Journal of
Mathematical
Inequalities
Volume 10, Number 2 (2016), 415–421 doi:10.7153/jmi-10-32

WEIGHTED POINCARÉ INEQUALITIES ON HALF SPACES IN CARNOT GROUPS

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(*Communicated by M. Plum*)

Abstract. We prove some weighted Poincaré inequalities on half spaces for the sublaplacian in Carnot groups. Furthermore, the constants we obtain are sharp.

1. Introduction

The Hardy inequality in $\mathbb{R}^N_+$ reads as follows, for all $u \in C_0^\infty(\mathbb{R}^N_+)$ and $N \geqslant 3$,

$$\int_{\mathbb{R}^N_+} |\nabla u|^2 dx \geqslant \frac{N^2}{4} \int_{\mathbb{R}^N_+} \frac{u^2}{|x|^2} dx, \tag{1}$$

where $\mathbb{R}^N_+ = \{(x_1, \cdots, x_n) | x_1 > 0\}$, and the constant $\frac{N^2}{4}$ in (1) is sharp (see [5] or [4]). This shows that the Hardy constant jumps from $\frac{(N-2)^2}{4}$ to $\frac{N^2}{4}$ when the singularity of the potential reaches the boundary since the Hardy inequality in $\mathbb{R}^N$ is

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geqslant \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx. \tag{2}$$

Inequality (1) has been generalized by Su et al([7]) to the cone $\mathbb{R}^N_{k_+} := \mathbb{R}^{N-k} \times (\mathbb{R}_+)^k = \{(x_1, \cdots, x_N) | x_{N-k+1} > 0, \cdots, x_N > 0\}$.

Recently, F. Ferrari and E. Valdinoci ([3]) proved few Poincaré inequalities through the stable solutions of suitable PDEs. By the choice of suitable stable solutions, they obtain some weighted Poincaré inequalities, among other results, for the Kohn's sub-laplace operator in the Heisenberg group $\mathbb{H}^n$ and the sublaplace operator in the Engel group. The sharp constants had been shown by Yang and the first author ([9], see also [8]).

The aim of this note is to prove similar results for the sublaplacian on half space in Carnot groups G. Recall that a Carnot group G is a stratified, simply connected nilpotent Lie group with the Lie algebra $\mathfrak{g} = \bigoplus_{i=1}^r V_i$ satisfying $[V_1, V_j] = V_{j+1}$ for all $1 \leqslant j \leqslant r-1$. The integer r is called the step of the group G. Set $n_j = \dim V_j$ $(1 \leqslant j \leqslant r)$. Let $\{X_1, \cdots, X_{n_1}\}$ be a basis of V_1 and denote by $\nabla_G = (X_1, \cdots, X_{n_1})$. Set $\xi^{(1)} = \xi_1 X_1 + \cdots + \xi_{n_1} X_{n_1}$ and $|\xi^{(1)}| = \sqrt{\xi_1^2 + \cdots + \xi_{n_1}^2}$. To this end, we have

Mathematics subject classification (2010): 26D15, 35H20.

Keywords and phrases: Poincaré inequalities, sharp constant, Carnot groups.

This research is supported by the Education Department of Hubei province science and technology research project (D20131108) and the National Natural Science Foundation of China (No. 11201346).

THEOREM 1. *Let $\alpha \geqslant 0$. If G is a Carnot group of step r, then for $\alpha \geqslant 0$ and $\phi \in C_0^\infty(G_+)$, there holds,*

$$\left(\frac{(n_1+2+\alpha)^2}{4}-\alpha-2\right)\int_{G_+}|\phi|^2|\xi^{(1)}|^\alpha \leqslant \int_{G_+}|\nabla_G\phi|^2|\xi^{(1)}|^{\alpha+2}, \tag{3}$$

where $G_+ = \{\xi \in G : \xi_1 > 0\}$ is the half space of G. Furthermore, the constant $\frac{(n_1+2+\alpha)^2}{4}-\alpha-2$ in (3) is sharp.

In order to prove Theorem 1.1, we use a new technique which is different from that in [4, 5, 7]. In fact, it seems that the method used in [4, 5, 7] can not be applied to the sublaplacian on Carnot groups. Our result also shows that the sharp constant jumps since the sharp Poincaré inequalities with weights on Carnot groups is (see [8])

$$\left(\frac{n_1+\alpha}{2}\right)^2\int_G|\phi|^2|\xi^{(1)}|^\alpha \leqslant \int_G|\nabla_G\phi|^2|\xi^{(1)}|^{\alpha+2}. \tag{4}$$

2. The proof

We begin by quoting some preliminary facts which will be needed in the sequel and refer to [1] and [2] for more precise information about Carnot group. Let G be a Carnot groups. The Lie algebra $\mathfrak{g} = \bigoplus_{i=1}^r V_i$ of G satisfies $[V_1,V_j] = V_{j+1}$ for all $1 \leqslant j \leqslant r-1$. As a simply connected nilpotent group, G is differential with $\mathbb{R}^N$, $N = \sum_{i=1}^r \dim V_i = \sum_{i=1}^r n_i$, via the exponential map $\exp : \mathfrak{g} \to G$. The Haar measure on G is induced by the exponential mapping from the Lebesgue measure on $\mathfrak{g} = \mathbb{R}^N$ and coincides with the Lebesgue measure on $\mathbb{R}^N$.

EXAMPLE 1. The Heisenberg group $\mathbb{H}^n$ is the Carnot group of step two whose group structure is given by

$$(x,t)\circ(x',t') = (x+x', t+t'+2\sum_{j=1}^n (x_{2j}x'_{2j-1}-x_{2j-1}x'_{2j})).$$

The vector fields

$$X_{2j-1} = \frac{\partial}{\partial x_{2j-1}} + 2x_{2j}\frac{\partial}{\partial t},$$
$$X_{2j} = \frac{\partial}{\partial x_{2j}} - 2x_{2j-1}\frac{\partial}{\partial t},$$

$(j = 1,\cdots,n)$ are left invariant and generate the Lie algebra of $\mathbb{H}^n$. The horizontal gradient on $\mathbb{H}^n$ is the $(2n)$-dimensional vector given by

$$\nabla_{\mathbb{H}} = (X_1,\cdots,X_{2n}) = \nabla_x + 2\Lambda x\frac{\partial}{\partial t},$$

where $\nabla_x = (\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{2n}})$, Λ is a skew symmetric and orthogonal matrix given by

$$\Lambda = \mathrm{diag}(J_1,\cdots,J_n),\ \ J_1 = \cdots = J_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Consider $\xi = (\xi^{(1)}, \cdots, \xi^{(r)}) \in \mathbb{R}^N$ with $\xi^{(i)} = (\xi_1^{(i)}, \cdots, \xi_{n_i}^{(i)}) \in \mathbb{R}^{n_i}$. For $j = 1, \cdots, n_1$, let X_j be the unique vector field in $\mathfrak{g}$ that coincides with $\partial/\partial\xi_j^{(1)}$ at the origin. The second order differential operator

$$\Delta_G = -\sum_{j=1}^{n_1} X_j^* X_j = \sum_{j=1}^{n_1} X_j^2$$

is called a sub-Laplacian on G. We shall denote by the $\nabla_G = (X_1, \cdots, X_{n_1})$ the related subelliptic gradient. By the Campbell-Hausdorff formula (see e.g. [6], page 2-4), $X_j\,(1 \leqslant j \leqslant n_1)$ can be expressed as the following

$$X_j = \frac{\partial}{\partial\xi_j^{(1)}} + \sum_{k=2}^{r}\sum_{s=1}^{n_k} p_{k,s}^j(\xi^{(1)}, \cdots, \xi^{(k-1)}) \frac{\partial}{\partial\xi_s^{(k)}}, \tag{5}$$

where $p_{k,s}^j(\xi^{(1)}, \cdots, \xi^{(k-1)})$ is a polynomial of $\xi^{(1)}, \cdots, \xi^{(k-1)}$. Therefore, for $|\xi^{(1)}| \neq 0$, we have

$$\begin{aligned}
\Delta_G(\xi_1|\xi^{(1)}|^\alpha) =& \sum_{j=1}^{n_1}\left(\frac{\partial}{\partial\xi_j^{(1)}}\right)^2 (\xi_1|\xi^{(1)}|^\alpha)\\
=&\xi_1 \sum_{j=1}^{n_1}\left(\frac{\partial}{\partial\xi_j^{(1)}}\right)^2 |\xi^{(1)}|^\alpha + 2\frac{\partial|\xi^{(1)}|^\alpha}{\partial\xi_1^{(1)}}\\
=&\alpha(n_1+\alpha-2)\xi_1|\xi^{(1)}|^{\alpha-2} + 2\alpha\xi_1|\xi^{(1)}|^{\alpha-2}\\
=&\alpha(n_1+\alpha)\xi_1|\xi^{(1)}|^{\alpha-2}.
\end{aligned} \tag{6}$$

Before the proof of main results, we need the following Lemma.

LEMMA 1. *Let $f \in C^\infty(\mathbb{R}_+^N)$ and $\alpha \geqslant 0$. There holds*

$$\left(\frac{(N+2+\alpha)^2}{4} - \alpha - 2\right)\int_{\mathbb{R}_+^N} |x|^\alpha f^2 dx \leqslant \int_{\mathbb{R}_+^N} |\nabla f|^2 |x|^{\alpha+2} dx. \tag{7}$$

Furthermore, the constant $\frac{(N+2+\alpha)^2}{4} - \alpha - 2$ is sharp.

Proof. Replacing u by $|x|^{\frac{\alpha+2}{2}} f$ in (1), we have

$$\int_{\mathbb{R}_+^N} |\nabla(|x|^{\frac{\alpha+2}{2}} f)|^2 dx \geqslant \frac{N^2}{4}\int_{\mathbb{R}_+^N} f^2|x|^\alpha dx. \tag{8}$$

We compute

$$\begin{aligned}\int_{\mathbb{R}^N_+}|\nabla(|x|^{\frac{\alpha+2}{2}}f)|^2dx&=\int_{\mathbb{R}^N_+}\left(|\nabla f|^2|x|^{\alpha+2}+f^2|\nabla|x|^{\frac{\alpha+2}{2}}|^2+\frac{1}{2}\langle\nabla|x|^{\alpha+2},\nabla f^2\rangle\right)dx\\&=\int_{\mathbb{R}^N_+}\left(|\nabla f|^2|x|^{\alpha+2}+\frac{(\alpha+2)^2}{4}f^2|x|^{\alpha}\right)dx\\&\quad-\frac{1}{2}\int_{\mathbb{R}^N_+}f^2\Delta|x|^{\alpha+2}dx\\&=\int_{\mathbb{R}^N_+}|\nabla f|^2|x|^{\alpha+2}dx-\frac{(\alpha+2)(2N+\alpha+2)}{4}\int_{\mathbb{R}^N_+}f^2|x|^{\alpha}dx.\end{aligned}\tag{9}$$

Combing (8) and (9) yields (7). the constant $\frac{(N+2+\alpha)^2}{4}-\alpha-2$ is sharp since the constant $\frac{N^2}{4}$ in (1) is sharp. $\square$

Now we can prove Theorem 1. Using the substitution $u=|\xi^{(1)}|^{\frac{\alpha+2}{2}}\phi$, we get

$$\begin{aligned}&\int_{G_+}|\nabla_G\phi|^2|\xi^{(1)}|^{\alpha+2}=\int_G\left|\nabla_G(|\xi^{(1)}|^{-\frac{\alpha+2}{2}}u)\right|^2|\xi^{(1)}|^{\alpha+2}\\&=\int_{G_+}\left(|\nabla_Gu|^2|\xi^{(1)}|^{-\alpha-2}+u^2|\nabla_G|\xi^{(1)}|^{-\frac{\alpha+2}{2}}|^2+\frac{1}{2}\left\langle\nabla_G|\xi^{(1)}|^{-\alpha-2},\nabla_Gu^2\right\rangle\right)|\xi^{(1)}|^{\alpha+2}\\&=\int_{G_+}\left(|\nabla_Gu|^2+\frac{(\alpha+2)^2}{4}\frac{u^2}{|\xi^{(1)}|^2}\right)-\frac{\alpha+2}{2}\int_{G_+}\langle\nabla_G\ln|\xi^{(1)}|,\nabla_Gu^2\rangle\\&=\int_{G_+}\left(|\nabla_Gu|^2+\frac{(\alpha+2)^2}{4}\frac{u^2}{|\xi^{(1)}|^2}\right)+\frac{\alpha+2}{2}\int_{G_+}u^2\Delta_G\ln|\xi^{(1)}|\\&=\int_{G_+}\left(|\nabla_Gu|^2+\frac{(\alpha+2)^2}{4}\frac{u^2}{|\xi^{(1)}|^2}+\frac{(\alpha+2)(n_1-2)}{2}\frac{u^2}{|\xi^{(1)}|^2}\right).\end{aligned}\tag{10}$$

To get the last equation, we use the fact

$$\Delta_G\ln|\xi^{(1)}|=\sum_{j=1}^{n_1}\left(\frac{\partial}{\partial\xi_j^{(1)}}\right)^2\ln|\xi^{(1)}|=\frac{n_1-2}{2}\frac{1}{|\xi^{(1)}|^2}.$$

On the other hand, using the substitution $u=\xi_1|\xi^{(1)}|^{-\frac{n_1}{2}}v$, we have

$$\begin{aligned}\int_{G_+}|\nabla_Gu|^2&=\int_{G_+}\left(|\nabla_Gv|^2\xi_1^2|\xi^{(1)}|^{-n_1}+v^2\left|\nabla_G\left(\xi_1|\xi^{(1)}|^{-\frac{n_1}{2}}\right)\right|^2\right.\\&\qquad\left.+\frac{1}{2}\left\langle\nabla_Gv^2,\nabla_G\left(\xi_1^2|\xi^{(1)}|^{-n_1}\right)\right\rangle\right)\\&\geqslant\int_{G_+}\left(v^2\left|\nabla_G\left(\xi_1|\xi^{(1)}|^{-\frac{n_1}{2}}\right)\right|^2-\frac{1}{2}v^2\Delta_G\left(\xi_1^2|\xi^{(1)}|^{-n_1}\right)\right)\\&=\int_{G_+}v^2\left(\left|\nabla_G\left(\xi_1|\xi^{(1)}|^{-\frac{n_1}{2}}\right)\right|^2-\frac{1}{2}\Delta_G\left(\xi_1^2|\xi^{(1)}|^{-n_1}\right)\right).\end{aligned}\tag{11}$$

Notice that, for $g \in C^2(G)$,

$$\Delta_G g^2 = \sum_{j=1}^{n_1} X_j^2 g^2 = 2g \sum_{j=1}^{n_1} X_j^2 g + 2\sum_{j=1}^{n_1} |X_j g|^2 = 2g\Delta_G g + 2|\nabla_G g|^2.$$

We have, by (6),

$$\begin{aligned}
&\left|\nabla_G\left(\xi_1|\xi^{(1)}|^{-\frac{n_1}{2}}\right)\right|^2 - \frac{1}{2}\Delta_G\left(\xi_1^2|\xi^{(1)}|^{-n_1}\right)\\
=&-\xi_1|\xi^{(1)}|^{-\frac{n_1}{2}}\Delta_G(\xi_1|\xi^{(1)}|^{-\frac{n_1}{2}})\\
=&\frac{n_1^2}{4}\xi_1^2|\xi^{(1)}|^{-n_1-2}.
\end{aligned} \tag{12}$$

Combining (11) and (12) yields

$$\int_{G_+} |\nabla_G u|^2 \geqslant \frac{n_1^2}{4}\int_{G_+} \xi_1^2|\xi^{(1)}|^{-n_1-2}v^2 = \frac{n_1^2}{4}\int_{G_+} \frac{u^2}{|\xi^{(1)}|^2}. \tag{13}$$

Therefore, we have, by (10) and (13),

$$\begin{aligned}
\int_{G_+} |\nabla_G \phi|^2|\xi^{(1)}|^{\alpha+2} \geqslant& \int_{G_+}\left(\frac{n_1^2}{4}\frac{u^2}{|\xi^{(1)}|^2} + \frac{(\alpha+2)^2}{4}\frac{u^2}{|\xi^{(1)}|^2} + \frac{(\alpha+2)(n_1-2)}{2}\frac{u^2}{|\xi^{(1)}|^2}\right).\\
=&\left(\frac{(n_1+2+\alpha)^2}{4} - \alpha - 2\right)\int_{G_+}\frac{u^2}{|\xi^{(1)}|^2}\\
=&\left(\frac{(n_1+2+\alpha)^2}{4} - \alpha - 2\right)\int_{G_+}|\phi|^2|\xi^{(1)}|^{\alpha}.
\end{aligned} \tag{14}$$

To finish the proof, we need to show that the constant $\frac{(n_1+2+\alpha)^2}{4} - \alpha - 2$ in (3) is sharp. Following [8], we choose the test function $f(\xi) = u(\xi^{(1)})g(\xi^{(2)},\cdots,\xi^{(r)})$, where $u(\cdot) \in C_0^\infty(\mathbb{R}^{n_1})$ and $g = \prod_{k=2}^{r}\prod_{s=1}^{n_k} w_{k,s}(\xi_s^{(k)})$ with $w_{k,s}(\cdot) \in C_0^\infty(\mathbb{R})$ for all $2 \leqslant k \leqslant r$ and $1 \leqslant s \leqslant n_l$. For convenience, we set $w_{k,s} \equiv 1$ for all $1 \leqslant s \leqslant n_k$ if $k = 1$. By (5),

$$\begin{aligned}
\int_{G_+} |\nabla_G f|^2|\xi^{(1)}|^{\alpha+2}d\xi = \sum_{j=1}^{n_1}\Bigg\{&\int_{G_+}\left(\frac{\partial u}{\partial \xi_j^{(1)}}\right)^2 g^2|\xi^{(1)}|^{\alpha+2}d\xi +\\
&\int_{G_+} u^2\left|\sum_{k=2}^{r}\sum_{s=1}^{n_k} p_{k,s}^j\frac{\partial g}{\partial \xi_s^{(k)}}\right|^2|\xi^{(1)}|^{\alpha+2}d\xi + (*)\Bigg\},
\end{aligned}$$

where

$$\begin{aligned}
(*) =& \int_{G_+}\frac{\partial u^2}{\partial \xi_j^{(1)}} g \sum_{k=2}^{r}\sum_{s=1}^{n_k} p_{k,s}^j \frac{\partial g}{\partial \xi_s^{(k)}}|\xi^{(1)}|^{\alpha+2}d\xi\\
=&\frac{1}{2}\sum_{k=2}^{r}\sum_{s=1}^{n_k}\int_{G_+}\frac{\partial u^2}{\partial \xi_j^{(1)}} p_{k,s}^j(\xi^{(1)},\cdots,\xi^{(k-1)})\frac{\partial g^2}{\partial \xi_s^{(k)}}|\xi^{(1)}|^{\alpha+2}d\xi.
\end{aligned}$$

Since $w_{k,s}(\cdot) \in C_0^\infty(\mathbb{R})$ for all $2 \leqslant k \leqslant r$ and $1 \leqslant s \leqslant n_k$, we have

$$\int_{\mathbb{R}} \frac{\partial w_{k,s}^2(\xi_s^{(k)})}{\partial \xi_s^{(k)}} d\xi_s^{(k)} = 0.$$

Therefore

$$\int_{\mathbb{R}} \frac{\partial g^2}{\partial \xi_s^{(k)}} d\xi_s^{(k)} = 0.$$

Thus $(*) = 0$ and

$$\begin{aligned}\int_{G_+} |\nabla_G f|^2 |\xi^{(1)}|^{\alpha+2} d\xi = \sum_{j=1}^{n_1} \Bigg\{ & \int_{G_+} \left(\frac{\partial u}{\partial \xi_j^{(1)}} \right)^2 g^2 |\xi^{(1)}|^{\alpha+2} d\xi + \\ & \int_{G_+} u^2 \left| \sum_{k=2}^{r} \sum_{s=1}^{n_k} p_{k,s}^j \frac{\partial g}{\partial \xi_s^{(k)}} \right|^2 |\xi^{(1)}|^{\alpha+2} d\xi \Bigg\}.\end{aligned} \tag{15}$$

Let C_G be a positive constant such that

$$\left| \sum_{k=2}^{r} \sum_{s=1}^{n_k} p_{k,s}^j \frac{\partial g}{\partial \xi_s^{(k)}} \right|^2 \leqslant C_G \sum_{k=2}^{r} \sum_{s=1}^{n_k} \left| p_{k,s}^j \frac{\partial g}{\partial \xi_s^{(k)}} \right|^2.$$

We have, by (15),

$$\begin{aligned}&\frac{\int_{G_+} |\nabla_G f|^2 |\xi^{(1)}|^{\alpha+2} d\xi}{\int_{G_+} f^2 |\xi^{(1)}|^{\alpha} d\xi} \\ \leqslant & \frac{\int_{\mathbb{R}_+^{n_1}} \sum_{j=1}^{n_1} \left(\frac{\partial u}{\partial \xi_j^{(1)}} \right)^2 |\xi^{(1)}|^{\alpha+2} d\xi^{(1)}}{\int_{\mathbb{R}_+^{n_1}} u^2 |\xi^{(1)}|^{\alpha} d\xi^{(1)}} + C_G \sum_{k=2}^{r} \sum_{s=1}^{n_k} \frac{\int_G u^2 \left| p_{k,s}^j \frac{\partial g}{\partial \xi_s^{(k)}} \right|^2 |\xi^{(1)}|^{\alpha+2} d\xi}{\int_G f^2 |\xi^{(1)}|^{\alpha} d\xi},\end{aligned}$$

Where $\mathbb{R}_+^{n_1} = \{(\xi_1, \cdots, \xi_{n_1}) \in \mathbb{R}^{n_1} : \xi_1 > 0\}$.

Since for all $2 \leqslant k \leqslant r$ and $1 \leqslant s \leqslant n_l$,

$$\inf_{w_{k,s} \in C_0^\infty(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} |w'_{k,s}|^2 d\xi_s^{(k)}}{\int_{\mathbb{R}} |w_{k,s}|^2 d\xi_s^{(k)}} = 0,$$

we have

$$\begin{aligned}&\inf_{w_{k,s} \in C_0^\infty(\mathbb{R}) \setminus \{0\}} \frac{\int_{G_+} u^2 \left| p_{k,s}^j \frac{\partial g}{\partial \xi_s^{(k)}} \right|^2 |\xi^{(1)}|^{\alpha+2} d\xi}{\int_{G_+} f^2 |\xi^{(1)}|^{\alpha} d\xi} \\ = & \frac{\int_{\mathbb{R}_+^{n_1} \times \mathbb{R}^{n_2 + \cdots + n_{k-1}}} u^2 |p_{k,s}^j|^2 \prod_{l=1}^{k-1} \prod_{i=1}^{n_l} w_{l,i}^2 |\xi^{(1)}|^{\alpha+2}}{\int_{\mathbb{R}_+^{n_1} \times \mathbb{R}^{n_2 + \cdots + n_{k-1}}} u^2 \prod_{l=1}^{k-1} \prod_{i=1}^{n_l} w_{l,i}^2 |\xi^{(1)}|^{\alpha}} \cdot \inf_{w_{k,s} \in C_0^\infty(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} |w'_{k,s}|^2 d\xi_s^{(k)}}{\int_{\mathbb{R}} |w_{k,s}|^2 d\xi_s^{(k)}} = 0.\end{aligned}$$

Therefore, by Lemma 1,

$$\inf_{\phi\in C_0^\infty(G_+)\setminus\{0\}} \frac{\int_{G_+}|\nabla_G\phi|^p|\xi^{(1)}|^{\alpha+2}}{\int_{G_+}|\phi|^p|\xi^{(1)}|^{\alpha}} \leqslant \inf_{u\in C_0^\infty(\mathbb{R}_+^{n_1})\setminus\{0\}} \frac{\int_{\mathbb{R}_+^{n_1}}\sum_{j=1}^{n_1}\left(\frac{\partial u}{\partial \xi_j^{(1)}}\right)^2|\xi^{(1)}|^{\alpha+2}d\xi^{(1)}}{\int_{\mathbb{R}_+^{n_1}}u^2|\xi^{(1)}|^{\alpha}d\xi^{(1)}}$$
$$= \left(\frac{(n_1+2+\alpha)^2}{4}-\alpha-2\right).$$

The proof of Theorem 1 is thereby completed.

REFERENCES

[1] A. BONFIGLIOLI, E. LANCONELLI AND F. UGUZZONI, *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*, Springer Monographs in Mathematics, Springer, Berlin, 2007.

[2] L. CAPOGNA, D. DANIELLI, S. PAULS, J. TYSON, *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, in: Progress in Mathematics, vol. 259, Birkhäser Verlag, Basel, 2007.

[3] F. FERRARI AND E. VALDINOCI, *Some weighted Poincaré inequalities*, Indiana Univ. Math. J., **58**, 4 (2009), 1619–1637.

[4] S. FILIPPAS, A. TERTIKAS AND J. TIDBLOM, *On the structure of Hardy-Sobolev-Maz'ya inequalities*, J. Eur. Math. Soc., **11**, 6 (2009), 1165–1185.

[5] A. I. NAZAROV, *Hardy-Sobolev inequalities in a cone*, J. Math. Sci., **132**, 4 (2006), 419–427.

[6] G. B. FOLLAND AND E. M. STEIN, *Hardy spaces on homogeneous groups*, Princeton University Press, Princeton, NJ, 1982.

[7] D. SU, Q. YANG, *On the best constants of Hardy inequality in $\mathbb{R}^{n-k}\times(\mathbb{R}_+)^k$ and related improvements*, J. Math. Anal. Appl., **389**, (2012), 48–53.

[8] Y. XIAO, *Some sharp Poincaré inequalities with weights on Carnot groups*, Bull. Malays. Math. Sci. Soc., **37**, 3 (2014), 661–669.

[9] Q. YANG, B. LIAN, *On the best constant of weighted Poincaré inequalities*, J. Math. Anal. Appl., **377**, (2011), 207–215.

(Received November 9, 2014)

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