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SHARP BOUNDS FOR NEUMAN MEANS IN TERMS OF GEOMETRIC, ARITHMETIC AND QUADRATIC MEANS

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Abstract. In this paper, we find the greatest values α_1, α_2, α_3, α_4, α_5, α_6, α_7, α_8 and the least values β_1, β_2, β_3, β_4, β_5, β_6, β_7, β_8 such that the double inequalities

$$A^{\alpha_1}(a,b)G^{1-\alpha_1}(a,b) < N_{GA}(a,b) < A^{\beta_1}(a,b)G^{1-\beta_1}(a,b),$$

$$\frac{\alpha_2}{G(a,b)} + \frac{1-\alpha_2}{A(a,b)} < \frac{1}{N_{GA}(a,b)} < \frac{\beta_2}{G(a,b)} + \frac{1-\beta_2}{A(a,b)},$$

$$A^{\alpha_3}(a,b)G^{1-\alpha_3}(a,b) < N_{AG}(a,b) < A^{\beta_3}(a,b)G^{1-\beta_3}(a,b),$$

$$\frac{\alpha_4}{G(a,b)} + \frac{1-\alpha_4}{A(a,b)} < \frac{1}{N_{AG}(a,b)} < \frac{\beta_4}{G(a,b)} + \frac{1-\beta_4}{A(a,b)},$$

$$Q^{\alpha_5}(a,b)A^{1-\alpha_5}(a,b) < N_{AQ}(a,b) < Q^{\beta_5}(a,b)A^{1-\beta_5}(a,b),$$

$$\frac{\alpha_6}{A(a,b)} + \frac{1-\alpha_6}{Q(a,b)} < \frac{1}{N_{AQ}(a,b)} < \frac{\beta_6}{A(a,b)} + \frac{1-\beta_6}{Q(a,b)},$$

$$Q^{\alpha_7}(a,b)A^{1-\alpha_7}(a,b) < N_{QA}(a,b) < Q^{\beta_7}(a,b)A^{1-\beta_7}(a,b),$$

$$\frac{\alpha_8}{A(a,b)} + \frac{1-\alpha_8}{Q(a,b)} < \frac{1}{N_{QA}(a,b)} < \frac{\beta_8}{A(a,b)} + \frac{1-\beta_8}{Q(a,b)}$$

hold for all $a,b>0$ with $a\neq b$, where G, A and Q are respectively the geometric, arithmetic and quadratic means, and N_{GA}, N_{AG}, N_{AQ} and N_{QA} are the Neuman means derived from the Schwab-Borchardt mean.

1. Introduction

For $a,b>0$ with $a\neq b$, the Schwab-Borchardt mean $SB(a,b)$ [1–3] of a and b is given by

$$SB(a,b) = \begin{cases} \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)}, & a<b, \\[2mm] \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & a>b, \end{cases}$$

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where $\cos^{-1}(x)$ and $\cosh^{-1}(x) = \log(x+\sqrt{x^2-1})$ are the inverse cosine and inverse hyperbolic cosine functions, respectively. Recently, the Schwab-Borchardt mean has been the subject of intensive research. In particular, many remarkable inequalities for the Schwab-Borchardt mean can be found in the literature [1–7]. Very recently, the Neuman mean $N(a,b) = (a+b^2/SB(a,b))/2$ derived from the Schwab-Borchardt was introduced and researched by Neuman in [8].

Let $N_{AG}(a,b) = N(A(a,b),G(a,b))$, $N_{GA}(a,b) = N(G(a,b),A(a,b))$, $N_{QA}(a,b) = N(Q(a,b),A(a,b))$ and $N_{AQ}(a,b) = N(A(a,b),Q(a,b))$ be the Neuman means, where $G(a,b) = \sqrt{ab}$, $A(a,b) = (a+b)/2$ and $Q(a,b) = \sqrt{(a^2+b^2)/2}$ are the classical geometric, arithmetic and quadratic means of a and b, respectively. Then Neuman [8] proved that the inequalities

$$G(a,b) < N_{AG}(a,b) < N_{GA}(a,b) < A(a,b) < N_{QA}(a,b) < N_{AQ}(a,b) < Q(a,b)$$

hold for all $a,b > 0$ with $a \neq b$.

Let $a > b > 0$ and $v = (a-b)/(a+b) \in (0,1)$. Then we clearly see that

$$G(a,b) = A(a,b)\sqrt{1-v^2}, \quad Q(a,b) = A(a,b)\sqrt{1+v^2}, \tag{1.1}$$

and the following explicit formulas for $N_{AG}(a,b)$, $N_{GA}(a,b)$, $N_{QA}(a,b)$ and $N_{AQ}(a,b)$ are given in [8]

$$N_{AG}(a,b) = \frac{1}{2}A(a,b)\left[1+(1-v^2)\frac{\tanh^{-1} v}{v}\right], \tag{1.2}$$

$$N_{GA}(a,b) = \frac{1}{2}A(a,b)\left[\sqrt{1-v^2}+\frac{\sin^{-1} v}{v}\right], \tag{1.3}$$

$$N_{QA}(a,b) = \frac{1}{2}A(a,b)\left[\sqrt{1+v^2}+\frac{\sinh^{-1} v}{v}\right], \tag{1.4}$$

$$N_{AQ}(a,b) = \frac{1}{2}A(a,b)\left[1+(1+v^2)\frac{\tan^{-1} v}{v}\right], \tag{1.5}$$

where $\tanh^{-1}(x) = \log[(1+x)/(1-x)]/2$, $\sin^{-1}(x)$, $\sinh^{-1}(x) = \log(x+\sqrt{1+x^2})$ and $\tan^{-1}(x)$ are the inverse hyperbolic tangent, inverse sine, inverse hyperbolic sine and inverse tangent functions, respectively.

In [8], Neuman also proved that the double inequalities

$$\alpha_1 A(a,b)+(1-\alpha_1)G(a,b) < N_{GA}(a,b) < \beta_1 A(a,b)+(1-\beta_1)G(a,b),$$

$$\alpha_2 Q(a,b)+(1-\alpha_2)A(a,b) < N_{AQ}(a,b) < \beta_2 Q(a,b)+(1-\beta_2)A(a,b),$$

$$\alpha_3 A(a,b)+(1-\alpha_3)G(a,b) < N_{AG}(a,b) < \beta_3 A(a,b)+(1-\beta_3)G(a,b),$$

$$\alpha_4 Q(a,b)+(1-\alpha_4)A(a,b) < N_{QA}(a,b) < \beta_4 Q(a,b)+(1-\beta_4)A(a,b)$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_1 \leqslant 2/3$, $\beta_1 \geqslant \pi/4$, $\alpha_2 \leqslant 2/3$, $\beta_2 \geqslant (\pi-2)/[4(\sqrt{2}-1)] = 0.689\ldots$, $\alpha_3 \leqslant 1/3$, $\beta_3 \geqslant 1/2$, $\alpha_4 \leqslant 1/3$ and $\beta_4 \geqslant [\log(1+\sqrt{2})+\sqrt{2}-2]/[2(\sqrt{2}-1)] = 0.356\ldots$.

In [9], the authors presented the best possible parameters $\alpha_1,\alpha_2,\beta_1,\beta_2 \in [0,1/2]$ and $\alpha_3,\alpha_4,\beta_3,\beta_4 \in [1/2,1]$ such that the double inequalities

$$G(\alpha_1 a+(1-\alpha_1)b,\alpha_1 b+(1-\alpha_1)a) < N_{AG}(a,b) < G(\beta_1 a+(1-\beta_1)b,\beta_1 b+(1-\beta_1)a),$$

$$G(\alpha_2 a+(1-\alpha_2)b,\alpha_2 b+(1-\alpha_2)a) < N_{GA}(a,b) < G(\beta_2 a+(1-\beta_2)b,\beta_2 b+(1-\beta_2)a),$$

$$Q(\alpha_3 a+(1-\alpha_3)b,\alpha_3 b+(1-\alpha_3)a) < N_{QA}(a,b) < Q(\beta_3 a+(1-\beta_3)b,\beta_3 b+(1-\beta_3)a),$$

$$Q(\alpha_4 a+(1-\alpha_4)b,\alpha_4 b+(1-\alpha_4)a) < N_{AQ}(a,b) < Q(\beta_4 a+(1-\beta_4)b,\beta_4 b+(1-\beta_4)a)$$

hold for all $a,b>0$ with $a\neq b$.

The main purpose of this paper is to find the greatest values α_1, α_2, α_3, α_4, α_5, α_6, α_7, α_8 and the least values β_1, β_2, β_3, β_4, β_5, β_6, β_7, β_8 such that the double inequalities

$$A^{\alpha_1}(a,b)G^{1-\alpha_1}(a,b) < N_{GA}(a,b) < A^{\beta_1}(a,b)G^{1-\beta_1}(a,b),$$

$$\frac{\alpha_2}{G(a,b)}+\frac{1-\alpha_2}{A(a,b)} < \frac{1}{N_{GA}(a,b)} < \frac{\beta_2}{G(a,b)}+\frac{1-\beta_2}{A(a,b)},$$

$$A^{\alpha_3}(a,b)G^{1-\alpha_3}(a,b) < N_{AG}(a,b) < A^{\beta_3}(a,b)G^{1-\beta_3}(a,b),$$

$$\frac{\alpha_4}{G(a,b)}+\frac{1-\alpha_4}{A(a,b)} < \frac{1}{N_{AG}(a,b)} < \frac{\beta_4}{G(a,b)}+\frac{1-\beta_4}{A(a,b)},$$

$$Q^{\alpha_5}(a,b)A^{1-\alpha_5}(a,b) < N_{AQ}(a,b) < Q^{\beta_5}(a,b)A^{1-\beta_5}(a,b),$$

$$\frac{\alpha_6}{A(a,b)}+\frac{1-\alpha_6}{Q(a,b)} < \frac{1}{N_{AQ}(a,b)} < \frac{\beta_6}{A(a,b)}+\frac{1-\beta_6}{Q(a,b)},$$

$$Q^{\alpha_7}(a,b)A^{1-\alpha_7}(a,b) < N_{QA}(a,b) < Q^{\beta_7}(a,b)A^{1-\beta_7}(a,b),$$

$$\frac{\alpha_8}{A(a,b)}+\frac{1-\alpha_8}{Q(a,b)} < \frac{1}{N_{QA}(a,b)} < \frac{\beta_8}{A(a,b)}+\frac{1-\beta_8}{Q(a,b)}$$

hold for all $a,b>0$ with $a\neq b$.

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

LEMMA 2.1. (See [10, Theorem 1.25]) *Let* $-\infty<a<b<\infty$, $f,g:[a,b]\to(-\infty,\infty)$ *be continuous on* $[a,b]$ *and differentiable on* (a,b), *and* $g'(x)\neq 0$ *on* (a,b). *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b), *then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, the the monotonicity in the conclusion is also strict.*

LEMMA 2.2. (See [11, Lemma 1.1]) *Suppose that the power series* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ *have the radius of convergence* $r > 0$ *with* $b_n > 0$ *for all* $n \geqslant 0$. *If the sequence* $\{a_n/b_n\}$ *is (strictly) increasing (decreasing) for all* $n \geqslant 0$, *then the function* $f(x)/g(x)$ *is also (strictly) increasing (decreasing) on* $(0,r)$.

LEMMA 2.3. *The function*

$$f_1(x) = \frac{\log[\sin(2x)] - \log[2x + \sin(2x)] + \log 2}{\log(\cos x)} \tag{2.1}$$

is strictly increasing from $(0, \pi/2)$ *onto* $(2/3, 1)$.

Proof. It follows from (2.1) that

$$f_1(0) = \frac{2}{3}, \tag{2.2}$$

$$f_1\left(\frac{\pi}{2}^-\right) = 1. \tag{2.3}$$

Let $g_1(x) = \log[\sin(2x)] - \log[2x + \sin(2x)] + \log 2$, $h_1(x) = \log(\cos x)$, $g_2(x) = \sin(2x) - 2x\cos(2x)$ and $h_2(x) = [2x + \sin(2x)]\sin^2 x$. Then simple computations lead to

$$g_1(0^+) = h_1(0) = g_2(0) = h_2(0) = 0, \tag{2.4}$$

$$f_1(x) = \frac{g_1(x)}{h_1(x)}, \quad \frac{g_1'(x)}{h_1'(x)} = \frac{g_2(x)}{h_2(x)}, \tag{2.5}$$

$$\frac{g_2'(x)}{h_2'(x)} = \frac{1}{\frac{1}{2} + \frac{\sin(2x)}{2x}}. \tag{2.6}$$

It is well known that the function $\sin x/x$ is strictly decreasing on $(0, \pi)$, hence equation (2.6) leads to the conclusion that the function $g_2'(x)/h_2'(x)$ is strictly increasing on $(0, \pi/2)$. Therefore, Lemma 2.3 follows from Lemma 2.1 and (2.2)–(2.5) together with the monotonicity of $g_2'(x)/h_2'(x)$. □

LEMMA 2.4. *The function*

$$f_2(x) = \frac{\log[2x + \sinh(2x)] - \log[\sinh(x)] - 2\log 2}{\log[\cosh(x)]} \tag{2.7}$$

is strictly increasing from $(0, \infty)$ *onto* $(1/3, 1)$.

Proof. It follows from (2.7) that

$$f_2(0^+) = \frac{1}{3}, \tag{2.8}$$

$$\lim_{x \to \infty} f_2(x) = 1. \tag{2.9}$$

Let $g_3(x)=\log[2x+\sinh(2x)]-\log[\sinh(x)]-2\log 2$ and $h_3(x)=\log[\cosh(x)]$. Then simple computations lead to

$$f_2(x)=\frac{g_3(x)}{h_3(x)},\quad g_3(0^+)=h_3(0)=0, \tag{2.10}$$

$$\begin{aligned}\frac{g_3'(x)}{h_3'(x)}&=\frac{\sinh(4x)-4x\cosh(2x)+2\sinh(2x)-4x}{\sinh(4x)+4x\cosh(2x)-2\sinh(2x)-4x}\\&=\frac{\sum_{n=1}^{\infty}\frac{(2^{2n}-2n)2^{2n+2}}{(2n+1)!}x^{2n+1}}{\sum_{n=1}^{\infty}\frac{(2^{2n}+2n)2^{2n+2}}{(2n+1)!}x^{2n+1}}\\&=\frac{\sum_{n=0}^{\infty}\frac{(2^{2n+2}-2n-2)2^{2n+4}}{(2n+3)!}x^{2n}}{\sum_{n=0}^{\infty}\frac{(2^{2n+2}+2n+2)2^{2n+4}}{(2n+3)!}x^{2n}}.\end{aligned} \tag{2.11}$$

Let

$$a_n=\frac{\left(2^{2n+2}-2n-2\right)2^{2n+4}}{(2n+3)!},\quad b_n=\frac{\left(2^{2n+2}+2n+2\right)2^{2n+4}}{(2n+3)!}. \tag{2.12}$$

Then

$$b_n>0,\quad \frac{a_{n+1}}{b_{n+1}}-\frac{a_n}{b_n}=\frac{(3n+2)2^{2n+2}}{(2^{2n+3}+n+2)(2^{2n+1}+n+1)}>0 \tag{2.13}$$

for all $n\geqslant 0$.

It follows from Lemma 2.2 and (2.11)–(2.13) that the function $g_3'(x)/h_3'(x)$ is strictly increasing on $(0,\infty)$. Therefore, Lemma 2.4 follows from Lemma 2.1 and (2.8)–(2.10) together with the monotonicity of $g_3'(x)/h_3'(x)$. □

LEMMA 2.5. *The function*

$$f_3(x)=\frac{2x-\sin(2x)}{(1-\cos x)[2x+\sin(2x)]} \tag{2.14}$$

is strictly increasing from $(0,\pi/2)$ *onto* $(2/3,1)$.

Proof. It follows from (2.14) that

$$f_3\left(0^+\right)=\frac{2}{3}, \tag{2.15}$$

$$f_3(\pi/2)=1. \tag{2.16}$$

Let $g_4(x)=2x-\sin(2x)$ and $h_4(x)=(1-\cos x)[2x+\sin(2x)]$. Then simple computations lead to

$$f_3(x)=\frac{g_4(x)}{h_4(x)},\quad g_4(0)=h_4(0)=0, \tag{2.17}$$

$$g_4'(x) = 4\sin^2 x,$$
$$h_4'(x) = 2\sin^2 x\cos x - 4\cos^3 x + 4\cos^2 x + 2x\sin x,$$
$$g_4'(0) = h_4'(0) = 0, \tag{2.18}$$
$$\frac{g_4''(x)}{h_4''(x)} = \frac{4}{9\cos x + \frac{x}{\sin x} - 4}, \tag{2.19}$$
$$\left(9\cos x + \frac{x}{\sin x}\right)' = -8\sin x - \frac{[2x - \sin(2x)]\cos x}{2\sin^2 x} < 0 \tag{2.20}$$

for $x \in (0, \pi/2)$.

Therefore, Lemma 2.5 follows easily from Lemma 2.1 and (2.15)–(2.20). □

LEMMA 2.6. *The function*

$$f_4(x) = \frac{\sinh(x)\cosh^2(x) - 2\sinh(x)\cosh(x) + x\cosh(x)}{\sinh(x)\cosh^2(x) - \sinh(x)\cosh(x) + x\cosh(x) - x} \tag{2.21}$$

is strictly increasing from $(0, \infty)$ *onto* $(1/3, 1)$.

Proof. It follows from (2.21) that

$$\begin{aligned} f_4(x) &= \frac{\frac{1}{4}\sinh(3x) + \frac{1}{4}\sinh(x) - \sinh(2x) + x\cosh(x)}{\frac{1}{4}\sinh(3x) + \frac{1}{4}\sinh(x) - \frac{1}{2}\sinh(2x) + x\cosh(x) - x} \\ &= \frac{\sum_{n=1}^{\infty} \frac{3^{2n+1} - 2^{2n+3} + 8n + 5}{4[(2n+1)!]} x^{2n+1}}{\sum_{n=1}^{\infty} \frac{3^{2n+1} - 2^{2n+2} + 8n + 5}{4[(2n+1)!]} x^{2n+1}} \\ &= \frac{\sum_{n=0}^{\infty} \frac{3^{2n+3} - 2^{2n+5} + 8n + 13}{4[(2n+3)!]} x^{2n}}{\sum_{n=0}^{\infty} \frac{3^{2n+3} - 2^{2n+4} + 8n + 13}{4[(2n+3)!]} x^{2n}}. \end{aligned} \tag{2.22}$$

Let

$$a_n = \frac{3^{2n+3} - 2^{2n+5} + 8n + 13}{4[(2n+3)!]}, \quad b_n = \frac{3^{2n+3} - 2^{2n+4} + 8n + 13}{4[(2n+3)!]}. \tag{2.23}$$

Then simple computations lead to

$$b_n > \frac{3^{2n+3} - 2^{2n+4}}{4[(2n+3)!]} = \frac{2^{2n+3}\left[\left(\frac{3}{2}\right)^{2n+3} - 2\right]}{4[(2n+3)!]} > 0 \tag{2.24}$$

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = \frac{\left(135 \times 3^{2n} - 24n - 31\right)2^{2n+4}}{(3^{2n+3} - 2^{2n+4} + 8n + 13)(3^{2n+5} - 2^{2n+6} + 8n + 21)} > 0 \tag{2.25}$$

for all $n \geqslant 0$.

Note that

$$f_4(0^+) = \frac{a_0}{b_0} = \frac{1}{3}, \quad \lim_{x\to\infty} f_4(x) = \lim_{n\to\infty} \frac{a_n}{b_n} = 1. \tag{2.26}$$

Therefore, Lemma 2.6 follows easily from Lemma 2.2 and (2.22)–(2.26). □

3. Main results

THEOREM 3.1. *The double inequalities*

$$A^{\alpha_1}(a,b)G^{1-\alpha_1}(a,b) < N_{GA}(a,b) < A^{\beta_1}(a,b)G^{1-\beta_1}(a,b), \tag{3.1}$$

$$\frac{\alpha_2}{G(a,b)} + \frac{1-\alpha_2}{A(a,b)} < \frac{1}{N_{GA}(a,b)} < \frac{\beta_2}{G(a,b)} + \frac{1-\beta_2}{A(a,b)} \tag{3.2}$$

hold for all $a,b>0$ *with* $a \neq b$ *if and only if* $\alpha_1 \leqslant 2/3$, $\beta_1 \geqslant 1$, $\alpha_2 \leqslant 0$ *and* $\beta_2 \geqslant 1/3$.

Proof. We clearly see that inequalities (3.1) and (3.2) can be rewritten as

$$\left(\frac{A(a,b)}{G(a,b)}\right)^{\alpha_1} < \frac{N_{GA}(a,b)}{G(a,b)} < \left(\frac{A(a,b)}{G(a,b)}\right)^{\beta_1} \tag{3.3}$$

and

$$1-\beta_2 < \frac{\frac{1}{G(a,b)} - \frac{1}{N_{GA}(a,b)}}{\frac{1}{G(a,b)} - \frac{1}{A(a,b)}} < 1-\alpha_2, \tag{3.4}$$

respectively.

Since both the geometric mean $G(a,b)$ and arithmetic mean $A(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a>b$. Let $v=(a-b)/(a+b)\in(0,1)$. Then from (1.1) and (1.3) we know that inequalities (3.3) and (3.4) are equivalent to

$$\alpha_1 < \frac{\log\left[\frac{1}{2}\left(1+\frac{\sin^{-1}(v)}{v\sqrt{1-v^2}}\right)\right]}{\log\frac{1}{\sqrt{1-v^2}}} < \beta_1 \tag{3.5}$$

and

$$1-\beta_2 < \frac{\sin^{-1}v - v\sqrt{1-v^2}}{(1-\sqrt{1-v^2})(v\sqrt{1-v^2}+\sin^{-1}v)} < 1-\alpha_2, \tag{3.6}$$

respectively.

Let $x=\sin^{-1}(v)$. Then $x\in(0,\pi/2)$,

$$\frac{\log\left[\frac{1}{2}\left(1+\frac{\sin^{-1}(v)}{v\sqrt{1-v^2}}\right)\right]}{\log\frac{1}{\sqrt{1-v^2}}} = \frac{\log[\sin(2x)] - \log[2x+\sin(2x)] + \log 2}{\log(\cos x)}, \tag{3.7}$$

$$\frac{\sin^{-1}v - v\sqrt{1-v^2}}{(1-\sqrt{1-v^2})(v\sqrt{1-v^2}+\sin^{-1}v)} = \frac{2x-\sin(2x)}{(1-\cos x)[2x+\sin(2x)]}. \tag{3.8}$$

Therefore, inequality (3.1) holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1 \leqslant 2/3$ and $\beta_1 \geqslant 1$ follows from (3.5) and (3.7) together with Lemma 2.3, and inequality (3.2) holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_2 \leqslant 0$ and $\beta_2 \geqslant 1/3$ follows from (3.6) and (3.8) together with Lemma 2.5. □

THEOREM 3.2. *The double inequalities*

$$A^{\alpha_3}(a,b)G^{1-\alpha_3}(a,b) < N_{AG}(a,b) < A^{\beta_3}(a,b)G^{1-\beta_3}(a,b), \tag{3.9}$$

$$\frac{\alpha_4}{G(a,b)} + \frac{1-\alpha_4}{A(a,b)} < \frac{1}{N_{AG}(a,b)} < \frac{\beta_4}{G(a,b)} + \frac{1-\beta_4}{A(a,b)} \tag{3.10}$$

hold for all $a,b>0$ *with* $a \neq b$ *if and only if* $\alpha_3 \leqslant 1/3$, $\beta_3 \geqslant 1$, $\alpha_4 \leqslant 0$ *and* $\beta_4 \geqslant 2/3$.

Proof. We clearly see that inequalities (3.9) and (3.10) can be rewritten as

$$\left(\frac{A(a,b)}{G(a,b)}\right)^{\alpha_3} < \frac{N_{AG}(a,b)}{G(a,b)} < \left(\frac{A(a,b)}{G(a,b)}\right)^{\beta_3} \tag{3.11}$$

and

$$1-\beta_4 < \frac{\frac{1}{G(a,b)} - \frac{1}{N_{AG}(a,b)}}{\frac{1}{G(a,b)} - \frac{1}{A(a,b)}} < 1-\alpha_4, \tag{3.12}$$

respectively.

Without loss of generality, we assume that $a>b$. Let $v=(a-b)/(a+b)\in(0,1)$. Then it follows from (1.1) and (1.2) that inequalities (3.11) and (3.12) are equivalent to

$$\alpha_3 < \frac{\log\left[\frac{1}{\sqrt{1-v^2}} + \frac{\sqrt{1-v^2}}{v}\tanh^{-1}(v)\right] - \log 2}{\log\frac{1}{\sqrt{1-v^2}}} < \beta_3 \tag{3.13}$$

and

$$1-\beta_4 < \frac{v+(1-v^2)\tanh^{-1}(v) - 2v\sqrt{1-v^2}}{(1-\sqrt{1-v^2})[v+(1-v^2)\tanh^{-1}(v)]} < 1-\beta_4, \tag{3.14}$$

respectively. Let $x=\tanh^{-1}(v)\in(0,\infty)$. Then simple computations lead to

$$\begin{aligned} &\frac{\log\left[\frac{1}{\sqrt{1-v^2}} + \frac{\sqrt{1-v^2}}{v}\tanh^{-1}(v)\right] - \log 2}{\log\frac{1}{\sqrt{1-v^2}}} \\ &= \frac{\log[2x+\sinh(2x)] - \log[\sinh(x)] - 2\log 2}{\log[\cosh(x)]} \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &\frac{v+(1-v^2)\tanh^{-1}(v) - 2v\sqrt{1-v^2}}{(1-\sqrt{1-v^2})[v+(1-v^2)\tanh^{-1}(v)]} \\ &= \frac{\sinh(x)\cosh^2(x) - 2\sinh(x)\cosh(x) + x\cosh(x)}{\sinh(x)\cosh^2(x) - \sinh(x)\cosh(x) + x\cosh(x) - x}. \end{aligned} \tag{3.16}$$

Therefore, inequality (3.9) holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_3 \leqslant 1/3$ and $\beta_3 \geqslant 1$ follows from (3.13) and (3.15) together with Lemma 2.4, and inequality (3.10) holds for all $a,b>0$ with $a \neq b$ if and only if $\alpha_4 \leqslant 0$ and $\beta_4 \geqslant 2/3$ follows from (3.14) and (3.16) together with Lemma 2.6. $\square$

THEOREM 3.3. *The double inequalities*

$$Q^{\alpha_5}(a,b)A^{1-\alpha_5}(a,b) < N_{AQ}(a,b) < Q^{\beta_5}(a,b)A^{1-\beta_5}(a,b), \tag{3.17}$$

$$\frac{\alpha_6}{A(a,b)} + \frac{1-\alpha_6}{Q(a,b)} < \frac{1}{N_{AQ}(a,b)} < \frac{\beta_6}{A(a,b)} + \frac{1-\beta_6}{Q(a,b)} \tag{3.18}$$

hold for all $a,b > 0$ *with* $a \neq b$ *if and only if* $\alpha_5 \leqslant 2/3$, $\beta_5 \geqslant 2\log(\pi+2)/\log 2 - 4 = 0.7244\ldots$, $\alpha_6 \leqslant [6+2\sqrt{2}-(1+\sqrt{2})\pi]/(\pi+2) = 0.2419\ldots$ *and* $\beta_6 \geqslant 1/3$.

Proof. We clearly see that inequalities (3.17) and (3.18) can be rewritten as

$$\left(\frac{Q(a,b)}{A(a,b)}\right)^{\alpha_5} < \frac{N_{AQ}(a,b)}{A(a,b)} < \left(\frac{Q(a,b)}{A(a,b)}\right)^{\beta_5} \tag{3.19}$$

and

$$1-\beta_6 < \frac{\frac{1}{A(a,b)} - \frac{1}{N_{AQ}(a,b)}}{\frac{1}{A(a,b)} - \frac{1}{Q(a,b)}} < 1-\alpha_6, \tag{3.20}$$

respectively.

Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b) \in (0,1)$. Then from (1.1) and (1.5) we clearly see that inequalities (3.19) and (3.20) are equivalent to

$$\alpha_5 < \frac{2\log(1+\frac{1+v^2}{v}\tan^{-1}(v)) - 2\log 2}{\log(1+v^2)} < \beta_5 \tag{3.21}$$

and

$$1-\beta_6 < \frac{\left[\left(1+v^2\right)\tan^{-1}(v) - v\right]\sqrt{1+v^2}}{[(1+v^2)\tan^{-1}(v)+v](\sqrt{1+v^2}-1)} < 1-\alpha_6, \tag{3.22}$$

respectively.

Let $x = \tan^{-1}(v)$. Then $x \in (0, \pi/4)$,

$$\frac{2\log(1+\frac{1+v^2}{v}\tan^{-1}(v)) - 2\log 2}{\log(1+v^2)} \tag{3.23}$$

$$= \frac{\log[\sin(2x)] - \log[2x+\sin(2x)] + \log 2}{\log(\cos x)} = f_1(x)$$

and

$$\frac{\left[\left(1+v^2\right)\tan^{-1}(v) - v\right]\sqrt{1+v^2}}{[(1+v^2)\tan^{-1}(v)+v](\sqrt{1+v^2}-1)} \tag{3.24}$$

$$= \frac{2x - \sin(2x)}{(1-\cos x)[2x+\sin(2x)]} = f_3(x).$$

Note that

$$f_1\left(\frac{\pi}{4}\right)=\frac{2\log(\pi+2)}{\log 2}-4, \tag{3.25}$$

$$f_3\left(\frac{\pi}{4}\right)=\frac{(2+\sqrt{2})(\pi-2)}{\pi+2}=1-\frac{6+2\sqrt{2}-(1+\sqrt{2})\pi}{\pi+2}. \tag{3.26}$$

Therefore, inequality (3.17) holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_5\leqslant 2/3$ and $\beta_5\geqslant 2\log(\pi+2)/\log 2-4$ follows from (3.21), (3.23), (3.25) and Lemma 2.3, and inequality (3.18) holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_6\leqslant[6+2\sqrt{2}-(1+\sqrt{2})\pi]/(\pi+2)$ and $\beta_6\geqslant 1/3$ follows from (3.22), (3.24), (3.26) and Lemma 2.5. □

THEOREM 3.4. *The double inequalities*

$$Q^{\alpha_7}(a,b)A^{1-\alpha_7}(a,b)<N_{QA}(a,b)<Q^{\beta_7}(a,b)A^{1-\beta_7}(a,b), \tag{3.27}$$

$$\frac{\alpha_8}{A(a,b)}+\frac{1-\alpha_8}{Q(a,b)}<\frac{1}{N_{QA}(a,b)}<\frac{\beta_8}{A(a,b)}+\frac{1-\beta_8}{Q(a,b)} \tag{3.28}$$

hold for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_7\leqslant 1/3$, $\beta_7\geqslant 2\log[\sqrt{2}+\log(1+\sqrt{2})]/\log 2-2=0.3977\ldots$, $\alpha_8\leqslant[2+\sqrt{2}-(1+\sqrt{2})\log(1+\sqrt{2})]/[\sqrt{2}+\log(1+\sqrt{2})]=0.5603\ldots$ *and* $\beta_8\geqslant 2/3$.

Proof. We clearly see that inequalities (3.27) and (3.28) can be rewritten as

$$\left(\frac{Q(a,b)}{A(a,b)}\right)^{\alpha_7}<\frac{N_{QA}(a,b)}{A(a,b)}<\left(\frac{Q(a,b)}{A(a,b)}\right)^{\beta_7} \tag{3.29}$$

and

$$1-\beta_8<\frac{\frac{1}{A(a,b)}-\frac{1}{N_{QA}(a,b)}}{\frac{1}{A(a,b)}-\frac{1}{Q(a,b)}}<1-\alpha_8, \tag{3.30}$$

respectively.

Without loss of generality, we assume that $a>b$. Let $v=(a-b)/(a+b)\in(0,1)$. Then from (1.1) and (1.4) we clearly see that inequalities (3.29) and (3.30) are equivalent to

$$\alpha_7<\frac{2\log\left[\sqrt{1+v^2}+\frac{\sinh^{-1}(v)}{v}\right]-2\log 2}{\log(1+v^2)}<\beta_7 \tag{3.31}$$

and

$$1-\beta_8<\frac{\left[v\left(1+v^2\right)+\sqrt{1+v^2}\sinh^{-1}(v)\right]-2v\sqrt{1+v^2}}{(\sqrt{1+v^2}-1)\left[v\sqrt{1+v^2}+\sinh^{-1}(v)\right]}<1-\alpha_8, \tag{3.32}$$

respectively.

Let $x=\sinh^{-1}(v)$. Then $x\in(0,\log(1+\sqrt{2}))$,

$$\frac{2\log\left[\sqrt{1+v^2}+\frac{\sinh^{-1}(v)}{v}\right]-2\log 2}{\log(1+v^2)} = \frac{\log[2x+\sinh(2x)]-\log[\sinh(x)]-2\log 2}{\log[\cosh(x)]}=f_2(x), \tag{3.33}$$

$$\frac{\left[v\left(1+v^2\right)+\sqrt{1+v^2}\sinh^{-1}(v)\right]-2v\sqrt{1+v^2}}{(\sqrt{1+v^2}-1)\left[v\sqrt{1+v^2}+\sinh^{-1}(v)\right]} \tag{3.34}$$
$$=\frac{\sinh(x)\cosh^2(x)-2\sinh(x)\cosh(x)+x\cosh(x)}{\sinh(x)\cosh^2(x)-\sinh(x)\cosh(x)+x\cosh(x)-x}=f_4(x).$$

Note that

$$f_2[\log(1+\sqrt{2})]=\frac{2\log[\sqrt{2}+\log(1+\sqrt{2})]}{\log 2}-2, \tag{3.35}$$

$$f_4[\log(1+\sqrt{2})]=\frac{(2+\sqrt{2})\log(1+\sqrt{2})-2}{\sqrt{2}+\log(1+\sqrt{2})} \tag{3.36}$$
$$=1-\frac{2+\sqrt{2}-(1+\sqrt{2})\log(1+\sqrt{2})}{\sqrt{2}+\log(1+\sqrt{2})}.$$

Therefore, inequality (3.27) holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_7\leqslant 1/3$ and $\beta_7\geqslant 2\log[\sqrt{2}+\log(1+\sqrt{2})]/\log 2-2$ follows from (3.31), (3.33), (3.35) and Lemma 2.4, and inequality (3.28) holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_8\leqslant[2+\sqrt{2}-(1+\sqrt{2})\log(1+\sqrt{2})]/[\sqrt{2}+\log(1+\sqrt{2})]$ and $\beta_8\geqslant 2/3$ follows from (3.32), (3.34), (3.36) and Lemma 2.6. □

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