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PARAMETRIC MARCINKIEWICZ INTEGRALS ON THE WEIGHTED HARDY AND WEAK HARDY SPACES

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Abstract. Let $0<\rho<n$ and μ_Ω^ρ be the parametric Marcinkiewicz integral. In this paper, by using the atomic decomposition theory of weighted Hardy and weak Hardy spaces, we will obtain the boundedness properties of μ_Ω^ρ on these spaces, under the Lipschitz condition imposed on the kernel Ω.

1. Introduction

Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ $(n\geqslant 2)$ equipped with the normalized Lebesgue measure $d\sigma$. Let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying $\Omega\in L^1(S^{n-1})$ and

$$\int_{S^{n-1}}\Omega(x')\,d\sigma(x')=0, \tag{1.1}$$

where $x'=x/|x|$ for any $x\neq 0$. For $0<\rho<n$, in 1960, Hörmander [16] defined the parametric Marcinkiewicz integral operator μ_Ω^ρ of higher dimension as follows.

$$\mu_\Omega^\rho(f)(x)=\left(\int_0^\infty\left|F_{\Omega,t}^\rho(x)\right|^2\frac{dt}{t^{2\rho+1}}\right)^{1/2}, \tag{1.2}$$

where

$$F_{\Omega,t}^\rho(x)=\int_{|x-y|\leqslant t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f(y)\,dy. \tag{1.3}$$

When $\rho=1$, we shall denote μ_Ω^1 simply by μ_Ω. This operator μ_Ω was first introduced by Stein in [30]. He proved that if $\Omega\in Lip_\alpha(S^{n-1})$ $(0<\alpha\leqslant 1)$, then μ_Ω is the operator of strong type (p,p) for $1<p\leqslant 2$ and of weak type $(1,1)$. Here, we say that $\Omega\in Lip_\alpha(S^{n-1})$ if

$$\left|\Omega(x')-\Omega(y')\right|\leqslant C|x'-y'|^\alpha,\quad x',y'\in S^{n-1}. \tag{1.4}$$

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In [2], Benedek, Calderón and Panzone showed that if Ω is continuously differentiable on S^{n-1}, then μ_Ω is of strong type (p,p) for all $1<p<\infty$. In 1990, Torchinsky and Wang [34] considered the weighted case and proved that if $\Omega \in Lip_\alpha(S^{n-1})$, $0<\alpha\leqslant 1$, then for all $1<p<\infty$ and $w\in A_p$ (Muckenhoupt weight class), μ_Ω is bounded on $L^p_w(\mathbb{R}^n)$. On the other hand, in 1960, Hörmander [16] showed that if $\Omega\in Lip_\alpha(S^{n-1})$ $(0<\alpha\leqslant 1)$, then for $0<\rho<n$, μ^ρ_Ω is of strong type (p,p) for all $1<p<\infty$. It is well known that the Littlewood–Paley g-function is a very important tool in harmonic analysis and the parametric Marcinkiewicz integral is essentially a Littlewood–Paley g-function. Therefore, many authors have been interested in studying the boundedness properties of μ^ρ_Ω on various function spaces, one can see [1, 9, 11, 27] and the references therein for further details.

In [28], Sato established the following weighted L^p boundedness of μ^ρ_Ω for all $0<\rho<n$ (see also [29]).

THEOREM A. *Let $0<\rho<n$ and $\Omega\in L^\infty(S^{n-1})$. If $w\in A_p$, $1<p<\infty$, then there exists a constant $C>0$ independent of f such that*

$$\big\|\mu^\rho_\Omega(f)\big\|_{L^p_w}\leqslant C\|f\|_{L^p_w}.$$

The main purpose of this paper is to discuss the boundedness properties of parametric Marcinkiewicz integrals μ^ρ_Ω $(0<\rho<n)$ on the weighted Hardy and weak Hardy spaces (see Section 2 for the definitions). We now present our main results as follows.

THEOREM 1.1. *Let $0<\rho<n$, $0<\alpha\leqslant 1$, $\Omega\in Lip_\alpha(S^{n-1})$ and $\beta=\min\{\alpha,1/2\}$. If $n/(n+\beta)<p\leqslant 1$ and $w\in A_{p(1+\frac{\beta}{n})}$, then there exists a constant $C>0$ independent of f such that*

$$\big\|\mu^\rho_\Omega(f)\big\|_{L^p_w}\leqslant C\|f\|_{H^p_w}.$$

THEOREM 1.2. *Let $0<\rho<n$, $0<\alpha\leqslant 1$, $\Omega\in Lip_\alpha(S^{n-1})$ and $\beta=\min\{\alpha,1/2\}$. If $p=n/(n+\beta)$ and $w\in A_1$, then there exists a constant $C>0$ independent of f such that*

$$\big\|\mu^\rho_\Omega(f)\big\|_{WL^p_w}\leqslant C\|f\|_{H^p_w}.$$

THEOREM 1.3. *Let $0<\rho<n$, $0<\alpha\leqslant 1$, $\Omega\in Lip_\alpha(S^{n-1})$ and $\beta=\min\{\alpha,1/2\}$. If $n/(n+\beta)<p\leqslant 1$ and $w\in A_{p(1+\frac{\beta}{n})}$, then there exists a constant $C>0$ independent of f such that*

$$\big\|\mu^\rho_\Omega(f)\big\|_{WL^p_w}\leqslant C\|f\|_{WH^p_w}.$$

2. Notations and preliminaries

Let us first recall some standard definitions and notations of A_p weights. The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy–Littlewood maximal functions in [25]. Let w be a nonnegative, locally integrable function defined on $\mathbb{R}^n$; all cubes are assumed to have their sides parallel to the coordinate axes. For $1 < p < \infty$, a weight function w is said to belong to A_p, if there is a constant $C > 0$ such that for every cube $Q \subseteq \mathbb{R}^n$,

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}\,dx\right)^{p-1} \leqslant C, \tag{2.1}$$

where $|Q|$ denotes the Lebesgue measure of Q. For the case $p = 1$, $w \in A_1$, if there is a constant $C > 0$ such that for every cube $Q \subseteq \mathbb{R}^n$,

$$\frac{1}{|Q|}\int_Q w(x)\,dx \leqslant C \cdot \operatorname*{ess\,inf}_{x\in Q} w(x). \tag{2.2}$$

A weight function $w \in A_\infty$ if it satisfies the A_p condition for some $1 < p < \infty$. It is well known that if $w \in A_p$ with $1 < p < \infty$, then $w \in A_r$ for all $r > p$, and $w \in A_q$ for some $1 < q < p$. We thus write $q_w \equiv \inf\{q > 1 : w \in A_q\}$ to denote the critical index of w. Given a cube Q and $\lambda > 0$, λQ stands for the cube with the same center as Q whose side length is λ times that of Q. $Q = Q(x_0, r)$ denotes the cube centered at x_0 with side length r. For a weight function w and a measurable set E, we set the weighted measure of E by $w(E)$, where $w(E) = \int_E w(x)\,dx$.

We state the following results that will be used later on.

LEMMA 2.1. ([15]) *Let $w \in A_q$ with $q \geqslant 1$. Then, for any cube Q, there exists an absolute constant $C > 0$ such that*

$$w(2Q) \leqslant C\,w(Q).$$

In general, for any $\lambda > 1$, we have

$$w(\lambda Q) \leqslant C \cdot \lambda^{nq} w(Q),$$

where C does not depend on Q or λ.

LEMMA 2.2. ([15]) *Let $w \in A_q$ with $q > 1$. Then, for all $r > 0$, there exists a constant $C > 0$ independent of r such that*

$$\int_{|x|\geqslant r} \frac{w(x)}{|x|^{nq}}\,dx \leqslant C \cdot r^{-nq} w\big(Q(0,2r)\big).$$

LEMMA 2.3. ([15]) *Let $w \in A_q$ with $q \geqslant 1$. Then there exists an absolute constant $C > 0$ such that*

$$C \cdot \left(\frac{|E|}{|Q|}\right)^q \leqslant \frac{w(E)}{w(Q)}$$

for any measurable subset E of a cube Q.

Given a weight function w on $\mathbb{R}^n$, for $0 < p < \infty$, we denote by $L^p_w(\mathbb{R}^n)$ the weighted space of all functions f satisfying

$$\|f\|_{L^p_w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p} < \infty. \tag{2.3}$$

When $p=\infty$, $L_w^\infty(\mathbb{R}^n)$ will be taken to mean $L^\infty(\mathbb{R}^n)$, and we set

$$\|f\|_{L_w^\infty} = \|f\|_{L^\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)| < \infty. \tag{2.4}$$

We also let $WL_w^p(\mathbb{R}^n)$ denote the weighted weak L^p space of all those measurable functions f which satisfy

$$\|f\|_{WL_w^p} = \sup_{\lambda>0} \lambda \cdot w\big(\{x\in\mathbb{R}^n : |f(x)|>\lambda\}\big)^{1/p} < \infty. \tag{2.5}$$

We write $\mathscr{S}(\mathbb{R}^n)$ to denote the Schwartz space of all rapidly decreasing infinitely differentiable functions and $\mathscr{S}'(\mathbb{R}^n)$ to denote the space of all tempered distributions, i.e., the topological dual of $\mathscr{S}(\mathbb{R}^n)$. As we know, for any $0<p\leqslant 1$, the weighted Hardy spaces $H_w^p(\mathbb{R}^n)$ can be defined in terms of maximal functions. Let φ be a function in $\mathscr{S}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n}\varphi(x)\,dx = 1$. Set

$$\varphi_t(x) = t^{-n}\varphi(x/t), \quad t>0,\ x\in\mathbb{R}^n.$$

We will define the radial maximal function $M_\varphi f(x)$ by

$$M_\varphi f(x) = \sup_{t>0}\big|(\varphi_t * f)(x)\big|.$$

Then the weighted Hardy space $H_w^p(\mathbb{R}^n)$ consists of those tempered distributions $f\in\mathscr{S}'(\mathbb{R}^n)$ for which $M_\varphi f\in L_w^p(\mathbb{R}^n)$ with $\|f\|_{H_w^p} = \|M_\varphi f\|_{L_w^p}$. The real-variable theory of weighted Hardy spaces has been extensively investigated by many authors. For example, Garcia-Cuerva [14] studied the atomic decomposition and the dual spaces of H_w^p for $0<p\leqslant 1$. The molecular characterization of H_w^p for $0<p\leqslant 1$ was given by Lee and Lin [18]. For more information about the continuity properties of some operators on weighted Hardy spaces, the reader is referred to [3, 4, 17, 19, 20, 21].

In this article, we will use Garcia-Cuerva's atomic decomposition theory for weighted Hardy spaces in [14, 32]. We characterize weighted Hardy spaces in terms of atoms in the following way.

Let $0<p\leqslant 1\leqslant q\leqslant\infty$ and $p\neq q$ such that $w\in A_q$ with critical index q_w. Set $[\,\cdot\,]$ the greatest integer function. For $s\in\mathbb{Z}_+$ satisfying $s\geqslant N=[n(q_w/p-1)]$, a real-valued function $a(x)$ is called a (p,q,s)-atom centered at x_0 with respect to w (or a w-(p,q,s)-atom centered at x_0) if the following conditions are satisfied:

(a) $a\in L_w^q(\mathbb{R}^n)$ and is supported in a cube Q centered at x_0;

(b) $\|a\|_{L_w^q}\leqslant w(Q)^{1/q-1/p}$;

(c) $\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for every multi-index α with $|\alpha|\leqslant s$.

THEOREM 2.4. *Let $0<p\leqslant 1\leqslant q\leqslant\infty$ and $p\neq q$ such that $w\in A_q$ with critical index q_w. For each $f\in H_w^p(\mathbb{R}^n)$, there exist a sequence $\{a_j\}$ of w-(p,q,s)-atoms and a sequence $\{\lambda_j\}$ of real numbers with $\sum_j|\lambda_j|^p\leqslant C\|f\|_{H_w^p}^p$ such that $f=\sum_j\lambda_j a_j$ both in the sense of distributions and in the H_w^p norm.*

Let us now turn to the weighted weak Hardy spaces, which are good substitutes for the weighted Hardy spaces in the study of the boundedness of some operators. The (unweighted) weak H^p spaces have first appeared in the work of Fefferman, Rivière and Sagher [12], which are the intermediate spaces between two Hardy spaces through the real method of interpolation. The atomic decomposition characterization of weak H^1 space on $\mathbb{R}^n$ was given by Fefferman and Soria in [13]. Later, Liu [22] established the weak H^p spaces on homogeneous groups for the whole range $0<p\leqslant 1$. The corresponding results related to $\mathbb{R}^n$ can be found in [24]. For the boundedness properties of some operators on weak Hardy spaces, we refer the readers to [5, 6, 7, 8, 9, 10, 23, 33]. In 2000, Quek and Yang [26] introduced the weighted weak Hardy spaces $WH^p_w(\mathbb{R}^n)$ and established their atomic decompositions. Moreover, by using the atomic decomposition theory of $WH^p_w(\mathbb{R}^n)$, Quek and Yang [26] also obtained the boundedness of Calderón–Zygmund type operators on these weighted spaces.

Let $w\in A_\infty$, $0<p\leqslant 1$ and $N=[n(q_w/p-1)]$. Define

$$\mathscr{A}_{N,w}=\Big\{\varphi\in\mathscr{S}(\mathbb{R}^n):\sup_{x\in\mathbb{R}^n}\sup_{|\alpha|\leqslant N+1}(1+|x|)^{N+n+1}\big|D^\alpha\varphi(x)\big|\leqslant 1\Big\},$$

where $\alpha=(\alpha_1,\dots,\alpha_n)\in(\mathbb{N}\cup\{0\})^n$, $|\alpha|=\alpha_1+\ldots+\alpha_n$, and

$$D^\alpha\varphi=\frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}.$$

For any given $f\in\mathscr{S}'(\mathbb{R}^n)$, the grand maximal function of f is defined by

$$G_wf(x)=\sup_{\varphi\in\mathscr{A}_{N,w}}\sup_{|y-x|<t}\big|(\varphi_t*f)(y)\big|.$$

Then we can define the weighted weak Hardy space $WH^p_w(\mathbb{R}^n)$ by $WH^p_w(\mathbb{R}^n)=\big\{f\in\mathscr{S}'(\mathbb{R}^n):G_wf\in WL^p_w(\mathbb{R}^n)\big\}$. Moreover, we set $\|f\|_{WH^p_w}=\|G_wf\|_{WL^p_w}$.

THEOREM 2.5. ([26]) *Let $0<p\leqslant 1$ and $w\in A_\infty$. For every $f\in WH^p_w(\mathbb{R}^n)$, there exists a sequence of bounded measurable functions $\{f_k\}_{k=-\infty}^\infty$ such that*

(i) $f=\sum_{k=-\infty}^\infty f_k$ in the sense of distributions.

(ii) Each f_k can be further decomposed into $f_k=\sum_i b_i^k$, where $\{b_i^k\}$ satisfies

(a) Each b_i^k is supported in a cube Q_i^k with $\sum_i w\big(Q_i^k\big)\leqslant c2^{-kp}$, and $\sum_i\chi_{Q_i^k}(x)\leqslant c$. Here χ_E denotes the characteristic function of the set E and $c\sim\big\|f\big\|^p_{WH^p_w}$;

(b) $\big\|b_i^k\big\|_{L^\infty}\leqslant C2^k$, where $C>0$ is independent of i and k;

(c) $\int_{\mathbb{R}^n}b_i^k(x)x^\alpha\,dx=0$ for every multi-index α with $|\alpha|\leqslant[n(q_w/p-1)]$.

Conversely, if $f\in\mathscr{S}'(\mathbb{R}^n)$ has a decomposition satisfying (i) and (ii), then $f\in WH^p_w(\mathbb{R}^n)$. Moreover, we have $\big\|f\big\|^p_{WH^p_w}\sim c$.

Throughout this article C always denotes a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. Moreover, we use $A\sim B$ to mean the equivalence of A and B; that is, there exist two positive constants C_1 and C_2 independent of A, B such that $C_1A\leqslant B\leqslant C_2A$.

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Set $q = p(1+\frac{\beta}{n})$ with $\beta = \min\{\alpha, 1/2\}$. Then by our assumption, we have $[n(q_w/p-1)] = 0$ provided that $w \in A_q$. In view of Theorem 2.4 and Theorem A, it suffices to show that for any w-$(p,q,0)$-atom a, there exists a constant $C > 0$ independent of a such that $\|\mu_\Omega^\rho(a)\|_{L_w^p} \leqslant C$. Let $a(x)$ be a w-$(p,q,0)$-atom with supp $a \subseteq Q = Q(x_0, r_Q)$, and let $Q^* = 2\sqrt{n}Q$. We write

$$\begin{aligned}\|\mu_\Omega^\rho(a)\|_{L_w^p}^p &= \int_{Q^*} |\mu_\Omega^\rho(a)(x)|^p w(x)\,dx + \int_{(Q^*)^c} |\mu_\Omega^\rho(a)(x)|^p w(x)\,dx \\ &= I_1 + I_2.\end{aligned}$$

For the term I_1, by using Hölder's inequality with exponent $s = q/p > 1$, the size condition of atom a, Lemma 2.1 and Theorem A, we have

$$\begin{aligned}I_1 &\leqslant \left(\int_{Q^*} |\mu_\Omega^\rho(a)(x)|^q w(x)\,dx\right)^{p/q} \left(\int_{Q^*} w(x)\,dx\right)^{1-p/q} \\ &\leqslant \|\mu_\Omega^\rho(a)\|_{L_w^q}^p \left[w(Q^*)\right]^{1-p/q} \\ &\leqslant C \cdot \|\mu_\Omega^\rho(a)\|_{L_w^q}^p \left[w(Q)\right]^{1-p/q} \\ &\leqslant C \cdot \|a\|_{L_w^q}^p \left[w(Q)\right]^{1-p/q} \\ &\leqslant C.\end{aligned}$$

Let us now turn to estimate the other term I_2. For $0 < \rho < n$ and for any $x \in (Q^*)^c$, we write

$$\begin{aligned}\mu_\Omega^\rho(a)(x) &= \left(\int_0^\infty \left|\int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} a(y)\,dy\right|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2} \\ &= \left(\int_0^{|x-x_0|+(\sqrt{n})r_Q} \left|\int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} a(y)\,dy\right|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2} \\ &\quad + \left(\int_{|x-x_0|+(\sqrt{n})r_Q}^\infty \left|\int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} a(y)\,dy\right|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2} \\ &= \text{I+II}.\end{aligned}$$

Clearly, the condition $\Omega \in Lip_\alpha(S^{n-1})$ $(0 < \alpha \leqslant 1)$ implies that $\Omega \in L^\infty(S^{n-1})$. Notice also that when $y \in Q$ and $x \in (Q^*)^c$, then we get $|x-y| \sim |x-x_0| \sim |x-x_0| + (\sqrt{n})r_Q$. Thus, for $0 < \rho < n$, we apply the mean value theorem to obtain

$$\left|\frac{1}{|x-y|^{2\rho}} - \frac{1}{[|x-x_0|+(\sqrt{n})r_Q]^{2\rho}}\right| \leqslant C \cdot \frac{r_Q}{|x-y|^{2\rho+1}}. \tag{3.1}$$

The above estimate (3.1) together with Minkowski's inequality for integrals yields

$$
\begin{aligned}
\mathrm{I} &\leqslant \int_Q \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}\big|a(y)\big|\left(\int_{|x-y|}^{|x-x_0|+(\sqrt{n})r_Q}\frac{dt}{t^{2\rho+1}}\right)^{1/2} dy\\
&\leqslant C\cdot\|\Omega\|_{L^\infty}\int_Q \frac{|a(y)|}{|x-y|^{n-\rho}}\left(\frac{r_Q}{|x-y|^{2\rho+1}}\right)^{1/2} dy\\
&\leqslant C\cdot\frac{(r_Q)^{1/2}}{|x-x_0|^{n+\frac{1}{2}}}\int_Q\big|a(y)\big|\,dy.
\end{aligned}
$$

Denote the conjugate exponent of $q>1$ by $q'=q/(q-1)$. Then it follows from Hölder's inequality, the A_q condition and the size condition of atom a that

$$
\begin{aligned}
\int_Q\big|a(y)\big|\,dy &\leqslant \left(\int_Q\big|a(y)\big|^q w(y)\,dy\right)^{1/q}\left(\int_Q w(y)^{-q'/q}\,dy\right)^{1/q'}\\
&\leqslant C\cdot\|a\|_{L^q_w}\left(\frac{|Q|^q}{w(Q)}\right)^{1/q}\\
&\leqslant C\cdot\frac{|Q|}{[w(Q)]^{1/p}}. \qquad (3.2)
\end{aligned}
$$

Hence, we have

$$
\mathrm{I}\leqslant C\cdot\frac{(r_Q)^{n+\frac{1}{2}}}{|x-x_0|^{n+\frac{1}{2}}}\cdot\frac{1}{[w(Q)]^{1/p}}.
$$

On the other hand, if $t\geqslant|x-x_0|+(\sqrt{n})r_Q$, then we can easily see that $Q\subset\{y:|x-y|\leqslant t\}$, and $Q\cap\{y:|x-y|\leqslant t\}=Q$. Thus, by the vanishing moment condition of atom a, we can see that

$$
\begin{aligned}
\mathrm{II} &= \left(\int_{|x-x_0|+(\sqrt{n})r_Q}^{\infty}\left|\int_Q\frac{\Omega(x-y)}{|x-y|^{n-\rho}}a(y)\,dy\right|^2\frac{dt}{t^{2\rho+1}}\right)^{1/2}\\
&= \left(\int_{|x-x_0|+(\sqrt{n})r_Q}^{\infty}\left|\int_Q\left[\frac{\Omega(x-y)}{|x-y|^{n-\rho}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}}\right]a(y)\,dy\right|^2\frac{dt}{t^{2\rho+1}}\right)^{1/2}\\
&\leqslant C\left(\int_{|x-x_0|+(\sqrt{n})r_Q}^{\infty}\left[\int_Q\left|\frac{1}{|x-y|^{n-\rho}}-\frac{1}{|x-x_0|^{n-\rho}}\right|\big|a(y)\big|\,dy\right]^2\frac{dt}{t^{2\rho+1}}\right)^{1/2}\\
&\quad+\left(\int_{|x-x_0|+(\sqrt{n})r_Q}^{\infty}\left[\int_Q\frac{|\Omega(x-y)-\Omega(x-x_0)|}{|x-x_0|^{n-\rho}}\big|a(y)\big|\,dy\right]^2\frac{dt}{t^{2\rho+1}}\right)^{1/2}\\
&= \mathrm{III}+\mathrm{IV}.
\end{aligned}
$$

When $x\in(Q^*)^c$ and $y\in Q$, then $|x-y|\sim|x-x_0|$. Using the mean value theorem and

(3.2), we obtain

$$\begin{aligned} \mathrm{III} &\leqslant C\left(\int_{|x-x_0|+(\sqrt{n})r_Q}^{\infty}\left[\int_Q \frac{|y-x_0|}{|x-x_0|^{n-\rho+1}}\big|a(y)\big|\,dy\right]^2\frac{dt}{t^{2\rho+1}}\right)^{1/2} \\ &\leqslant C\cdot\frac{r_Q}{|x-x_0|^{n-\rho+1}}\int_Q\big|a(y)\big|\,dy\times\left(\int_{|x-x_0|+(\sqrt{n})r_Q}^{\infty}\frac{dt}{t^{2\rho+1}}\right)^{1/2} \\ &\leqslant C\cdot\frac{r_Q}{|x-x_0|^{n+1}}\int_Q\big|a(y)\big|\,dy \\ &\leqslant C\cdot\frac{(r_Q)^{n+1}}{|x-x_0|^{n+1}}\cdot\frac{1}{[w(Q)]^{1/p}}. \end{aligned}$$

In addition, from the definition of $\Omega \in Lip_\alpha(S^{n-1})\ (0<\alpha\leqslant 1)$, we can easily check that

$$\begin{aligned} \big|\Omega(x-y)-\Omega(x-x_0)\big| &= \left|\Omega\Big(\frac{x-y}{|x-y|}\Big)-\Omega\Big(\frac{x-x_0}{|x-x_0|}\Big)\right| \\ &\leqslant C\left|\frac{x-y}{|x-y|}-\frac{x-x_0}{|x-x_0|}\right|^{\alpha} \\ &\leqslant C\left(\frac{|y-x_0|}{|x-x_0|}\right)^{\alpha}. \end{aligned} \tag{3.3}$$

Substituting the above inequality (3.3) into the term IV and then using (3.2), we can deduce that

$$\begin{aligned} \mathrm{IV} &\leqslant C\left(\int_{|x-x_0|+(\sqrt{n})r_Q}^{\infty}\left[\int_Q \frac{|y-x_0|^{\alpha}}{|x-x_0|^{n-\rho+\alpha}}\big|a(y)\big|\,dy\right]^2\frac{dt}{t^{2\rho+1}}\right)^{1/2} \\ &\leqslant C\cdot\frac{(r_Q)^{\alpha}}{|x-x_0|^{n-\rho+\alpha}}\int_Q\big|a(y)\big|\,dy\times\left(\int_{|x-x_0|+(\sqrt{n})r_Q}^{\infty}\frac{dt}{t^{2\rho+1}}\right)^{1/2} \\ &\leqslant C\cdot\frac{(r_Q)^{n+\alpha}}{|x-x_0|^{n+\alpha}}\cdot\frac{1}{[w(Q)]^{1/p}}. \end{aligned}$$

Therefore, combining the above estimates for I, III and IV, we have for any fixed $x \in (Q^*)^c$,

$$\big|\mu_\Omega^\rho(a)(x)\big| \leqslant C\cdot\frac{1}{[w(Q)]^{1/p}}\left[\frac{(r_Q)^{n+\frac12}}{|x-x_0|^{n+\frac12}}+\frac{(r_Q)^{n+1}}{|x-x_0|^{n+1}}+\frac{(r_Q)^{n+\alpha}}{|x-x_0|^{n+\alpha}}\right]. \tag{3.4}$$

Hence,

$$\begin{aligned} I_2 &\leqslant C\cdot\frac{(r_Q)^{(n+\frac12)p}}{w(Q)}\int_{(Q^*)^c}\frac{w(x)}{|x-x_0|^{(n+\frac12)p}}\,dx + C\cdot\frac{(r_Q)^{(n+1)p}}{w(Q)}\int_{(Q^*)^c}\frac{w(x)}{|x-x_0|^{(n+1)p}}\,dx \\ &\quad + C\cdot\frac{(r_Q)^{(n+\alpha)p}}{w(Q)}\int_{(Q^*)^c}\frac{w(x)}{|x-x_0|^{(n+\alpha)p}}\,dx \\ &= I_3+I_4+I_5. \end{aligned}$$

Observe that $\beta=\min\{\alpha,1/2\}$ and $n/(n+\beta)<p\leqslant 1$, then $n/(n+\alpha)<p\leqslant 1$ and $n/(n+\frac{1}{2})<p\leqslant 1$. Since $w\in A_{p(1+\frac{\beta}{n})}$ and $\beta=\min\{\alpha,1/2\}$, then we have $w\in A_{p(1+\frac{\alpha}{n})}$ and $w\in A_{p(1+\frac{1}{2n})}$. By using Lemma 2.1 and Lemma 2.2, we thus obtain

$$\begin{aligned} I_3 &= C\cdot\frac{(r_Q)^{(n+\frac{1}{2})p}}{w(Q)}\int_{|y|\geqslant(\sqrt{n})r_Q}\frac{w_1(y)}{|y|^{n\cdot p(1+\frac{1}{2n})}}\,dy\\ &\leqslant C\cdot\frac{(r_Q)^{(n+\frac{1}{2})p}}{w(Q)}\cdot(r_Q)^{-n\cdot p(1+\frac{1}{2n})}w_1\big(Q(0,r_Q)\big)\\ &= C\cdot\frac{(r_Q)^{(n+\frac{1}{2})p}}{w(Q)}\cdot(r_Q)^{-n\cdot p(1+\frac{1}{2n})}w(Q)\leqslant C \end{aligned}$$

and

$$\begin{aligned} I_5 &= C\cdot\frac{(r_Q)^{(n+\alpha)p}}{w(Q)}\int_{|y|\geqslant(\sqrt{n})r_Q}\frac{w_1(y)}{|y|^{n\cdot p(1+\frac{\alpha}{n})}}\,dy\\ &\leqslant C\cdot\frac{(r_Q)^{(n+\alpha)p}}{w(Q)}\cdot(r_Q)^{-n\cdot p(1+\frac{\alpha}{n})}w_1\big(Q(0,r_Q)\big)\leqslant C, \end{aligned}$$

where $w_1(x)=w(x+x_0)$ is the translation of $w(x)$. It is obvious that $w_1\in A_s$ for $w\in A_s$, $s>1$ ($s=p(1+\frac{1}{2n})$ or $s=p(1+\frac{\alpha}{n})$), and $q_{w_1}=q_w$. By using the same arguments as above, we can also prove that $I_4\leqslant C$. Summing up the above estimates for I_1 and I_2, we then complete the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

In order to prove our main result of this section, we shall need the following superposition principle on the weighted weak type estimates.

LEMMA 4.1. *Let $w\in A_1$ and $0<p<1$. If a sequence of measurable functions $\{f_j\}$ satisfy*

$$w\big(\{x\in\mathbb{R}^n:|f_j(x)|>\alpha\}\big)\leqslant\alpha^{-p}\quad\textit{for all } j\in\mathbb{Z}$$

and

$$\sum_{j\in\mathbb{Z}}|\lambda_j|^p\leqslant 1,$$

then we obtain that $\sum_j\lambda_j f_j(x)$ is absolutely convergent almost everywhere and

$$w\Big(\Big\{x\in\mathbb{R}^n:\Big|\sum_j\lambda_j f_j(x)\Big|>\alpha\Big\}\Big)\leqslant\frac{2-p}{1-p}\cdot\alpha^{-p}.$$

Proof. The proof of this lemma is similar to the corresponding result for the unweighted case which can be found in [31]. See also [24, p. 123]. □

We are now ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. We first observe that for $w \in A_1$ and $p = n/(n+\beta)$, then $[n(q_w/p-1)] = [\beta] = 0$. According to Theorem 2.4 and Lemma 4.1, it is enough for us to show that for any w-$(p,q,0)$-atom $a(x)$, there exists a constant $C > 0$ independent of a such that $\big\|\mu_\Omega^\rho(a)\big\|_{WL_w^p} \leqslant C$. Let $a(x)$ be a w-$(p,q,0)$-atom centered at x_0 with supp $a \subseteq Q = Q(x_0, r_Q)$, and let $Q^* = 2\sqrt{n}Q$. Then for any fixed $\lambda > 0$, we write

$$\begin{aligned}
&\lambda^p \cdot w\big(\big\{x \in \mathbb{R}^n : \big|\mu_\Omega^\rho(a)(x)\big| > \lambda\big\}\big)\\
\leqslant&\lambda^p \cdot w\big(\big\{x \in Q^* : \big|\mu_\Omega^\rho(a)(x)\big| > \lambda\big\}\big) + \lambda^p \cdot w\big(\big\{x \in (Q^*)^c : \big|\mu_\Omega^\rho(a)(x)\big| > \lambda\big\}\big)\\
=&J_1 + J_2.
\end{aligned}$$

Since $w \in A_1$, then $w \in A_q$ for any $1 < q < \infty$. Applying Chebyshev's inequality, Hölder's inequality, Lemma 2.1, Theorem A and the size condition of atom a, we have

$$\begin{aligned}
J_1 &\leqslant \int_{Q^*} \big|\mu_\Omega^\rho(a)(x)\big|^p w(x)\,dx\\
&\leqslant \bigg(\int_{Q^*} \big|\mu_\Omega^\rho(a)(x)\big|^q w(x)\,dx\bigg)^{p/q}\bigg(\int_{Q^*} w(x)\,dx\bigg)^{1-p/q}\\
&\leqslant C \cdot \big\|\mu_\Omega^\rho(a)\big\|_{L_w^q}^p \big[w(Q)\big]^{1-p/q}\\
&\leqslant C \cdot \|a\|_{L_w^q}^p \big[w(Q)\big]^{1-p/q}\\
&\leqslant C.
\end{aligned}$$

For any $x \in (Q^*)^c$, in the proof of Theorem 1.1, we have already obtained the following pointwise inequality (see (3.4))

$$\big|\mu_\Omega^\rho(a)(x)\big| \leqslant C \cdot \frac{1}{[w(Q)]^{1/p}}\left[\frac{(r_Q)^{n+\alpha}}{|x-x_0|^{n+\alpha}} + \frac{(r_Q)^{n+1}}{|x-x_0|^{n+1}} + \frac{(r_Q)^{n+\frac{1}{2}}}{|x-x_0|^{n+\frac{1}{2}}}\right].$$

Setting

$$\begin{aligned}
F(x) &= \frac{(r_Q)^{n+\alpha}}{|x-x_0|^{n+\alpha}[w(Q)]^{1/p}},\\
G(x) &= \frac{(r_Q)^{n+1}}{|x-x_0|^{n+1}[w(Q)]^{1/p}},\\
H(x) &= \frac{(r_Q)^{n+\frac{1}{2}}}{|x-x_0|^{n+\frac{1}{2}}[w(Q)]^{1/p}}.
\end{aligned}$$

Thus, in order to complete the proof of Theorem 1.2, we only need to prove that the following three inequalities hold.

$$\lambda^p \cdot w\big(\big\{x \in (Q^*)^c : \big|F(x)\big| > \lambda\big\}\big) \leqslant C, \tag{4.1}$$

$$\lambda^p \cdot w\big(\big\{x \in (Q^*)^c : \big|G(x)\big| > \lambda\big\}\big) \leqslant C, \tag{4.2}$$

and

$$\lambda^p\cdot w\big(\big\{x\in(Q^*)^c:\big|H(x)\big|>\lambda\big\}\big)\leqslant C. \tag{4.3}$$

Let us start with the inequality (4.1). For any given $\lambda>0$, we are going to consider two cases. If

$$\lambda\geqslant\frac{(r_Q)^{n+\alpha}}{(\sqrt{n}r_Q)^{n+\alpha}[w(Q)]^{1/p}},$$

then for any $x\in(Q^*)^c$, we have $|x-x_0|\geqslant\sqrt{n}r_Q$. Hence, we can easily verify that

$$\big\{x\in(Q^*)^c:\big|F(x)\big|>\lambda\big\}=\emptyset.$$

Therefore, in this case, the inequality

$$\lambda^p\cdot w\big(\big\{x\in(Q^*)^c:\big|F(x)\big|>\lambda\big\}\big)\leqslant C$$

holds trivially. Now suppose that

$$\lambda<\frac{(r_Q)^{n+\alpha}}{(\sqrt{n}r_Q)^{n+\alpha}[w(Q)]^{1/p}}.$$

If we take $R=\dfrac{r_Q}{\lambda^{\frac{1}{n+\alpha}}[w(Q)]^{\frac{1}{p(n+\alpha)}}}$, then it is not difficult to check that $R\geqslant\sqrt{n}r_Q\geqslant r_Q$ and

$$\big\{x\in(Q^*)^c:\big|F(x)\big|>\lambda\big\}\subseteq\big\{x\in\mathbb{R}^n:|x-x_0|<R\big\}\subseteq Q(x_0,2R). \tag{4.4}$$

Since $w\in A_1$, then by Lemma 2.3, we can get (below, $\widetilde{C}$ is an absolute constant)

$$\widetilde{C}\cdot\frac{|Q(x_0,r_Q)|}{|Q(x_0,2R)|}\leqslant\frac{w(Q(x_0,r_Q))}{w(Q(x_0,2R))},$$

which implies

$$w\big(Q(x_0,2R)\big)\leqslant\frac{(2R)^n\cdot w(Q)}{\widetilde{C}\cdot(r_Q)^n}\leqslant\frac{2^n\cdot w(Q)}{\widetilde{C}\cdot\lambda^{\frac{n}{n+\alpha}}[w(Q)]^{\frac{n}{p(n+\alpha)}}}. \tag{4.5}$$

Noting that $\lambda<\dfrac{1}{(\sqrt{n})^{n+\alpha}[w(Q)]^{1/p}}$ and $p-\dfrac{n}{n+\alpha}>0$. Hence, it follows directly from (4.4) and (4.5) that

$$\begin{aligned}\lambda^p\cdot w\big(\big\{x\in(Q^*)^c:\big|F(x)\big|>\lambda\big\}\big)&\leqslant\lambda^p\cdot w\big(Q(x_0,2R)\big)\\&\leqslant C\cdot\lambda^p\cdot\frac{w(Q)}{\lambda^{\frac{n}{n+\alpha}}[w(Q)]^{\frac{n}{p(n+\alpha)}}}\\&\leqslant C.\end{aligned}$$

We now prove the inequality (4.2). Similarly, for any given $\lambda>0$, we will consider the following two cases. If

$$\lambda\geqslant\frac{(r_Q)^{n+1}}{(\sqrt{n}r_Q)^{n+1}[w(Q)]^{1/p}},$$

then as before, we can also show that

$$\big\{x\in(Q^*)^c:\big|G(x)\big|>\lambda\big\}=\varnothing,$$

and so the following estimate holds trivially.

$$\lambda^p\cdot w\big(\big\{x\in(Q^*)^c:\big|G(x)\big|>\lambda\big\}\big)\leqslant C.$$

Now if instead we assume that

$$\lambda<\frac{(r_Q)^{n+1}}{(\sqrt{n}r_Q)^{n+1}[w(Q)]^{1/p}}.$$

In this case, if we take $R'=\dfrac{r_Q}{\lambda^{\frac{1}{n+1}}[w(Q)]^{\frac{1}{p(n+1)}}}$, then it is not difficult to verify that $R'\geqslant\sqrt{n}r_Q\geqslant r_Q$ and

$$\big\{x\in(Q^*)^c:\big|G(x)\big|>\lambda\big\}\subseteq\big\{x\in\mathbb{R}^n:|x-x_0|<R'\big\}\subseteq Q(x_0,2R'). \tag{4.6}$$

Recall that $p=n/(n+\beta)$ and $\beta=\min\{\alpha,1/2\}<1$, then $1<p(1+\frac{1}{n})$. Since $w\in A_1$, then $w\in A_{p(1+\frac{1}{n})}$. Furthermore, by using Lemma 2.3 again, we can get (below, $\widetilde{\widetilde{C}}$ is an absolute constant)

$$\widetilde{\widetilde{C}}\cdot\left(\frac{|Q(x_0,r_Q)|}{|Q(x_0,2R')|}\right)^{p(1+\frac{1}{n})}\leqslant\frac{w(Q(x_0,r_Q))}{w(Q(x_0,2R'))},$$

which in turn gives

$$w\big(Q(x_0,2R')\big)\leqslant\frac{(2R')^{p(n+1)}\cdot w(Q)}{\widetilde{\widetilde{C}}\cdot(r_Q)^{p(n+1)}}\leqslant\frac{2^{p(n+1)}}{\widetilde{\widetilde{C}}\cdot\lambda^p}. \tag{4.7}$$

Therefore, by (4.6) and (4.7), we obtain

$$\lambda^p\cdot w\big(\big\{x\in(Q^*)^c:\big|G(x)\big|>\lambda\big\}\big)\leqslant\lambda^p\cdot w\big(Q(x_0,2R')\big)\leqslant C.$$

By using the same arguments as above, we can also prove the inequality (4.3). Collecting all these estimates and then taking the supremum over all $\lambda>0$, we conclude the proof of Theorem 1.2. □

5. Proof of Theorem 1.3

Proof of Theorem 1.3. The basic idea of the proof is taken from [26]. For any given $\lambda>0$, we may choose $k_0\in\mathbb{Z}$ such that $2^{k_0}\leqslant\lambda<2^{k_0+1}$. For every $f\in WH^p_w(\mathbb{R}^n)$, then by Theorem 2.5, we can write

$$f=\sum_{k=-\infty}^{\infty}f_k=\sum_{k=-\infty}^{k_0}f_k+\sum_{k=k_0+1}^{\infty}f_k:=F_1+F_2,$$

where $F_1 = \sum_{k=-\infty}^{k_0} f_k = \sum_{k=-\infty}^{k_0} \sum_i b_i^k$, $F_2 = \sum_{k=k_0+1}^{\infty} f_k = \sum_{k=k_0+1}^{\infty} \sum_i b_i^k$ and $\{b_i^k\}$ satisfies (a)–(c) in Theorem 2.5. Then we have

$$\begin{aligned}
&\lambda^p \cdot w\big(\{x \in \mathbb{R}^n : |\mu_\Omega^\rho(f)(x)| > \lambda\}\big) \\
\leqslant &\lambda^p \cdot w\big(\{x \in \mathbb{R}^n : |\mu_\Omega^\rho(F_1)(x)| > \lambda/2\}\big) + \lambda^p \cdot w\big(\{x \in \mathbb{R}^n : |\mu_\Omega^\rho(F_2)(x)| > \lambda/2\}\big) \\
= &K_1 + K_2.
\end{aligned}$$

First we claim that the following inequality holds:

$$\|F_1\|_{L^2_w} \leqslant C \cdot \lambda^{1-p/2} \|f\|_{WH^p_w}^{p/2}. \tag{5.1}$$

In fact, since $\operatorname{supp} b_i^k \subseteq Q_i^k = Q(x_i^k, r_i^k)$ and $\|b_i^k\|_{L^\infty} \leqslant C2^k$ by Theorem 2.5, then it follows from Minkowski's integral inequality that

$$\begin{aligned}
\|F_1\|_{L^2_w} &\leqslant \sum_{k=-\infty}^{k_0} \sum_i \|b_i^k\|_{L^2_w} \\
&\leqslant \sum_{k=-\infty}^{k_0} \sum_i \|b_i^k\|_{L^\infty} w(Q_i^k)^{1/2}.
\end{aligned}$$

For each $k \in \mathbb{Z}$, by using the bounded overlapping property of the cubes $\{Q_i^k\}$ and the fact that $1 - p/2 > 0$, we thus obtain

$$\begin{aligned}
\|F_1\|_{L^2_w} &\leqslant C \sum_{k=-\infty}^{k_0} 2^k \Big(\sum_i w(Q_i^k)\Big)^{1/2} \\
&\leqslant C \sum_{k=-\infty}^{k_0} 2^{k(1-p/2)} \|f\|_{WH^p_w}^{p/2} \\
&\leqslant C \sum_{k=-\infty}^{k_0} 2^{(k-k_0)(1-p/2)} \cdot \lambda^{1-p/2} \|f\|_{WH^p_w}^{p/2} \\
&\leqslant C \cdot \lambda^{1-p/2} \|f\|_{WH^p_w}^{p/2}.
\end{aligned}$$

By the hypothesis $w \in A_{p(1+\frac{\beta}{n})}$ and $\beta = \min\{\alpha, 1/2\}$, then we have $1 < p(1+\frac{\beta}{n}) \leqslant 1 + \frac{\beta}{n} \leqslant 2$ and $w \in A_2$. Applying Chebyshev's inequality, Theorem A and the inequality (5.1), we can deduce that

$$\begin{aligned}
K_1 &\leqslant \lambda^p \cdot \frac{4}{\lambda^2} \|\mu_\Omega^\rho(F_1)\|_{L^2_w}^2 \\
&\leqslant C \cdot \lambda^{p-2} \|F_1\|_{L^2_w}^2 \leqslant C \|f\|_{WH^p_w}^p.
\end{aligned}$$

Now we turn our attention to the estimate of K_2. We set

$$A_{k_0} = \bigcup_{k=k_0+1}^{\infty} \bigcup_i \widetilde{Q_i^k},$$

where $\widetilde{Q_i^k} = Q\big(x_i^k, \tau^{(k-k_0)/(n+\beta)}(2\sqrt{n})r_i^k\big)$ and τ is a fixed positive number such that $1 < \tau < 2$. Thus, we can further decompose K_2 as

$$\begin{aligned} K_2 &\leqslant \lambda^p \cdot w\big(\big\{x \in A_{k_0} : \big|\mu_\Omega^\rho(F_2)(x)\big| > \lambda/2\big\}\big) \\ &\quad + \lambda^p \cdot w\big(\big\{x \in (A_{k_0})^c : \big|\mu_\Omega^\rho(F_2)(x)\big| > \lambda/2\big\}\big) \\ &= K_2' + K_2''. \end{aligned}$$

Let us first deal with the term K_2'. Since $w \in A_{p(1+\frac{\beta}{n})}$, then by Lemma 2.1, we can get

$$\begin{aligned} K_2' &\leqslant \lambda^p \sum_{k=k_0+1}^{\infty} \sum_i w\Big(\widetilde{Q_i^k}\Big) \\ &\leqslant C \cdot \lambda^p \sum_{k=k_0+1}^{\infty} \tau^{(k-k_0)p} \sum_i w\big(Q_i^k\big) \\ &\leqslant C\|f\|_{WH_w^p}^p \sum_{k=k_0+1}^{\infty} \Big(\frac{\tau}{2}\Big)^{(k-k_0)p} \leqslant C\|f\|_{WH_w^p}^p. \end{aligned}$$

On the other hand, an application of Chebyshev's inequality leads to that

$$\begin{aligned} K_2'' &\leqslant 2^p \int_{(A_{k_0})^c} \big|\mu_\Omega^\rho(F_2)(x)\big|^p w(x)\,dx \\ &\leqslant 2^p \sum_{k=k_0+1}^{\infty} \sum_i \int_{\big(\widetilde{Q_i^k}\big)^c} \big|\mu_\Omega^\rho\big(b_i^k\big)(x)\big|^p w(x)\,dx. \end{aligned}$$

As before, for $0 < \rho < n$ and for any $x \in \big(\widetilde{Q_i^k}\big)^c$, we write

$$\begin{aligned} \mu_\Omega^\rho\big(b_i^k\big)(x) &= \left(\int_0^\infty \left|\int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} b_i^k(y)\,dy\right|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2} \\ &= \left(\int_0^{|x-x_i^k|+(\sqrt{n})r_i^k} \left|\int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} b_i^k(y)\,dy\right|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2} \\ &\quad + \left(\int_{|x-x_i^k|+(\sqrt{n})r_i^k}^\infty \left|\int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} b_i^k(y)\,dy\right|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2} \\ &= \text{I+II}. \end{aligned}$$

If $y \in Q_i^k$ and $x \in \big(\widetilde{Q_i^k}\big)^c$, then we have $|x-y| \sim |x-x_i^k| \sim |x-x_i^k| + (\sqrt{n})r_i^k$ for all i and k. Thus, for $0 < \rho < n$,

$$\left|\frac{1}{|x-y|^{2\rho}} - \frac{1}{[|x-x_i^k| + (\sqrt{n})r_i^k]^{2\rho}}\right| \leqslant C \cdot \frac{r_i^k}{|x-y|^{2\rho+1}}. \tag{5.2}$$

The above estimate (5.2) together with Minkowski's inequality for integrals implies

$$\begin{aligned}
\mathrm{I} &\leqslant \int_{Q_i^k} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} \big|b_i^k(y)\big| \left(\int_{|x-y|}^{|x-x_i^k|+(\sqrt{n})r_i^k} \frac{dt}{t^{2\rho+1}}\right)^{1/2} dy \\
&\leqslant C \int_{Q_i^k} \frac{|b_i^k(y)|}{|x-y|^{n-\rho}} \left(\frac{r_i^k}{|x-y|^{2\rho+1}}\right)^{1/2} dy \\
&\leqslant C \cdot \frac{(r_i^k)^{1/2}}{|x-x_i^k|^{n+\frac{1}{2}}} \int_{Q_i^k} \big|b_i^k(y)\big|\, dy \\
&\leqslant C \cdot \big\|b_i^k\big\|_{L^\infty} \cdot \frac{(r_i^k)^{n+\frac{1}{2}}}{|x-x_i^k|^{n+\frac{1}{2}}}.
\end{aligned}$$

Moreover, if $t \geqslant |x-x_i^k| + (\sqrt{n})r_i^k$, then we can easily see that $Q_i^k \subset \{y : |x-y| \leqslant t\}$, and $Q_i^k \cap \{y : |x-y| \leqslant t\} = Q_i^k$. Let $q = p(1+\frac{\beta}{n})$ for simplicity. Then for any $n/(n+\beta) < p \leqslant 1$ and $w \in A_q$ with $q > 1$, we can see that $[n(q_w/p-1)] = 0$. Hence, for any $x \in \big(\widetilde{Q_i^k}\big)^c$, by the vanishing moment condition of $b_i^k \in L^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}
\mathrm{II} &= \left(\int_{|x-x_i^k|+(\sqrt{n})r_i^k}^{\infty} \left|\int_{Q_i^k} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} b_i^k(y)\, dy\right|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2} \\
&= \left(\int_{|x-x_i^k|+(\sqrt{n})r_i^k}^{\infty} \left|\int_{Q_i^k} \left[\frac{\Omega(x-y)}{|x-y|^{n-\rho}} - \frac{\Omega(x-x_i^k)}{|x-x_i^k|^{n-\rho}}\right] b_i^k(y)\, dy\right|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2} \\
&\leqslant C\left(\int_{|x-x_i^k|+(\sqrt{n})r_i^k}^{\infty} \left[\int_{Q_i^k} \left|\frac{1}{|x-y|^{n-\rho}} - \frac{1}{|x-x_i^k|^{n-\rho}}\right| \big|b_i^k(y)\big|\, dy\right]^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2} \\
&\quad + \left(\int_{|x-x_i^k|+(\sqrt{n})r_i^k}^{\infty} \left[\int_{Q_i^k} \frac{|\Omega(x-y)-\Omega(x-x_i^k)|}{|x-x_i^k|^{n-\rho}} \big|b_i^k(y)\big|\, dy\right]^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2} \\
&= \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

Note that for any $y \in Q_i^k$ and $x \in \big(\widetilde{Q_i^k}\big)^c$, then $|x-y| \sim |x-x_i^k|$ for all i and k. This fact together with the mean value theorem yields

$$\begin{aligned}
\mathrm{III} &\leqslant C\left(\int_{|x-x_i^k|+(\sqrt{n})r_i^k}^{\infty} \left[\int_{Q_i^k} \frac{|y-x_i^k|}{|x-x_i^k|^{n-\rho+1}} \big|b_i^k(y)\big|\, dy\right]^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2} \\
&\leqslant C \cdot \frac{r_i^k}{|x-x_i^k|^{n-\rho+1}} \int_{Q_i^k} \big|b_i^k(y)\big|\, dy \times \left(\int_{|x-x_i^k|+(\sqrt{n})r_i^k}^{\infty} \frac{dt}{t^{2\rho+1}}\right)^{1/2} \\
&\leqslant C \cdot \frac{r_i^k}{|x-x_i^k|^{n+1}} \cdot \int_{Q_i^k} \big|b_i^k(y)\big|\, dy \\
&\leqslant C \cdot \big\|b_i^k\big\|_{L^\infty} \cdot \frac{(r_i^k)^{n+1}}{|x-x_i^k|^{n+1}}.
\end{aligned}$$

In addition, from the definition of $\Omega \in Lip_\alpha(S^{n-1})$, we can easily see that

$$\begin{aligned}
\left|\Omega(x-y)-\Omega(x-x_i^k)\right| &= \left|\Omega\Big(\frac{x-y}{|x-y|}\Big)-\Omega\Big(\frac{x-x_i^k}{|x-x_i^k|}\Big)\right| \\
&\leqslant C\left|\frac{x-y}{|x-y|}-\frac{x-x_i^k}{|x-x_i^k|}\right|^\alpha \\
&\leqslant C\left(\frac{|y-x_i^k|}{|x-x_i^k|}\right)^\alpha. \qquad (5.3)
\end{aligned}$$

Substituting the above inequality (5.3) into the term IV, then we can get

$$\begin{aligned}
\mathrm{IV} &\leqslant C\left(\int_{|x-x_i^k|+(\sqrt{n})r_i^k}^\infty\left[\int_{Q_i^k}\frac{|y-x_i^k|^\alpha}{|x-x_i^k|^{n-\rho+\alpha}}\left|b_i^k(y)\right|dy\right]^2\frac{dt}{t^{2\rho+1}}\right)^{1/2} \\
&\leqslant C\cdot\frac{(r_i^k)^\alpha}{|x-x_i^k|^{n-\rho+\alpha}}\int_{Q_i^k}\left|b_i^k(y)\right|dy\times\left(\int_{|x-x_i^k|+(\sqrt{n})r_i^k}^\infty\frac{dt}{t^{2\rho+1}}\right)^{1/2} \\
&\leqslant C\cdot\left\|b_i^k\right\|_{L^\infty}\cdot\frac{(r_i^k)^{n+\alpha}}{|x-x_i^k|^{n+\alpha}}.
\end{aligned}$$

Summarizing the above estimates for I, III and IV, for any $x\in\big(\widetilde{Q_i^k}\big)^c$, we have

$$\left|\mu_\Omega^\rho(b_i^k)(x)\right|\leqslant C\cdot\left\|b_i^k\right\|_{L^\infty}\left[\frac{(r_i^k)^{n+\frac12}}{|x-x_i^k|^{n+\frac12}}+\frac{(r_i^k)^{n+1}}{|x-x_i^k|^{n+1}}+\frac{(r_i^k)^{n+\alpha}}{|x-x_i^k|^{n+\alpha}}\right].$$

Note that $\left\|b_i^k\right\|_{L^\infty}\leqslant C2^k$. Therefore, from the above pointwise estimate, it follows that

$$\begin{aligned}
K_2'' &\leqslant C\sum_{k=k_0+1}^\infty\sum_i 2^{kp}\left(r_i^k\right)^{(n+\alpha)p}\int_{|x-x_i^k|\geqslant\tau^{(k-k_0)/(n+\beta)}\sqrt{n}r_i^k}\frac{w(x)}{|x-x_i^k|^{(n+\alpha)p}}\,dx \\
&\quad+C\sum_{k=k_0+1}^\infty\sum_i 2^{kp}\left(r_i^k\right)^{(n+1)p}\int_{|x-x_i^k|\geqslant\tau^{(k-k_0)/(n+\beta)}\sqrt{n}r_i^k}\frac{w(x)}{|x-x_i^k|^{(n+1)p}}\,dx \\
&\quad+C\sum_{k=k_0+1}^\infty\sum_i 2^{kp}\left(r_i^k\right)^{(n+\frac12)p}\int_{|x-x_i^k|\geqslant\tau^{(k-k_0)/(n+\beta)}\sqrt{n}r_i^k}\frac{w(x)}{|x-x_i^k|^{(n+\frac12)p}}\,dx \\
&= K_3+K_4+K_5.
\end{aligned}$$

Let us consider the term K_3. Recall that $\beta=\min\{\alpha,1/2\}$ and $n/(n+\beta)<p\leqslant 1$, then $n/(n+\alpha)<p\leqslant 1$. Since $w\in A_{p(1+\frac{\beta}{n})}$ and $\beta=\min\{\alpha,1/2\}\leqslant\alpha$, then we have

$w \in A_{p(1+\frac{\alpha}{n})}$. By using Lemma 2.1 and Lemma 2.2, then we can deduce

$$\begin{aligned} K_3 &= C\sum_{k=k_0+1}^{\infty}\sum_i 2^{kp}\left(r_i^k\right)^{(n+\alpha)p}\int_{|y|\geqslant \tau^{(k-k_0)/(n+\beta)}\sqrt{n}r_i^k}\frac{w_i^k(y)}{|y|^{(n+\alpha)p}}\,dy \\ &\leqslant C\sum_{k=k_0+1}^{\infty}\sum_i 2^{kp}\left(\tau^{(k-k_0)/(n+\beta)}\right)^{-(n+\alpha)p} w_i^k\Big(Q\big(0,\tau^{(k-k_0)/(n+\beta)}\cdot r_i^k\big)\Big) \\ &= C\sum_{k=k_0+1}^{\infty}\sum_i 2^{kp}\left(\tau^{(k-k_0)/(n+\beta)}\right)^{-(n+\alpha)p} w\Big(Q\big(x_i^k,\tau^{(k-k_0)/(n+\beta)}\cdot r_i^k\big)\Big), \end{aligned}$$

where $w_i^k(x) = w(x+x_i^k)$ is the translation of $w(x)$. It is obvious that $w_i^k \in A_{p(1+\frac{\alpha}{n})}$ whenever $w \in A_{p(1+\frac{\alpha}{n})}$, and $q_{w_i^k} = q_w$. In addition, for $w \in A_{p(1+\frac{\alpha}{n})}$ with $p(1+\frac{\alpha}{n}) > 1$, then we can take a sufficiently small number $\varepsilon > 0$ such that $p(1+\frac{\alpha}{n}) - \varepsilon \geqslant 1$ and $w \in A_{p(1+\frac{\alpha}{n})-\varepsilon}$. Therefore, by using Lemma 2.1 again, we eventually obtain

$$\begin{aligned} K_3 &\leqslant C\sum_{k=k_0+1}^{\infty}\sum_i 2^{kp}\left(\tau^{(k-k_0)/(n+\beta)}\right)^{-n\varepsilon} w\big(Q_i^k\big) \\ &\leqslant C\big\|f\big\|_{WH_w^p}^p \sum_{k=k_0+1}^{\infty}\left(\tau^{(k-k_0)/(n+\beta)}\right)^{-n\varepsilon} \\ &\leqslant C\big\|f\big\|_{WH_w^p}^p. \end{aligned}$$

For the last two terms K_4 and K_5, since $w \in A_{p(1+\frac{\beta}{n})}$ with $\beta = \min\{\alpha, 1/2\}$ and $0 < \alpha \leqslant 1$, then we have $w \in A_{p(1+\frac{1}{n})}$ and $w \in A_{p(1+\frac{1}{2n})}$. Thus, by using the same arguments as above, we can also show that $K_4 \leqslant C\big\|f\big\|_{WH_w^p}^p$ and $K_5 \leqslant C\big\|f\big\|_{WH_w^p}^p$. Combining the above estimates for K_1 and K_2, and then taking the supremum over all $\lambda > 0$, we conclude the proof of Theorem 1.3. □

We finally remark that for any function f, a straightforward computation shows that the grand maximal function of f is pointwise dominated by $M(f)$, where M denotes the standard Hardy–Littlewood maximal operator. Hence, by the weighted weak (1,1) estimate of M, it is easy to see that the space L_w^1 is continuously embedded as a subspace of WH_w^1 whenever $w \in A_1$, and we have $\|f\|_{WH_w^1} \leqslant C\|f\|_{L_w^1}$ provided that $w \in A_1$. As a direct consequence of Theorem 1.3, we immediately obtain the following result.

COROLLARY 5.1. *Let $0 < \rho < n$, $0 < \alpha \leqslant 1$ and $\Omega \in Lip_\alpha(S^{n-1})$. If $p = 1$ and $w \in A_1$, then there exists a constant $C > 0$ independent of f such that*

$$\big\|\mu_\Omega^\rho(f)\big\|_{WL_w^1} \leqslant C\|f\|_{L_w^1}.$$

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