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INEQUALITIES WITH CURVATURE AND THEIR STABILITY ESTIMATES FOR CONVEX CURVES

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Abstract. In this paper, we deal with a geometric inequality for a closed convex plane curve γ involving the area, the perimeter, the curvature of γ. Together with an inequality obtained by Lin and Tsai in [12], we conclude the upper and lower bound estimates for the integration of the squared curvature radius of γ and show the stability results of the corresponding inequalities.

1. Introduction

There are many inequalities in convex geometry and differential geometry, such as the classical isoperimetric inequality (see [18]) in $\mathbb{R}^2$, given by: $p(K)^2 - 4\pi a(K) \geqslant 0$, here $a(K)$ and $p(K)$ denote the area and perimeter of the convex domain K, and the equality holds if and only if K is a circular disc. This fact was known to the ancient Greeks, and the first mathematical proof was only given in the 19th century by Steiner in [18]. However, for dealing with the geometric flow problems, geometric inequalities play a crucial role, especially, geometric inequalities with curvature are very important in the curve evolution problem (see [3, 4, 5, 6, 12, 14]), such as the well known Gage's inequality [3] and so on. In [14], to estimate the isoperimetric deficit of a evolving curve γ, Pan and Yang [14] established a reverse isoperimetric inequality:

$$\int_0^{2\pi} \frac{1}{\kappa^2(\theta)} \mathrm{d}\theta \geqslant \frac{p(K)^2 - 2\pi a(K)}{\pi}, \tag{1}$$

where K is a convex domain enclosed by γ, κ, p and a are the signed curvature, the length of γ and the area it bounds, and the equality in (1) holds if and only if K is a circular disc.

Furthermore, in [12], to study a nonlocal flow of convex plane curve, Lin and Tsai established a new stronger reverse isoperimetric inequality:

$$\int_0^{2\pi} \frac{1}{\kappa^2(\theta)} \mathrm{d}\theta \geqslant \frac{2p(K)^2 - 6\pi a(K)}{\pi}, \tag{2}$$

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and the equality in (2) holds if and only if the Minkowski support function of K is of the form

$$H_K(\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta, \quad \forall\theta \in [0,2\pi] \tag{3}$$

for some constants a_0, a_1, b_1, a_2, b_2 satisfying

$$H_K''(\theta) + H_K(\theta) > 0, \quad \forall\theta \in [0,2\pi].$$

Here the variable θ is the outward normal angle of K.

In this paper, we estimate the upper bound of the integration of the squared curvature radius of curve γ. Together with inequalities (1) and (2), we obtain the upper and lower bound estimates for the integration of the squared curvature radius of γ. To discuss the stability of the corresponding inequalities, we have to deal with the equality cases between (1) and (2) as the following theorem.

THEOREM 1. *Let K be a convex domain enclosed by a C^{k+2} closed and strictly convex plane curve γ with area $a(K)$ and perimeter $p(K)$. Then*

$$\frac{p(K)^2 - 2\pi a(K) + \varepsilon(p(K)^2 - 4\pi a(K))}{\pi} \leqslant \int_0^{2\pi} \frac{1}{\kappa^2(\theta)}\mathrm{d}\theta \leqslant \frac{p(K)^2}{2\pi} + \frac{1}{4^k}\int_0^{2\pi} (\rho^{(k)}(\theta))^2 \mathrm{d}\theta \tag{4}$$

satisfied for any $0 \leqslant \varepsilon \leqslant 1$, where κ and ρ are the curvature and curvature radius of γ respectively, $\rho^{(k)}$ denotes the $k-$order derivative of ρ with respect to θ. Moreover,

i) $0 \leqslant \varepsilon < 1$, *the equality on the left-hand side of (4) holds if and only if K is a circular disc; $\varepsilon = 1$, the equality on the left-hand side of (4) holds if and only if the Minkowski support function of K is of the form in (3);*

ii) *the equality on the right-hand side of (4) holds if and only if the Minkowski support function of K is of the form in (3).*

The next goal is to deal with the stability of inequalities in (4). Recently, the stability of geometric inequalities have been extensively investigated, see [1, 2, 7, 8, 9, 10, 11, 13, 16, 17]. Roughly speaking, these investigations focus on the geometric implications if the inequalities are in a certain sense close equalities. For more information of the stability problem one may consult [7, 9].

Generally, the stability of inequalities depends on various measures. For convex domains K, L with respective support functions $H_K(\theta)$, $H_L(\theta)$, the most frequently used function to measure the deviation between K and L is the Hausdorff distance

$$h_1(K,L) = \max_\theta \mid H_K(\theta) - H_L(\theta) \mid .$$

Another such measure with respect to stability problem is L^2-metric, which is defined by

$$h_2(K,L) = \left(\int_0^{2\pi} \mid H_K(\theta) - H_L(\theta) \mid^2 \mathrm{d}\theta\right)^{\frac{1}{2}}.$$

It is obvious that $h_1(K,L)=0$ (or $h_2(K,L)=0$) if and only if $K=L$.

By Hausdorff distance h_1 and L^2-metric h_2, we conclude the stability theorems of inequalities in (4), see Theorems 2–4 below.

The contents of this paper are as follows. In section 2 we give some preliminaries about convex domains. We prove Theorem 1 in section 3 and conclude the rest theorems in section 4 by Fourier series of the support function.

2. Preliminaries

In this section, we recall some basic facts about plane convex geometry, which will be used later. In this paper, we always assume that K is a convex domain in the plane and is enclosed by a C^{k+2}, closed and strictly convex plane curve γ with area $a(K)$ and perimeter $p(K)$ such that the curvature of γ is positive everywhere and the Fourier series needed in the proof converges uniformly. The details can be found in [13].

We assume that the origin o of $\mathbb{R}^2$ lies in the interior of K. Let $H_K(\theta)$ denote the Minkowski support function of K (or γ), where θ is the angle between x-axis and pointing outward normal vector $\vec{u}$ along curve γ. It is clear that $H_K(\theta)$ is a continuous 2π-periodic function. Then the curvature κ and the curvature radius ρ of $\gamma(\theta)$ are given by

$$\rho(\theta)=\frac{1}{\kappa(\theta)}=H_K(\theta)+H_K''(\theta)>0, \tag{5}$$

where $'$ denotes the derivative with respect to θ. $p(K)$ and $a(K)$ can be calculated by

$$p(K)=\int_0^{2\pi}H_K(\theta)\mathrm{d}\theta, \tag{6}$$

and

$$a(K)=\frac{1}{2}\int_0^{2\pi}\left(H_K^2(\theta)-H_K'^{\,2}(\theta)\right)\mathrm{d}\theta. \tag{7}$$

The support function $H_K(\theta)$ of K has a Fourier series of the form

$$H_K(\theta)=a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta), \tag{8}$$

where

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}H_K(\theta)\,\mathrm{d}\theta=\frac{p(K)}{2\pi}$$

and

$$a_n=\frac{1}{\pi}\int_0^{2\pi}H_K(\theta)\cos n\theta\,\mathrm{d}\theta,\quad b_n=\frac{1}{\pi}\int_0^{2\pi}H_K(\theta)\sin n\theta\,\mathrm{d}\theta,\quad n\in\mathbb{Z}^+.$$

Differentiation of this with respect to θ gives us

$$H_K'(\theta)=-\sum_{n=1}^{\infty}n(a_n\sin n\theta-b_n\cos n\theta),\quad H_K''(\theta)=-\sum_{n=1}^{\infty}n^2(a_n\cos n\theta+b_n\sin n\theta).$$

From (5) and (8), it follows immediately that

$$\rho(\theta) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) - \sum_{n=1}^{\infty} n^2(a_n \cos n\theta + b_n \sin n\theta), \tag{9}$$

and

$$\rho^{(k)}(\theta) = \begin{cases} (-1)^{\frac{k-1}{2}} \sum_{n=2}^{\infty} n^k(n^2-1)(a_n \sin n\theta - b_n \cos n\theta), \, k \text{ is odd}, \\ (-1)^{\frac{k}{2}-1} \sum_{n=2}^{\infty} n^k(n^2-1)(a_n \cos n\theta + b_n \sin n\theta), \, k \text{ is even}, \end{cases} \tag{10}$$

where $k \in \mathbb{Z}^+$. By the Parseval equality, one can get

$$\int_0^{2\pi} H_K^2(\theta)\mathrm{d}\theta = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty}(a_n^2 + b_n^2),$$

$$\int_0^{2\pi} H_K'^{\,2}(\theta)\mathrm{d}\theta = \pi \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2), \qquad \int_0^{2\pi} H_K''^{\,2}(\theta)\mathrm{d}\theta = \pi \sum_{n=1}^{\infty} n^4(a_n^2 + b_n^2)$$

and

$$a(K) = \pi a_0^2 - \frac{\pi}{2} \sum_{n=2}^{\infty}(n^2-1)(a_n^2 + b_n^2). \tag{11}$$

The steiner disc of K, denoted by $S(K)$, is the circular disc with radius $\frac{p(K)}{2\pi}$ and center at the steiner point which can be defined in terms of the Minkowski support function

$$\vec{s}(K) = \frac{1}{\pi} \int_0^{2\pi} \vec{u}(\theta) H_K(\theta)\mathrm{d}\theta.$$

From (8) and the definition of the Fourier coefficients, we have

$$\vec{s}(K) = (a_1, b_1). \tag{12}$$

The steiner disc will play a role in our stability statement in section 4 below. For more information on the steiner point of a convex body one may consult [16]. Let $S_2(K)$ be a convex domain enclosed by a closed and strictly convex plane curve, and its the support function is of the form

$$H_{S_2(K)}(\theta) = a_0 + a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta. \tag{13}$$

3. The proof of Theorem 1

In this section, we prove Theorem 1 by using Fourier series.

Proof of Theorem 1. By (9), one can easily get

$$\int_0^{2\pi} \frac{1}{\kappa^2(\theta)}\mathrm{d}\theta = 2\pi\left(a_0^2 + \frac{1}{2}\sum_{n=2}^{\infty}(n^2-1)^2(a_n^2 + b_n^2)\right). \tag{14}$$

It follows from (10) that

$$\int_0^{2\pi}(\rho^{(k)}(\theta))^2\mathrm{d}\theta = \pi\sum_{n=2}^{\infty} n^{2k}(n^2-1)^2(a_n^2+b_n^2). \tag{15}$$

From (14) and (15), we can get

$$\begin{aligned}&\frac{p(K)^2}{2\pi}+\frac{1}{4^k}\int_0^{2\pi}(\rho^{(k)}(\theta))^2\mathrm{d}\theta-\int_0^{2\pi}\frac{1}{\kappa^2(\theta)}\mathrm{d}\theta\\ =&\pi\sum_{n=2}^{\infty}(n^2-1)^2\Big(\big(\frac{n^2}{4}\big)^k-1\Big)(a_n^2+b_n^2).\end{aligned} \tag{16}$$

Hence, $\frac{p(K)^2}{2\pi}+\frac{1}{4^k}\int_0^{2\pi}(\rho^{(k)}(\theta))^2\mathrm{d}\theta-\int_0^{2\pi}\frac{1}{\kappa^2(\theta)}\mathrm{d}\theta \geqslant 0$. Clearly, the equality on the right hand in (4) holds if and only if $a_n=b_n=0, n\geqslant 3, n\in\mathbb{N}^+$, then the Minkowski support function of K is of the form in (3), which completes the proof of inequality on the right hand in (4).

On the other hand, let

$$f(K,\varepsilon)=\int_0^{2\pi}\frac{1}{\kappa^2(\theta)}\mathrm{d}\theta-\frac{p^2(K)-2\pi a(K)+\varepsilon(p^2(K)-4\pi a(K))}{\pi}.$$

It follows from (11) and (14) that

$$\begin{aligned}f(K,\varepsilon)&=\frac{1}{\pi}\left\{\pi^2\sum_{n=2}^{\infty}\big((n^2-1)^2-(n^2-1)\big)(a_n^2+b_n^2)-2\pi^2\varepsilon\sum_{n=2}^{\infty}(n^2-1)(a_n^2+b_n^2)\right\}\\ &=\pi\sum_{n=2}^{\infty}(n^2-1)\big(n^2-2(1+\varepsilon)\big)(a_n^2+b_n^2),\end{aligned} \tag{17}$$

observing that $f(K,\varepsilon)$ is a linear function with respect to ε such that

$$f(K,1)=\pi\sum_{n=2}^{\infty}(n^2-1)(n^2-4)(a_n^2+b_n^2)\geqslant 0, \tag{18}$$

thus for any $0\leqslant\varepsilon\leqslant 1$ we have

$$f(K,\varepsilon)\geqslant f(K,1)\geqslant 0.$$

In particular, case i). when $0\leqslant\varepsilon<1$, the equality on the left-hand side of (4) holds if and only if $f(K,\varepsilon)=0$, i.e.

$$\sum_{n=2}^{\infty}(n^2-1)[n^2-2(1+\varepsilon)](a_n^2+b_n^2)=0$$

which leads to $a_n=b_n=0, n\geqslant 2, n\in\mathbb{N}^+$, then the Minkowski support function of K is of the form

$$H_K(\theta)=a_0+a_1\cos\theta+b_1\sin\theta,$$

which implies that K is a circular disc.

Case ii). when $\varepsilon = 1$, the equality on the left-hand side of (4) holds if and only if equality holds in (18), which leads to $a_n = b_n = 0, n \geqslant 3, n \in \mathbb{N}^+$, then the Minkowski support function of K is of the form in (3). This completes the proof of Theorem 1. □

From (15), it is not difficult to obtain the following corollary.

COROLLARY 1. *Under the same assumptions as in Theorem 1, we have that*

$$\frac{1}{4^k}\int_0^{2\pi}(\rho^{(k)}(\theta))^2 \mathrm{d}\theta \geqslant \frac{3}{2\pi}(p(K)^2 - 4\pi a(K)),$$

with equality if and only if the Minkowski support funtion of K is of the form in (3).

4. The stability results

In this section, we use the Hausdorff distance h_1 and the L^2-metric h_2 to build stability estimates of inequalities in (4), see theorems 2–4 below.

THEOREM 2. *Let K be a convex domain enclosed by a C^2 closed and strictly convex plane curve γ with area $a(K)$ and perimeter $p(K)$. Then*

$$\begin{aligned} &\int_0^{2\pi}\frac{1}{\kappa^2(\theta)}\mathrm{d}\theta - \frac{p(K)^2 - 2\pi a(K) + \varepsilon(p(K)^2 - 4\pi a(K))}{\pi} \\ \geqslant &\begin{cases} \frac{\pi}{C(\varepsilon)}h_1(K,S(K))^2, 0 \leqslant \varepsilon < 1; \\ \frac{144\pi}{5}h_1(K,S_2(K))^2, \varepsilon = 1, \end{cases} \end{aligned} \tag{19}$$

where κ is the curvature of γ and

$$C(\varepsilon) = \frac{1}{(1+2\varepsilon)^2} - \frac{3}{4(1+2\varepsilon)} + \frac{1-\sqrt{2(1+\varepsilon)}\pi\cot(\sqrt{2(1+\varepsilon)}\pi)}{4(1+\varepsilon)(1+2\varepsilon)}.$$

Moreover, when

i) $0 \leqslant \varepsilon < 1$*, the first equality in (19) holds if K is a circular disc;*

ii) $\varepsilon = 1$*, the second equality in (19) holds if the Minkowski support function of K is of the form in (3).*

Proof. Without loss of generality, we may assume $\vec{s}(K) = \vec{0}$, then $H_{S(K)}(\theta) = a_0$ and $H_{S_2(K)}(\theta) = a_0 + a_2\cos 2\theta + b_2 \sin 2\theta$, where $a_0 = \frac{p(K)}{2\pi}$. Since it is easy to see that

$$\begin{aligned} &| H_K(\theta) - H_{S(K)}(\theta) | \\ = &| \sum_{n=2}^{\infty}(a_n\cos n\theta + b_n \sin n\theta) | \leqslant \sum_{n=2}^{\infty} | (a_n\cos n\theta + b_n\sin n\theta) | \leqslant \sum_{n=2}^{\infty}\sqrt{a_n^2+b_n^2} \end{aligned} \tag{20}$$

and

$$| H_K(\theta) - H_{S_2(K)}(\theta) | \leqslant \sum_{n=3}^{\infty} \sqrt{a_n^2 + b_n^2}. \tag{21}$$

case i) When $0 \leqslant \varepsilon < 1$, by (20) and Hölder's inequality, we find

$$\begin{aligned} h_1(K,S(K)) &\leqslant \sum_{n=2}^{\infty} \sqrt{a_n^2 + b_n^2} \\ &\leqslant \left(\sum_{n=2}^{\infty} \frac{1}{(n^2-1)(n^2-2(1+\varepsilon))} \right)^{\frac{1}{2}} \left(\sum_{n=2}^{\infty} (n^2-1)(n^2-2(1+\varepsilon))(a_n^2+b_n^2) \right)^{\frac{1}{2}}, \end{aligned}$$

hence,

$$h_1(K,S(K))^2 \leqslant \sum_{n=2}^{\infty} \frac{1}{(n^2-1)(n^2-2(1+\varepsilon))} \sum_{n=2}^{\infty} (n^2-1)(n^2-2(1+\varepsilon))(a_n^2+b_n^2). \tag{22}$$

Recall that if p is not an integer, by Fourier series calculation we have

$$\pi \cot p\pi = \frac{1}{p} - 2p \sum_{n=1}^{\infty} \frac{1}{n^2 - p^2}.$$

Together with $0 \leqslant \varepsilon < 1$ then we can calculate that

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 2(1+\varepsilon)} = \frac{1}{1+2\varepsilon} + \frac{1}{4(1+\varepsilon)} - \frac{\sqrt{2(1+\varepsilon)}\pi \cot(\sqrt{2(1+\varepsilon)}\pi)}{4(1+\varepsilon)}.$$

Moreover,

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{1}{(n^2-2(1+\varepsilon))(n^2-1)} \\ =&\frac{1}{1+2\varepsilon} \sum_{n=2}^{\infty} \left(\frac{1}{n^2-2(1+\varepsilon)} - \frac{1}{n^2-1}\right) \\ =&\frac{1}{1+2\varepsilon} \sum_{n=2}^{\infty} \frac{1}{n^2-2(1+\varepsilon)} - \frac{1}{2(1+2\varepsilon)} \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1}\right) \\ =&\frac{1}{(1+2\varepsilon)^2} - \frac{3}{4(1+2\varepsilon)} + \frac{1 - \sqrt{2(1+\varepsilon)}\pi \cot(\sqrt{2(1+\varepsilon)}\pi)}{4(1+\varepsilon)(1+2\varepsilon)}. \end{aligned} \tag{23}$$

By (17), (22) and (23), we finally get

$$\begin{aligned} &\int_0^{2\pi} \frac{1}{\kappa^2(\theta)} \mathrm{d}\theta - \frac{p(K)^2 - 2\pi a(K) + \varepsilon(p(K)^2 - 4\pi a(K))}{\pi} \\ =&\pi \sum_{n=2}^{\infty} (n^2-1)(n^2-2(1+\varepsilon))(a_n^2+b_n^2) \\ \geqslant&\frac{\pi}{C(\varepsilon)} h_1(K,S(K))^2, \end{aligned}$$

where $C(\varepsilon)=\frac{1}{(1+2\varepsilon)^2}-\frac{3}{4(1+2\varepsilon)}+\frac{1-\sqrt{2(1+\varepsilon)}\pi\cot(\sqrt{2(1+\varepsilon)}\pi)}{4(1+\varepsilon)(1+2\varepsilon)}$. Furthermore, if K is a circular disc, the first equality in (19) holds clearly.

case ii) When $\varepsilon=1$, by (21) and Hölder's inequality, we obtain

$$
\begin{aligned}
&| H_K(\theta)-H_{S_2(K)}(\theta) |\\
\leqslant &\sum_{n=3}^{\infty}\sqrt{a_n^2+b_n^2}\leqslant\left\{\sum_{n=3}^{\infty}\frac{1}{(n^2-1)(n^2-4)}\right\}^{\frac{1}{2}}\left\{\sum_{n=3}^{\infty}(n^2-1)(n^2-4)(a_n^2+b_n^2)\right\}^{\frac{1}{2}},
\end{aligned}
$$

thus

$$
\begin{aligned}
h_1(K,S_2(K))^2 &=\left\{\max_{\theta} | H_K(\theta)-H_{S_2(K)}(\theta) |\right\}^2\\
\leqslant&\frac{1}{3}\left(\frac{1}{4}\sum_{n=3}^{\infty}(\frac{1}{n-2}-\frac{1}{n+2})-\frac{1}{2}\sum_{n=3}^{\infty}(\frac{1}{n-1}-\frac{1}{n+1})\right)\sum_{n=3}^{\infty}(n^2-1)(n^2-4)(a_n^2+b_n^2)\\
=&\frac{5}{144}\sum_{n=3}^{\infty}(n^2-1)(n^2-4)(a_n^2+b_n^2),
\end{aligned}
\tag{24}
$$

From (17) and (24) it follows that

$$
\begin{aligned}
\int_0^{2\pi}\frac{1}{\kappa^2(\theta)}\mathrm{d}\theta-\frac{2p(K)^2-6\pi a(K)}{\pi} &=\pi\sum_{n=3}^{\infty}(n^2-1)(n^2-4)(a_n^2+b_n^2)\\
&\geqslant\frac{144\pi}{5}h_1(K,S_2(K))^2.
\end{aligned}
\tag{25}
$$

Clearly, by (25), the second equality in (19) holds if K is $S_2(K)$. □

THEOREM 3. *Under the same assumptions as in Theorem 2, we have that*

$$
\begin{aligned}
&\int_0^{2\pi}\frac{1}{\kappa^2(\theta)}\mathrm{d}\theta-\frac{p(K)^2-2\pi a(K)+\varepsilon(p(K)^2-4\pi a(K))}{\pi}\\
\geqslant&\begin{cases} 6(1-\varepsilon)h_2(K,S(K))^2, 0\leqslant\varepsilon<1;\\ 40h_2(K,S_2(K))^2, \varepsilon=1.\end{cases}
\end{aligned}
\tag{26}
$$

Moreover, when

i) $0\leqslant\varepsilon<1$, *the first equality in (26) holds if K is a circular disc;*

ii) $\varepsilon=1$, *the second equality in (26) holds if the Minkowski support function of K is of the form in (3).*

Proof. As in the proof of Theorem 2, we use (8), (13) and Parseval's equality to deduce that

$$
h_2(K,S(K))^2=\int_0^{2\pi} | H_K(\theta)-H_{S(K)}(\theta) |^2\,\mathrm{d}\theta=\pi\sum_{n=2}^{\infty}(a_n^2+b_n^2),\tag{27}
$$

and

$$h_2(K,S_2(K))^2 = \int_0^{2\pi} |\, H_K(\theta) - H_{S_2(K)}(\theta)\,|^2 \,\mathrm{d}\theta = \pi \sum_{n=3}^{\infty}(a_n^2+b_n^2). \tag{28}$$

case i) when $0 \leqslant \varepsilon < 1$, together with (27) and (17) we have

$$\int_0^{2\pi} \frac{1}{\kappa^2(\theta)}\mathrm{d}\theta - \frac{p(K)^2 - 2\pi a(K) + \varepsilon(p(K)^2 - 4\pi a(K))}{\pi}$$
$$= \pi \sum_{n=2}^{\infty}(n^2-1)\big(n^2-2(1+\varepsilon)\big)(a_n^2+b_n^2) \geqslant 6(1-\varepsilon)h_2(K,S(K))^2.$$

It is clear that the equality in (26) holds if K is a circular disc when $0 \leqslant \varepsilon < 1$.

case ii) when $\varepsilon = 1$, together with (28) and (17) we get

$$\int_0^{2\pi} \frac{1}{\kappa^2(\theta)}\mathrm{d}\theta - \frac{2p(K)^2 - 6\pi a(K)}{\pi}$$
$$= \pi \sum_{n=3}^{\infty}(n^2-1)(n^2-4)(a_n^2+b_n^2) \geqslant 40h_2(K,S_2(K))^2.$$

It is clear that the second equality in (26) holds if the support function of K is of the form in (3). We complete the proof of Theorem 3. □

REMARK 1. By the proof of Theorem 3, it is clear that the equality in Theorem 3 holds if and only if the support function of K is of the form

$$H_K(\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta,\ 0 \leqslant \varepsilon < 1;$$

or

$$H_K(\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta + a_3\cos 3\theta + b_3\sin 3\theta,\ \varepsilon = 1.$$

THEOREM 4. *Let K be a convex domain enclosed by a C^3, closed and strictly convex plane curve γ with area $a(K)$ and perimeter $p(K)$. Then*

$$\frac{p(K)^2}{2\pi} + \frac{1}{4}\int_0^{2\pi}(\rho'(\theta))^2\mathrm{d}\theta - \int_0^{2\pi}\frac{1}{\kappa^2(\theta)}\mathrm{d}\theta \geqslant \begin{cases} \frac{9\pi}{10-\pi^2}h_1(K,S_2(K))^2, \\ 80h_2(K,S_2(K))^2. \end{cases} \tag{29}$$

and equalities holds if the Minkowski support function of K is of the form in (3).

Proof. As in the proof of Theorem 2, by (21) and Hölder's inequality, we find

$$h_1(K,S_2(K))^2 \leqslant \sum_{n=3}^{\infty}\frac{1}{(\frac{n^2}{4}-1)(n^2-1)^2}\sum_{n=3}^{\infty}(\frac{n^2}{4}-1)(n^2-1)^2(a_n^2+b_n^2). \tag{30}$$

By a direct calculation, one gets

$$\begin{aligned}&\sum_{n=3}^{\infty}\frac{1}{(\frac{n^2}{4}-1)(n^2-1)^2}\\ =&4\sum_{n=3}^{\infty}(\frac{1}{36(n-2)}-\frac{1}{36(n+2)}-\frac{1}{36(n+1)}+\frac{1}{36(n-1)}-\frac{1}{12(n+1)^2}-\frac{1}{12(n-1)^2})\\ =&\frac{10-\pi^2}{9},\end{aligned} \tag{31}$$

From (16), (30) and (31) it follows that

$$\frac{p(K)^2}{2\pi}+\frac{1}{4}\int_0^{2\pi}(\rho'(\theta))^2\mathrm{d}\theta-\int_0^{2\pi}\frac{1}{\kappa^2(\theta)}\mathrm{d}\theta \geqslant \frac{9\pi}{10-\pi^2}h_1(K,S_2(K))^2,$$

we complete the proof of the first inequality in Theorem 4.

From (16) and (27), one gets

$$\frac{p(K)^2}{2\pi}+\frac{1}{4}\int_0^{2\pi}(\rho'(\theta))^2\mathrm{d}\theta-\int_0^{2\pi}\frac{1}{\kappa^2(\theta)}\mathrm{d}\theta \geqslant 80h_2(K,S_2(K))^2.$$

It is easy to see that equalities in (29) hold if $a_n=b_n=0, n\geqslant 3, n\in\mathbb{Z}^+$. We complete the proof of Theorem 4. □

REMARK 2. By the proof of Theorem 4, it is easy to see that the second equality in Theorem 4 holds if and only if the support function of K is of the form

$$H_K(\theta)=a_0+a_1\cos\theta+b_1\sin\theta+a_2\cos 2\theta+b_2\sin 2\theta+a_3\cos 3\theta+b_3\sin 3\theta,$$

where a_2, b_2, a_3, b_3 are small in comparison with a_0.

REMARK 3. Obviously, when $\varepsilon=1$, Theorems 2 and 3 are stability estimates of the inequality (2). Meanwhile, the above three stability theorems are looked as stronger versions of inequalities in (4).

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