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AN INEQUALITY FOR INTEGRALS OF THE FORM $\int_x^\infty f(t)e^{it}\,dt$

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Abstract. It is shown that for a completely monotonic function f, the absolute value of $\int_x^\infty f(t)e^{it}\,dt$ is not greater than $f(x)$.

1. Introduction

We consider integrals of the form

$$I_f(x) = \int_x^\infty f(t)e^{it}\,dt,$$

with real and imaginary parts

$$C_f(x) = \int_x^\infty f(t)\cos t\,dt, \qquad S_f(x) = \int_x^\infty f(t)\sin t\,dt.$$

We assume that $f(t)$ is defined and non-negative for all $t > 0$ (and possibly also for $t = 0$), and that it satisfies a condition of the following type:

(CM): $\lim_{t\to\infty} f(t) = 0$ and $(-1)^n f^{(n)}(t) \geqslant 0$ for all $n \geqslant 0$ and $t > 0$.

(CM$_k$): the same restricted to $n \leqslant k$, with $f^{(k)}(t)$ continuous.

Functions satisfying (CM) are said to be *completely monotonic*, though the condition $\lim_{t\to\infty} f(t) = 0$ is not always included in the definition.

It follows easily from condition (CM$_k$) that for $1 \leqslant r \leqslant k-1$, the function $(-1)^r f^{(r)}(t)$ is decreasing and tends to 0 as $t \to \infty$, so itself satisfies (CM$_{k-r}$). Also, if f satisfies (CM$_k$), then so do $f(t+a)$ and $f(t) - f(t+a)$ for $a > 0$.

The most basic class of completely monotonic functions is $f(t) = 1/t^p$ for any $p > 0$. In the case $p = 1$, the functions S_f and C_f are the "sine" and "cosine" integrals (the established notation for this case is $\mathrm{si}(x) = -S_f(x)$ and $\mathrm{ci}(x) = -C_f(x)$). The case $p = \frac{1}{2}$ gives the Fresnel integrals.

Another function satisfying (CM) is $f(t) = e^{-at}$, where $a > 0$. Numerous further examples are given in [1] and references listed there.

For f satisfying (CM$_1$), we have $|f'(t)e^{it}| \leqslant -f'(t)$, hence

$$|I_{f'}(x)| \leqslant \int_x^\infty (-f'(t))\,dt = f(x). \tag{1}$$

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Now integrating by parts, we have

$$I_f(x) = \Big[-if(t)e^{it}\Big]_x^\infty + i\int_x^\infty f'(t)e^{it}\,dt$$
$$= if(x)e^{ix} + iI_{f'}(x). \tag{2}$$

So the integral defining $I_f(x)$ is convergent, and also

$$|I_f(x)| \leqslant 2f(x). \tag{3}$$

Our concern in this note is the best version of this inequality. It is arguable that the natural comparison is with $f(x)$ rather than $2f(x)$. Indeed, if f satisfies (CM_2), we can apply (3) to $-f'$ and substitute in (2) to obtain $|I_f(x)| = f(x) + r_1(x)$, where $|r_1(x)| \leqslant -2f'(x)$. If $f'(x) = o[f(x)]$ as $x \to \infty$, (which occurs in many cases, including $f(t) = t^{-p}$), it follows that $|I_f(x)| \sim f(x)$ as $x \to \infty$. Of course, this shows that if C is the best constant in the inequality $|I_f(x)| \leqslant Cf(x)$, then $C \geqslant 1$.

However, without any differentiability conditions, a trivial example shows that it is possible to have $|I_f(x)| > f(x)$, in fact the factor 2 in (3) can be attained. To see this, take $f(t)$ to be 1 for $0 \leqslant t \leqslant \pi$ and 0 for $t > \pi$: then $I_f(0) = 2i$. For an example satisfying (CM_2), take $f(t)$ to be $(2\pi - t)^2$ on $[0, 2\pi]$ and 0 for $t > 2\pi$: it is easily checked that $I_f(0) = 4\pi^2 i + 4\pi$. (Strictly, this example does not satisfy (CM_2) because f'' has a discontinuity at 2π; however, this can be corrected by a small perturbation.)

In the light of these examples, it is rather striking that the suspected inequality actually does hold under condition (CM_4). In fact, we will prove the following theorem.

THEOREM 1. *If f satisfies (CM_4), then $|I_f(x)|$ is decreasing and $|I_f(x)| \leqslant f(x)$ for all $x > 0$.*

Although these integrals are a highly classical topic, only limited attention seems to have been given to inequalities of this type. For the special case $f(t) = 1/t$, it is pointed out in [2, p. 123] that the usual contour integral method for the sine integral delivers the inequality $|S_f(x)| \leqslant (\pi/2x)$. For this case, a recent article [4] establishes the stronger bound $\frac{\pi}{2} - \tan^{-1} x$: this is less than $1/x$, and reproduces the exact value at 0.

We mention some further preliminary facts. If f satisfies (CM_2), by applying (2) to $f'(t)$ and substituting back (which is equivalent to repeating the integration by parts), we obtain

$$I_f(x) = if(x)e^{ix} - f'(x)e^{ix} - I_{f''}(x). \tag{4}$$

If f satisfies (CM_4), we can apply this to f'' and substitute again to obtain

$$I_f(x) = i\left[f(x) - f''(x)\right]e^{ix} - \left[f'(x) - f^{(3)}(x)\right]e^{ix} + I_{f^{(4)}}(x), \tag{5}$$

For (CM) functions, as is well known, repetition of the process generates an asymptotic expansion for $I_f(x)$ [3, p. 146, 150]. For the particular case $f(t) = 1/t^p$, one can apply (5) to establish $|I_f(x)| < f(x)$ (and indeed stronger inequalities with further terms) for sufficiently large x, but the method does not extend to the whole range $x > 0$. Details for the case $f(t) = 1/t$ can be seen in [4].

2. The auxiliary functions and the proof of Theorem 1

We consider the function $K_f(x) = e^{-ix}I_f(x)$. Note that

$$K_f(x) = \int_x^\infty f(t)e^{i(t-x)}\,dt = \int_0^\infty f(u+x)e^{iu}\,du. \tag{6}$$

Write $K_f(x) = V_f(x) + iU_f(x)$, so that

$$U_f(x) = S_f(x)\cos x - C_f(x)\sin x, \tag{7}$$

$$V_f(x) = C_f(x)\cos x + S_f(x)\sin x, \tag{8}$$

U_f and V_f are the "auxiliary functions". (We write U_f, V_f this way round in a gesture to customary notation.) Note that $U_f(n\pi) = (-1)^n S_f(n\pi)$ and $V_f(n\pi) = (-1)^n C_f(n\pi)$; these points are the successive maxima and minima of $S_f(x)$. Also, denoting $(n-\frac{1}{2})\pi$ by u_n, we have $U_f(u_n) = (-1)^n C_f(u_n)$ and $V_f(u_n) = (-1)^{n+1}S_f(u_n)$.

Stated for $K_f(x)$ and its components, identity (2) becomes

$$K_f(x) = if(x) + iK_{f'}(x), \tag{9}$$

$$U_f(x) = f(x) + V_{f'}(x), \qquad V_f(x) = -U_{f'}(x). \tag{10}$$

Similarly, (4) becomes

$$K_f(x) = if(x) - f'(x) - K_{f''}(x), \tag{11}$$

$$U_f(x) = f(x) - U_{f''}(x), \qquad V_f(x) = -f'(x) - V_{f''}(x), \tag{12}$$

Also, since $I'_f(x) = -e^{ix}f(x)$ (distinguish between $I'_f(x)$ and $I_{f'}(x)$!), we have

$$K'_f(x) = -ie^{ix}I_f(x) - e^{-ix}e^{ix}f(x) = -iK_f(x) - f(x), \tag{13}$$

hence

$$U'_f(x) = -V_f(x), \qquad V'_f(x) = U_f(x) - f(x). \tag{14}$$

By (9) and (13), we have $K'_f(x) = K_{f'}(x)$, and similarly for U_f and V_f: this also follows formally from (6) by differentiation under the integral sign. We remark (though we will not use this fact) that U_f and V_f satisfy the differential equations $U''_f + U_f = -f'$ and $V''_f + V_f = f$.

Some basic facts about $U_f(x)$ and $V_f(x)$ are summarised in the next result.

PROPOSITION 2. *If f satisfies (CM_4), then*

$$0 \leqslant U_f(x) \leqslant f(x), \tag{15}$$

$$0 \leqslant V_f(x) \leqslant -f'(x). \tag{16}$$

Further, $U_f(x)$, $V_f(x)$ and $f(x) - U_f(x)$ are decreasing.

Proof. For f satisfying (CM_1), we have, by (1), $|V_{f'}(x)| \leqslant |I_{f'}(x)| \leqslant f(x)$, so by (10), $U_f(x) \geqslant 0$. If f satisfies (CM_2), then $-f'$ satisfies (CM_1), so this gives

$U_{f'}(x) \leqslant 0$, and again by (10), we have $V_f(x) \geqslant 0$. Now assuming f satisfies (CM$_4$), we apply these statements to f'', concluding that $U_{f''}(x)$ and $V_{f''}(x)$ are non-negative. So by (12), we have $U_f(x) \leqslant f(x)$ and $V_f(x) \leqslant -f'(x)$.

By (14), we now have $U_f'(x) = -V_f(x) \leqslant 0$, also $V_f'(x) = U_f(x) - f(x) \leqslant 0$ and $f'(x) - U_f'(x) = f'(x) + V_f(x) \leqslant 0$. $\square$

Theorem 1 now follows in a simple and pleasant way.

Proof of Theorem 1. Let $M_f(x) = |I_f(x)|^2 = C_f(x)^2 + S_f(x)^2$. Then

$$\begin{aligned} M_f'(x) &= -2C_f(x)f(x)\cos x - 2S_f(x)f(x)\sin x \\ &= -2f(x)[C_f(x)\cos x + S_f(x)\sin x] \\ &= -2f(x)V_f(x). \end{aligned}$$

By (16), we have $0 \leqslant V_f(x) \leqslant -f'(x)$. Hence $M_f'(x) \leqslant 0$, so $M_f(x)$ is decreasing. Also, $M_f'(x) \geqslant 2f(x)f'(x)$, so $f(x)^2 - M_f(x)$ is decreasing. Since it tends to 0 as $x \to \infty$, it follows that it is non-negative, so $M_f(x) \leqslant f(x)^2$, hence $|I_f(x)| \leqslant f(x)$, for $x > 0$. $\square$

Of course, it follows that $|C_f(x)|$ and $|S_f(x)|$ are also bounded by $f(x)$. The upper bounds in (15) and (16), which were used in the proof, are now seen to be special cases of Theorem 1.

We derive a version of the result for integrals on intervals of length $2k\pi$. Write $I_f(x,y) = \int_x^y f(t)e^{it}\,dt$.

COROLLARY 1.1. *If f satisfies (CM$_4$) and $y = x + 2k\pi$, then $|I_f(x,y)| \leqslant f(x) - f(y)$.*

Proof. Clearly, $I_f(x+a) = e^{ia}I_g(x)$, where $g(t) = f(t+a)$. In particular, if $a = 2k\pi$, then $I_f(x+a) = I_g(x)$. Hence $I_f(x,y) = I_f(x) - I_f(x+a) = I_h(x)$, where $h(t) = f(t) - f(t+a)$. The statement follows by applying Theorem 1 to h. $\square$

EXAMPLE. (the Fresnel integral) Among the many particular cases of Theorem 1, we just mention the Fresnel integral, given by $f(t) = 1/t^{1/2}$. It is often presented in the following alternative form, given by the substitution $t = u^2$:

$$J_F(x) =: \int_x^\infty e^{iu^2}\,du = \tfrac{1}{2}I_f(x^2).$$

By Theorem 1, we have $|J_F(x)| \leqslant \frac{1}{2x}$.

3. An alternative proof for completely monotonic functions

We sketch a superficially very quick alternative proof of Theorem 1 for (CM) functions using a theorem of Bernstein [5, p. 160]. This theorem states that all such functions can be expressed as

$$f(t) = \int_0^\infty e^{-ut}\,d\mu(u)$$

for some non-negative measure μ. As an illustration, the expression for $1/t^p$ is

$$\frac{1}{t^p} = \frac{1}{\Gamma(p)} \int_0^\infty u^{p-1} e^{-ut}\, du.$$

Proof of Theorem 1 for (CM) functions. Assuming validity of the reversal of integration, we have

$$\begin{aligned} I_f(x) &= \int_x^\infty e^{it} \int_0^\infty e^{-ut}\, d\mu(u)\, dt \\ &= \int_0^\infty \int_x^\infty e^{-(u-i)t}\, dt\, d\mu(u) \\ &= \int_0^\infty \frac{e^{-(u-i)x}}{u-i}\, d\mu(u). \end{aligned}$$

Since $|u-i| \geqslant 1$, we deduce

$$|I_f(x)| \leqslant \int_0^\infty e^{-ux}\, d\mu(u) = f(x). \quad \square$$

However, this method only applies to (CM) functions, and it depends on heavy machinery, in the form of Bernstein's theorem. Also, the reversal of integration requires justification, at least by first considering the integral on a bounded interval $[x,y]$ for t.

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