Journal of
Mathematical
Inequalities
Volume 10, Number 2 (2016), 577–582 doi:10.7153/jmi-10-46

MULTI–PARAMETER GENERALIZATION OF RADO–POPOVICIU INEQUALITIES

JING ZHANG* AND HUAN-NAN SHI

(*Communicated by G. Toader*)

Abstract. By using methods on the theory of majorization, new generalizations of Rado's inequality and Popoviciu's inequality which involves multi-parameter are established.

1. Introduction

Throughout the article, $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, \cdots, x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\boldsymbol{x} = (x_1, \cdots, x_n) : x_i \in \mathbb{R}, i = 1, \cdots, n\},$$

$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, \cdots, x_n) : x_i \geqslant 0, i = 1, \cdots, n\},$$

$$\mathbb{R}^n_{++} = \{\boldsymbol{x} = (x_1, \cdots, x_n) : x_i > 0, i = 1, \cdots, n\}.$$

In particular, the notations $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote $\mathbb{R}^1$, $\mathbb{R}^1_+$ and $\mathbb{R}^1_{++}$, respectively.

Let $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$. The elementary symmetric functions are defined by

$$E_k(\boldsymbol{x}) = E_k(x_1, \cdots, x_n) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \prod_{j=1}^k x_{i_j}, \quad k = 1, \cdots, n,$$

$E_0(\boldsymbol{x}) = 1$, and $E_k(\boldsymbol{x}) = 0$ for $k < 0$ or $k > n$. The dual form of the elementary symmetric functions are

$$E^*_k(\boldsymbol{x}) = E^*_k(x_1, \cdots, x_n) = \prod_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \sum_{j=1}^k x_{i_j}, \quad k = 1, \cdots, n,$$

$E^*_0(\boldsymbol{x}) = 1$, and $E^*_k(\boldsymbol{x}) = 0$ for $k < 0$ or $k > n$.

For $\boldsymbol{x} \in \mathbb{R}^n_{++}$, let

$$A_n(\boldsymbol{x}) = \frac{\sum_{i=1}^n x_i}{n}, G_n(\boldsymbol{x}) = \sqrt[n]{\prod_{i=1}^n x_i}.$$

Mathematics subject classification (2010): Primary 26D15, 26E60, 26B25.
Keywords and phrases: Rado's inequality, Popoviciu's inequality, majorization, Schur-concavity, symmetric function, dual form, convex function.

* The corresponding author.

Paper JMI-10-46

The following inequalities:

$$n[A_n(\boldsymbol{x}) - G_n(\boldsymbol{x})] \geqslant (n-1)[A_{n-1}(\boldsymbol{x}) - G_{n-1}(\boldsymbol{x})] \tag{1}$$

$$\left[\frac{A_n(\boldsymbol{x})}{G_n(\boldsymbol{x})}\right]^n \geqslant \left[\frac{A_{n-1}(\boldsymbol{x})}{G_{n-1}(\boldsymbol{x})}\right]^{n-1} \tag{2}$$

are known in the bibliography as Rado's inequality and Popoviciu's inequality respectively (see Mitrinović and Vasić [1], p. 94). Inequalities (1) and (2) furnish a good route for joining arithmetic means and geometric means of positive numbers. This pair of inequalities has attracted considerable attention by many mathematicians, and has actuated a quantity of research articles giving their simple proofs providing diverse improvements, generalizations and analogs (see [1]–[8] and references therein).

The aim of this paper is to establish a new generalization of Rado's inequality and Popoviciu's inequality which involves multi-parameter, by Schur-concavity of the elementary symmetric function and its dual formula, as well as a simple majorization relation.

Our main results are as follows:

THEOREM 1. *Let* $\boldsymbol{x} \in \mathbb{R}^n_{++}$, $n \geqslant 2$, $0 < \alpha \leqslant 1$, $\lambda > 0$. *Then for* $k = 1, \cdots, n$, *we have*

$$\left[A_n(\boldsymbol{x}) + \frac{(\lambda-1)x_n}{n}\right]^\alpha \geqslant \left[\left(1-\frac{1}{k}\right)(A_{n-1}(\boldsymbol{x}))^\alpha + \frac{1}{k}\lambda^\alpha (x_n)^\alpha\right]^{\frac{k}{n}} [A_{n-1}(\boldsymbol{x})]^{(1-\frac{k}{n})\alpha}. \tag{3}$$

Taking $\alpha = \lambda = 1$ and $k = n$, from the inequality (3), it follows that

COROLLARY 1. *Let* $\boldsymbol{x} \in \mathbb{R}^n_{++}$, $n \geqslant 2$, *we have*

$$nA_n(\boldsymbol{x}) - (n-1)A_{n-1}(\boldsymbol{x}) \geqslant x_n. \tag{4}$$

REMARK 1. By the arithmetic-geometric mean inequality, it follows that

$$\frac{x_n + G_{n-1}(\boldsymbol{x}) + \cdots + G_{n-1}(\boldsymbol{x})}{n} \geqslant [x_n G_{n-1}(\boldsymbol{x}) \cdots G_{n-1}(\boldsymbol{x})]^{\frac{1}{n}},$$

i.e.

$$x_n + (n-1)G_{n-1}(\boldsymbol{x}) \geqslant n\left[x_n G^{n-1}_{n-1}(\boldsymbol{x})\right]^{\frac{1}{n}} = nG_n(\boldsymbol{x}),$$

i.e.

$$x_n \geqslant nG_n(\boldsymbol{x}) - (n-1)G_{n-1}(\boldsymbol{x}). \tag{5}$$

(4) and (5) together give

$$nA_n(\boldsymbol{x}) - (n-1)A_{n-1}(\boldsymbol{x}) \geqslant x_n \geqslant nG_n(\boldsymbol{x}) - (n-1)G_{n-1}(\boldsymbol{x}). \tag{6}$$

The inequality (6) refine the equivalent form of Rado's inequality (1).

Taking $\alpha=\lambda=1$ and $k=1$, from the inequality (3), it follows that

COROLLARY 2. *Let* $\boldsymbol{x}\in\mathbb{R}^n_{++}$, $n\geqslant 2$, *we have*

$$A_n(\boldsymbol{x})\geqslant (x_n)^{\frac{1}{n}}[A_{n-1}(\boldsymbol{x})]^{1-\frac{1}{n}}. \tag{7}$$

REMARK 2. It is clear that inequality (7) is equivalent to

$$[A_n(\boldsymbol{x})]^n\geqslant x_n[A_{n-1}(\boldsymbol{x})]^{n-1},$$

and then

$$\left[\frac{A_n(\boldsymbol{x})}{G_n(\boldsymbol{x})}\right]^n\geqslant\frac{x_n[A_{n-1}(\boldsymbol{x})]^{n-1}}{[G_n(\boldsymbol{x})]^n}=\left[\frac{A_{n-1}(\boldsymbol{x})}{G_{n-1}(\boldsymbol{x})}\right]^{n-1}.$$

This shows that the inequality (7) is equivalent to Popoviciu's inequality (2).

THEOREM 2. *Let* $\boldsymbol{x}\in\mathbb{R}^n_{++}$, $n\geqslant 2$, $0<\alpha\leqslant 1$, $\lambda>0$. *Then for* $k=1,\cdots,n$, *we have*

$$\frac{\left[A_n(\boldsymbol{x})+\frac{(\lambda-1)x_n}{n}\right]^{k\alpha}}{[G_n(\boldsymbol{x})]^{n\alpha}}\geqslant\lambda^\alpha\cdot\frac{k}{n}\cdot\frac{[A_{n-1}(\boldsymbol{x})]^{(k-1)\alpha}}{[G_{n-1}(\boldsymbol{x})]^{(n-1)\alpha}}+\left(1-\frac{k}{n}\right)\frac{[A_{n-1}(\boldsymbol{x})]^{k\alpha}}{[G_n(\boldsymbol{x})]^{n\alpha}}. \tag{8}$$

REMARK 3. When $\alpha=\lambda=1$ and $k=n$, the inequality (8) can be simplified to Popoviciu's inequality (2), and when $\alpha=\lambda=1$ and $k=1$, the inequality (8) can be simplified to (4).

2. Definitions and Lemmas

We need the following definitions and auxiliary lemmas.

DEFINITION 1. [9, 10] Let $\boldsymbol{x}=(x_1,\cdots,x_n)$ and $\boldsymbol{y}=(y_1,\cdots,y_n)\in\mathbb{R}^n$.

- (i) $\boldsymbol{x}\geqslant\boldsymbol{y}$ means $x_i\geqslant y_i$ for all $i=1,2,\cdots,n$.
- (ii) Let $\Omega\subset\mathbb{R}^n$, $\varphi\colon\ \Omega\to\mathbb{R}$ is said to be increasing if $\boldsymbol{x}\geqslant\boldsymbol{y}$ implies $\varphi(\boldsymbol{x})\geqslant\varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

DEFINITION 2. [9, 10] Let $\boldsymbol{x}=(x_1,\cdots,x_n)$ and $\boldsymbol{y}=(y_1,\cdots,y_n)\in\mathbb{R}^n$.

- (i) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x}\prec\boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]}\leqslant\sum_{i=1}^k y_{[i]}$ for $k=1,2,\cdots,n-1$ and $\sum_{i=1}^n x_i=\sum_{i=1}^n y_i$, where $x_{[i]}$ denotes the ith largest component of $\boldsymbol{x}$, $x_{[1]}\geqslant\cdots\geqslant x_{[n]}$ and $y_{[1]}\geqslant\cdots\geqslant y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.
- (ii) Let $\Omega\subset\mathbb{R}^n$, $\varphi:\Omega\to\mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x}\prec\boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x})\leqslant\varphi(\boldsymbol{y})$.
  φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function on Ω.

The Schur-convexity described the ordering of majorization, the order-preserving functions were first comprehensively studied by I. Schur in 1923. It has important applications in combinatorial analysis, geometric inequalities, matrix theory, numerical analysis, etc. See [9].

LEMMA 1. [9, 10, 11] *Let* $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ *and* $\bar{\boldsymbol{x}} = \frac{1}{n}\sum_{i=1}^{n} x_i$. *Then*

$$(\bar{\boldsymbol{x}}, \cdots, \bar{\boldsymbol{x}}) \prec (x_1, \cdots, x_n).$$

LEMMA 2. [9, p. 115] *The elementary symmetric functions* $E_k(\boldsymbol{x})$ *are increasing and Schur-concave on* $\mathbb{R}^n_+$.

LEMMA 3. [9, p. 123] *The dual form of the elementary symmetric functions* $E^*_k(\boldsymbol{x})$ *are increasing and Schur-concave on* $\mathbb{R}^n_+$.

LEMMA 4. [10, p. 64] *Let the set* $\mathbb{A} \subset \mathbb{R}$. *The function* $\varphi : \mathbb{R}^n_+ \to \mathbb{R}$ *is increasing and Schur-concave, and the function* $g : \mathbb{A} \to \mathbb{R}_+$ *is concave. Then the composite function* $\psi(x_1, \cdots, x_n) = \varphi(g(x_1), \cdots, g(x_n)) : \mathbb{A}^n \to \mathbb{R}$ *is Schur-concave.*

LEMMA 5. *Let* $n \geqslant 2$, $t, \lambda > 0$, $g : \mathbb{R}_{++} \to \mathbb{R}_+$ *is concave. Then for* $k = 1, \cdots, n$, *we have*

$$g\left(\frac{t+\lambda}{n}\right) \geqslant \left[\left(1-\frac{1}{k}\right) g\left(\frac{t}{n-1}\right) + \frac{1}{k}\cdot g(\lambda)\right]^{\frac{k}{n}} \left[g\left(\frac{t}{n-1}\right)\right]^{1-\frac{k}{n}} \tag{9}$$

Proof. By Lemma 1, it follows that

$$u = (\underbrace{\frac{t+\lambda}{n}, \cdots, \frac{t+\lambda}{n}}_{n}) \prec (\underbrace{\frac{t}{n-1}, \cdots, \frac{t}{n-1}}_{n-1}, \lambda) = v, \tag{10}$$

and from Lemma 3 and Lemma 4, it follows that $E^*_k(g(x_1), \cdots, g(x_n))$ is Schur-concave on $\mathbb{R}^n_{++}$, combining (10) we get

$$E^*_k\left[g\left(\frac{t+\lambda}{n}\right), \cdots, g\left(\frac{t+\lambda}{n}\right)\right] \geqslant E^*_k\left[g\left(\frac{t}{n-1}\right), \cdots, g\left(\frac{t}{n-1}\right), g(\lambda)\right],$$

i.e.

$$\left[kg\left(\frac{t+\lambda}{n}\right)\right]^{\binom{n}{k}} \geqslant \left[(k-1)g\left(\frac{t}{n-1}\right) + g(\lambda)\right]^{\binom{n-1}{k-1}} \left[kg\left(\frac{t}{n-1}\right)\right]^{\binom{n-1}{k}}. \tag{11}$$

Extracting root of $\binom{n}{k}$ the both sides in the inequality (11), we get

$$kg\left(\frac{t+\lambda}{n}\right) \geqslant \left[(k-1)g\left(\frac{t}{n-1}\right) + g(\lambda)\right]^{\frac{k}{n}} \left[kg\left(\frac{t}{n-1}\right)\right]^{1-\frac{k}{n}} \tag{12}$$

Dividing both sides in the inequality (12) by $k = k^{\frac{k}{n}} \cdot k^{1-\frac{k}{n}}$, we obtain the inequality (9). □

LEMMA 6. *Let $n \geqslant 2$, $t, \lambda > 0$, $g : \mathbb{R}_{++} \to \mathbb{R}_+$ is concave. Then for $k = 1, \cdots, n$, we have*

$$\left[g\left(\frac{t+\lambda}{n}\right)\right]^k \geqslant \frac{k}{n} \cdot g(\lambda)\left[g\left(\frac{t}{n-1}\right)\right]^{k-1} + \left(1-\frac{k}{n}\right)\left[g\left(\frac{t}{n-1}\right)\right]^k \tag{13}$$

Proof. From Lemma 2 and Lemma 4, it follows that $E_k(g(x_1), \cdots, g(x_n))$ is Schur-concave on $\mathbb{R}^n_{++}$, combining (10) we get

$$E_k\left[g\left(\frac{t+\lambda}{n}\right), \cdots, g\left(\frac{t+\lambda}{n}\right)\right] \geqslant E_k\left[g\left(\frac{t}{n-1}\right), \cdots, g\left(\frac{t}{n-1}\right), g(\lambda)\right],$$

i.e.

$$\binom{n}{k}\left[g\left(\frac{t+\lambda}{n}\right)\right]^k \geqslant \binom{n-1}{k-1} g(\lambda)\left[g\left(\frac{t}{n-1}\right)\right]^{k-1} + \binom{n-1}{k}\left[g\left(\frac{t}{n-1}\right)\right]^k. \tag{14}$$

Dividing both sides in the inequality (14) by $\binom{n}{k}$, we obtain the inequality (13). □

3. Proof of main results

Proof of Theorem 1. Taking $g(y) = y^\alpha$, $0 < \alpha \leqslant 1$ and $t = \frac{\sum_{i=1}^{n-1} x_i}{x_n}$, from (9) it is deduced that

$$\left[\frac{A_n(\boldsymbol{x}) + \frac{(\lambda-1)x_n}{n}}{x_n}\right]^\alpha \geqslant \left[\left(1-\frac{1}{k}\right)\left(\frac{A_{n-1}(\boldsymbol{x})}{x_n}\right)^\alpha + \frac{1}{k}\lambda^\alpha\right]^{\frac{k}{n}}\left[\frac{A_{n-1}(\boldsymbol{x})}{x_n}\right]^{(1-\frac{k}{n})\alpha}. \tag{15}$$

Multiplying both sides in the inequality (15) by $(x_n)^\alpha = (x_n)^{\frac{k}{n}\alpha}(x_n)^{(1-\frac{k}{n})\alpha}$, we obtain the inequality (3).

The proof of Theorem 1 is completed. □

Proof of Theorem 2. Taking $g(y) = y^\alpha$, $0 < \alpha \leqslant 1$ and $t = \frac{\sum_{i=1}^{n-1} x_i}{x_n}$, from (13) it is deduced that

$$\left(\frac{A_n(\boldsymbol{x}) + \frac{(\lambda-1)x_n}{n}}{x_n}\right)^{k\alpha} \geqslant \lambda^\alpha \frac{k}{n}\left(\frac{A_{n-1}(\boldsymbol{x})}{x_n}\right)^{(k-1)\alpha} + \left(1-\frac{k}{n}\right)\left(\frac{A_{n-1}(\boldsymbol{x})}{x_n}\right)^{k\alpha}. \tag{16}$$

Multiplying both sides in the inequality (16) by $(x_n)^{k\alpha}\left(\prod_{i=1}^{n} x_i^{\alpha}\right)^{-1}$, we obtain the inequality (8).

The proof of Theorem 2 is completed. □

Conflict of interests. The authors declare that there is no conflict of interests regarding the publication of this article.

Acknowledgement. The work was supported by the National Natural Science Foundation of China (Grant No. 11501030).

REFERENCES

[1] D. S. MITRINOVIĆ, P. M. VASIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin, Heidelberg, 1970, 74–94.
[2] D. S. MITRINOVIĆ, J. E. PEČARIĆ, A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993, 21–56.
[3] CHUNG-LIE WANG, *Inequalities of the Rado-Popoviciu type for functions and their applications*, J. Math. Anal. Appl., **100** (2) (1984), 436–446.
[4] HUAN-NAN SHI, *A new generalization of the Popoviciu's inequality*, J. Sichuan Norm. Univ. Nat. Sci. Ed., **25** (5) (2002), 510–511, (in Chinese).
[5] VASILE MIHEŞAN, *Rado and Popoviciu type inequalities for pseudo arithmetic and geometric means*, Int. J. Pure Appl. Math., **23** (3) (2005), 293–297.
[6] M. BENCZE, *A generalization of T. Popoviciu's inequality*, Gazeta Mat., **96** (1991), 159–161.
[7] H. ALZER, *Rado-type inequalities for geometric and harmonic means*, J. Pure Appl. Math. Sci., **24** (1989), 125–130.
[8] S. WU, L. DEBNATH, *Weighted generalization of Rado's inequality and Popoviciu's inequality*, Appl. Math. Lett., **21** (2008), 313–319.
[9] A. W. MARSHALL, I. OLKIN, AND B. C. ARNOLD, *Inequalities: Theory of Majorization and Its Application* (Second Edition), Springer, New York, 2011.
[10] B. Y. WANG, *Foundations of Majorization Inequalities*, Beijing Normal University Press, Beijing, 1990, (in Chinese).
[11] H.-N. SHI, *Theory of Majorization and Analytic Inequalities*, Harbin Institute of Technology Press, Harbin, 2012, (in Chinese).

(Received June 3, 2015)

Jing Zhang
Basic Courses Department, Beijing Union University
Beijing 100101, P. R. China
e-mail: zhang1jing4@outlook.com, ldtzhangjing1@buu.edu.cn

Huan-Nan Shi
Department of Electronic Information, Teacher's College
Beijing Union University
Beijing 100011, P. R. China
e-mail: sfthuannan2014@qq.com, sfthuannan@buu.edu.cn