}{1-r}} - 1\right)^{-\frac{1}{r}},$$

we have

$$\begin{aligned} &\lim_{r\to 0}\left[\frac{t}{1-t}t^{\frac{1}{r-1}}\mu_f^{\frac{r}{r-1}} - \frac{t}{1-t}\right]^{\frac{1}{r}} \\ &= \exp\left(\frac{t}{1-t}\lim_{r\to 0}\frac{d}{dr}\left(t^{\frac{1}{r-1}}\mu_f^{\frac{r}{1-r}}\right)\right) \\ &= \exp\left(\frac{t}{1-t}\lim_{r\to 0}\left[-\frac{1}{(r-1)^2}t^{\frac{1}{r-1}}\log t\,\mu_f^{\frac{r}{1-r}} + t^{\frac{1}{r-1}}\mu_f^{\frac{r}{r-1}}\right.\right. \\ &\qquad \left.\left.\times\left[\frac{1}{(r-1)^2}\log\mu_f + \frac{r}{r-1}\frac{1}{\mu_f}\frac{d\mu_f}{dr}\right]\right]\right) \\ &= \exp\left(\frac{t}{1-t}\left[-\frac{1}{t}\log t + \frac{1}{t}\log\mu_f(0)\right]\right) \\ &= t^{\frac{1}{t-1}}\mu_f(0)^{\frac{1}{1-t}} \end{aligned}$$

and hence it follows that

$$\begin{aligned} \lim_{r\to 0} C(m,M,r,t) &= \mu_f(0)\frac{1-t}{t}t^{\frac{1}{1-t}}\mu_f(0)^{\frac{1}{t-1}} - \nu_f(0) \\ &= (1-t)t^{\frac{t}{1-t}}\mu_f(0)^{\frac{t}{t-1}} - \nu_f(0) = C(m,M,t). \quad \square \end{aligned}$$

We show the following two reverse inequalities by virtue of two constants (3.3) and (3.4):

THEOREM 3.2. *Let A and B be positive invertible operators such that $mA \leqslant B \leqslant MA$ for some scalars $0 < m \leqslant M$, and Φ a positive linear map. Then the following inequalities hold:*

$$\begin{aligned} T_{r,t}(\Phi(A)|\Phi(B)) &\geqslant \Phi(T_{r,t}(A|B)) \\ &\geqslant T_{r,t}(\Phi(A)|\Phi(B)) + \frac{K(m,M,r,t)^{-1} - 1}{t}\Phi(A)\ m_{r,t}\ \Phi(B) \qquad (3.6) \end{aligned}$$

and

$$T_{r,t}(\Phi(A)|\Phi(B)) \geqslant \Phi(T_{r,t}(A|B)) \geqslant T_{r,t}(\Phi(A)|\Phi(B)) - \frac{C(m,M,r,t)}{t}\Phi(A) \tag{3.7}$$

for $t \in [0,1]$ *and* $r \in [-1,1]$.

Proof. By (3.1) in Theorem A, we have

$$\Phi(A)\ m_{r,t}\ \Phi(B) \leqslant K(m,M,r,t)\Phi(A\ m_{r,t}\ B)$$

for each $r \in [-1,1]$ and $t \in [0,1]$, where $K(m,M,r,t)$ is defined by (3.3). Hence it follows that

$$\begin{aligned}\frac{\Phi(A)\ m_{r,t}\ \Phi(B) - \Phi(A)}{t} &\geqslant \frac{\Phi(A\ m_{r,t}\ B) - \Phi(A)}{t} \\ &\geqslant \frac{K(m,M,r,t)^{-1}\Phi(A)\ m_{r,t}\ \Phi(B) - \Phi(A)}{t} \\ &= \frac{K(m,M,r,t)^{-1} - 1}{t}\Phi(A)\ m_{r,t}\ \Phi(B) + \frac{\Phi(A)\ m_{r,t}\ \Phi(B) - \Phi(A)}{t}\end{aligned}$$

and hence we have the desired inequality (3.6).

By (3.2) in Theorem A, we have

$$\Phi(A)\ m_{r,t}\ \Phi(B) - \Phi(A\ m_{r,t}\ B) \leqslant C(m,M,r,t)\Phi(A)$$

for each $r \in [-1,1]$ and $t \in [0,1]$, where $C(m,M,r,t)$ is defined by (3.4). Hence it follows that

$$\begin{aligned}\frac{\Phi(A)\ m_{r,t}\ \Phi(B) - \Phi(A)}{t} &\geqslant \frac{\Phi(A\ m_{r,t}\ B) - \Phi(A)}{t} \\ &\geqslant \frac{\Phi(A)\ m_{r,t}\ \Phi(B) - \Phi(A) - C(m,M,r,t)\Phi(A)}{t}\end{aligned}$$

and hence we have the desired inequality (3.7). □

REMARK 3.3. If we put $r \to 0$ in Theorem 3.2, then we have [12, Theorem 2.1] due to Furuta:

$$\frac{1 - K(m,M,t)}{t}\Phi(A)\ \sharp_t\ \Phi(B) + \Phi(T_t(A|B)) \geqslant T_t(\Phi(A)|\Phi(B)) \tag{3.8}$$

and

$$\frac{C(m,M,t)}{t}\Phi(A) + \Phi(T_t(A|B)) \geqslant T_t(\Phi(A)|\Phi(B)). \tag{3.9}$$

Hence Theorem 3.2 is two variable versions of (3.8) and (3.9).

4. Shannon type inequality

In this section, we discuss the Shannon type inequality and its reverse. The original Shannon inequality and its reverse asserts that

$$0 \geqslant \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j} \geqslant -\log \sum_{j=1}^{n} \frac{a_j^2}{b_j}$$

for $a_j, b_j > 0$ with $\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j = 1$, also see [11]. From the viewpoint of the relative operator entropy, Furuta [11] showed the operator version of the Shannon inequality:

$$0 \geqslant \sum_{j=1}^{n} S(A_j|B_j) \geqslant -\log \sum_{j=1}^{n} A_j B_j^{-1} A_j \tag{4.1}$$

for positive invertible operators A_j and B_j with $\sum_{j=1} A_j = \sum_{j=1}^{n} B_j = I$.

In [20], Yanagi, Kuriyama and Furuichi extended the Shannon type operator inequality (4.1) by virture of Tsallis relative operator entropy. Also, in [14, 16], Isa, Ito, Kamei, Tohyama and Watanabe showed the following generalization of Shannon type operator inequality: For positive invertible operators A_j and B_j for $j = 1, \ldots, n$ with $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$,

$$\sum_{j=1}^{n} S(A_j|B_j) \leqslant \frac{1}{t} \log \sum_{j=1}^{n} A_j \ m_{r,t} \ B_j \leqslant \sum_{j=1}^{n} T_{r,t}(A_j|B_j) \leqslant 0$$

for each $t \in [0,1]$ and $r \in [-1,1]$. By Theorem 3.2, we show another two variable Shannon type operator inequality and its reverse in terms of the generalized Tsallis relative operator entropy:

THEOREM 4.1. *Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of positive invertible operators. If $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$, then for each $t \in [0,1]$ and $r \in [-1,1]$*

$$0 \geqslant \sum_{j=1}^{n} T_{r,t}(A_j|B_j) \geqslant \frac{\left((1-t)I + t(\sum_{j=1}^{n} A_j B_j^{-1} A_j)^{-r}\right)^{1/r} - I}{t}.$$

Proof. For $\mathbb{X} = X_1 \oplus X_2 \oplus \cdots \oplus X_n$, put $\Phi(\mathbb{X}) = \sum_{j=1}^{n} X_j$, then Φ is a positive linear map from $B(H \oplus \cdots \oplus H)$ to $B(H)$. Since

$$T_{r,t}(\Phi(\mathbb{A})|\Phi(\mathbb{B})) = T_{r,t}(\sum_{j=1}^{n} A_j | \sum_{j=1}^{n} B_j) = T_{r,t}(I|I) = 0$$

and

$$\Phi(T_{r,t}(\mathbb{A}|\mathbb{B})) = \Phi(T_{r,t}(A_1|B_1) \oplus \cdots \oplus T_{r,t}(A_n|B_n)) = \sum_{j=1}^{n} T_{r,t}(A_j|B_j)$$

for $\mathbb{A} = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ and $\mathbb{B} = B_1 \oplus B_2 \oplus \cdots \oplus B_n$, we have the first inequality in Theorem 4.1. Put $\tilde{g}_{r,t}(x) = (1-t+tx^{-r})^{1/r} = ((1-t+tx^{-r})^{-1/r})^{-1}$ for $r \in [-1,1]$ and then $\tilde{g}_{r,t}$ is operator convex. Hence it follows from [13, Theorem 1.10] that

$$\begin{aligned}
\sum_{j=1}^{n} T_{r,t}(A_j|B_j) &= \sum_{j=1}^{n} \frac{A_j^{1/2} g_{r,t}(A_j^{-1/2} B_j A_j^{-1/2}) A_j^{1/2} - A_j}{t} \\
&= \frac{\sum_{j=1}^{n} A_j^{1/2} \tilde{g}_{r,t}\left(A_j^{1/2} B_j^{-1} A_j^{1/2}\right) A_j^{1/2} - I}{t} \\
&\geqslant \frac{\tilde{g}_{r,t}\left(\sum_{j=1}^{n} A_j B_j^{-1} A_j\right) - I}{t} \\
&= \frac{\left((1-t)I + t(\sum_{j=1}^{n} A_j B_j^{-1} A_j)^{-r}\right)^{1/r} - I}{t}. \qquad \square
\end{aligned}$$

By Theorem 3.2, we obtain the lower bound of the Shannon type operator inequality in terms of the spectra of given operators:

THEOREM 4.2. *Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of positive invertible operators such that $mA_j \leqslant B_j \leqslant MA_j$ for $j = 1, \ldots, n$ and some scalars $0 < m \leqslant M$. If $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$, then for each $t \in [0,1]$ and $r \in [-1,1]$*

$$0 \geqslant \sum_{j=1}^{n} T_{r,t}(A_j|B_j) \geqslant -\frac{C(m,M,r,t)}{t} I$$

where the constant $C(m,M,r,t)$ is defined by (3.4).

Proof. Since $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$, it follows from Theorem 3.2 that

$$\begin{aligned}
0 = T_{r,t}\Big(\sum_{j=1}^{n} A_j \Big| \sum_{j=1}^{n} B_j\Big) &\geqslant \sum_{j=1}^{n} T_{r,t}(A_j|B_j) \\
&\geqslant T_{r,t}\Big(\sum_{j=1}^{n} A_j \Big| \sum_{j=1}^{n} B_j\Big) - \frac{C(m,M,r,t)}{t} \sum_{j=1}^{n} A_j \\
&= -\frac{C(m,M,r,t)}{t} I. \qquad \square
\end{aligned}$$

REMARK 4.3. (1) Notice that the assumptions of Theorem 4.2 imply $m \leqslant 1 \leqslant M$.

(2) If we put $r \to 0$ in Theorem 4.2, then we obtain the lower bound of the Shannon type operator inequality by the Tsallis relative operator entropy: Let A_j, B_j, m, M $(j = 1, \ldots, n)$ be as in Theorem 4.2. If $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$, then for each $t \in [0,1]$

$$0 \geqslant \sum_{j=1}^{n} T_t(A_j|B_j) \geqslant -\frac{C(m,M,t)}{t} I, \tag{4.2}$$

where $C(m,M,t)$ is defined by (3.5). Moreover, if we put $t \to 0$ in (4.2), then

$$0 \geqslant \sum_{j=1}^{n} S(A_j|B_j) \geqslant -\log S(h)I,$$

where the Specht ratio $S(h)$ is defined by

$$S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e\log h} \quad \text{and} \quad h = \frac{M}{m}.$$

Finally, we obtain a simple lower bound for Shannon type operator inequality. For this, we need the following theorem due to Jakšić, Pečarić and Perić [17, Corollary 3.6]:

THEOREM B. Let A_j and B_j be positive invertible operators such that $mA_j \leqslant B_j \leqslant MA_j$ for some scalars $0 < m < M$ and $j = 1,\dots,n$. If $r \in [-1,1]$ and $t \in [0,1]$, then

$$\begin{aligned}\left(\sum_{j=1}^{n} A_j\right) m_{r,t} \left(\sum_{j=1}^{n} B_j\right) - \sum_{j=1}^{n} A_j\, m_{r,t}\, B_j \\ \leqslant t\,\frac{(t+(1-t)m^{-r})^{\frac{1-r}{r}} - (t+(1-t)M^{-r})^{\frac{1-r}{r}}}{M-m} \\ \times \left(M\sum_{j=1}^{n} A_j - \sum_{j=1}^{n} B_j\right)\left(\sum_{j=1}^{n} A_j\right)^{-1}\left(\sum_{j=1}^{n} B_j - m\sum_{j=1}^{n} A_j\right).\end{aligned}$$

COROLLARY 4.4. *Let A_j, B_j, m, M $(j = 1,\dots,n)$ be as in Theorem 4.2. If $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$, then for each $t \in [0,1]$ and $r \in [-1,1]$*

$$\begin{aligned}0 &\geqslant \sum_{j=1}^{n} T_{r,t}(A_j|B_j) \\ &\geqslant -(M-1)(1-m)\frac{(t+(1-t)m^{-r})^{\frac{1-r}{r}} - (t+(1-t)M^{-r})^{\frac{1-r}{r}}}{M-m} I.\end{aligned}$$

In particular, if $r \to 0$, then

$$0 \geqslant \sum_{j=1}^{n} T_t(A_j|B_j) \geqslant \frac{(M-1)(1-m)(M^{t-1} - m^{t-1})}{M-m} I$$

for each $t \in [0,1]$. Moreover, if $t \to 0$, then

$$0 \geqslant \sum_{j=1}^{n} S(A_j|B_j) \geqslant -\frac{(M-1)(1-m)}{Mm} I.$$

Proof. Since $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$, it follows from Theorem B that

$$I - \sum_{j=1}^{n} A_j\, m_{r,t}\, B_j \leqslant t\, \frac{(t+(1-t)m^{-r})^{\frac{1-r}{r}} - (t+(1-t)M^{-r})^{\frac{1-r}{r}}}{M-m}(M-1)(1-m)I,$$

and hence we have

$$\begin{aligned} 0 &\geqslant \sum_{j=1}^{n} T_{r,t}(A_j|B_j) \\ &= \frac{1}{t}\left(\sum_{j=1}^{n} A_j\, m_{r,t}\, B_j - I\right) \\ &\geqslant -(M-1)(1-m)\frac{(t+(1-t)m^{-r})^{\frac{1-r}{r}} - (t+(1-t)M^{-r})^{\frac{1-r}{r}}}{M-m}I. \quad \square \end{aligned}$$

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Rozarija Mikić
Faculty of Textile Technology
University of Zagreb
Zagreb, Croatia
e-mail: rozarija.jaksic@ttf.hr

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Zagreb, Croatia
e-mail: pecaric@element.hr

Ivan Perić
Faculty of Food Technology and Biotechnology
University of Zagreb
Zagreb, Croatia
e-mail: iperic@pbf.hr

Yuki Seo
Department of Mathematics Education
Osaka Kyoiku University
Asahigaoka, Kashiwara, Osaka 582-8582, Japan
e-mail: yukis@cc.osaka-kyoiku.ac.jp