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BOUNDEDNESS FROM BELOW OF COMPOSITION OPERATORS ON α–LOGARITHMIC BLOCH SPACES

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Abstract. In this paper, we characterize the boundedness from below of composition operator on α-logarithmic Bloch spaces.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ be its boundary. Let $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ be the set of analytic self-maps of $\mathbb{D}$. For $a \in \mathbb{D}$, let σ_a be the automorphism of $\mathbb{D}$ exchanging 0 for a, namely $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$, $z \in \mathbb{D}$. It is well known that $\sigma_a^{-1} = \sigma_a$,

$$\frac{|\sigma_a'(z)|}{1-|\sigma_a(z)|^2} = \frac{1}{1-|z|^2}, \qquad \frac{(1-|z|^2)(1-|a|^2)}{|1-\bar{a}z|} = 1-|\sigma_a(z)|^2. \tag{1}$$

Let ψ be a Möbius mapping such that $a = \psi(0)$. For $z \in \mathbb{D}$, by (1) we get

$$\frac{1-|a|}{1+|a|} \leqslant \frac{1-|\psi(z)|^2}{1-|z|^2} = |\psi'(z)| \leqslant \frac{1+|a|}{1-|a|}. \tag{2}$$

Moreover, let $\psi, \sigma \in S(\mathbb{D})$ such that $\sigma(0) = 0$ and $\psi = \sigma_a \circ \sigma$. Then by the Schwarz lemma and (2), we obtain

$$\frac{1-|z|^2}{1-|\psi(z)|^2} = \frac{1-|z|^2}{1-|\sigma(z)|^2}\frac{1-|\sigma(z)|^2}{1-|\sigma_a(\sigma(z))|^2} \leqslant \frac{1+|a|}{1-|a|}, \quad z \in \mathbb{D}. \tag{3}$$

For $z, w \in \mathbb{D}$, the pseudo distance on $\mathbb{D}$ is defined by $\rho(z,w) = |\frac{z-w}{1-\bar{w}z}|$. Let $E \subset \mathbb{D}$. For every $w \in \mathbb{D}$, if there exists a $z \in E$ such that $\rho(z,w) \leqslant r$, then E is called a pseudo r-net. For $w \in \mathbb{C}$ and $t > 0$, we denote by $D(w,t)$ the disk with radius t and centered at w. For $w \in \mathbb{D}$ and $r \in (0,1)$, we denote by $\Delta(w,r)$ the pseudo disk, where $\Delta(w,r) = \{z \in \mathbb{D} : \rho(z,w) < r\}$. Let $\overline{D}(w,t)$ and $\overline{\Delta}(w,r)$ denote the closures of $D(w,t)$ and $\Delta(w,r)$, respectively.

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An $f \in H(\mathbb{D})$ is said to belong to the α-logarithmic Bloch space, denoted by $\mathscr{L}\mathscr{B}^{\alpha} = \mathscr{L}\mathscr{B}^{\alpha}(\mathbb{D})$, if

$$\|f\|_{\mathscr{L}\mathscr{B}^{\alpha}} = \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha}\ln\frac{2}{1-|z|^2}|f'(z)| < \infty.$$

The space $\mathscr{L}\mathscr{B}^{\alpha}$ becomes a Banach space with the norm $\|f\|_{\mathscr{L}} = |f(0)| + \|f\|_{\mathscr{L}\mathscr{B}^{\alpha}}$. If $\alpha = 1$, the space $\mathscr{L}\mathscr{B}^{\alpha}$ is the logarithmic Bloch space, which will be denoted by $\mathscr{L}\mathscr{B}$. From [1] or [11], we see that $\mathscr{L}\mathscr{B}\bigcap H^{\infty}$ is the space of multipliers of the Bloch space $\mathscr{B}$. Here H^{∞} is the space consisting of all bounded analytic functions and the Bloch space is defined as follows.

$$\mathscr{B} = \{f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}}|f'(z)|(1-|z|^2) < \infty\}.$$

The composition operator C_{φ} defined by $C_{\varphi}f = f\circ\varphi$ for $f \in H(\mathbb{D})$. The main subject in the study of composition operators is to describe operator theoretic properties of C_{φ} in terms of function theoretic properties of φ. See [5] and the references therein for the study of the composition operator.

See [8, 9, 10] for the study of composition operators on the logarithmic Bloch space. In [8], Ye studied the boundedness and compactness of composition operator on α-logarithmic Bloch spaces. For example, he proved that C_{φ} is bounded on $\mathscr{L}\mathscr{B}^{\alpha}$ if and only if $\sup_{z\in\mathbb{D}}\tau_{\varphi}^{\alpha}(z) < \infty$, where

$$\tau_{\varphi}^{\alpha}(z) = \frac{(1-|z|^2)^{\alpha}\ln\frac{2}{1-|z|^2}}{(1-|\varphi(z)|^2)^{\alpha}\ln\frac{2}{1-|\varphi(z)|^2}}|\varphi'(z)|.$$

Recall that the operator $C_{\varphi}: \mathscr{L}\mathscr{B}^{\alpha} \to \mathscr{L}\mathscr{B}^{\alpha}$ is said to be bounded, if there exists a $C > 0$, such that $\|C_{\varphi}f\|_{\mathscr{L}\mathscr{B}^{\alpha}} \leqslant C\|f\|_{\mathscr{L}\mathscr{B}^{\alpha}}$ for all $f \in \mathscr{L}\mathscr{B}^{\alpha}$. A bounded composition operator C_{φ} on $\mathscr{L}\mathscr{B}^{\alpha}$ is said to be bounded below, if there exists a $\delta > 0$ such that

$$\|C_{\varphi}f\|_{\mathscr{L}\mathscr{B}^{\alpha}} \geqslant \delta\|f\|_{\mathscr{L}\mathscr{B}^{\alpha}}$$

for all $f \in \mathscr{L}\mathscr{B}^{\alpha}$.

The boundedness from below of composition operator C_{φ} on $\mathscr{B}$ was first studied by Gathage, Yan and Zheng in [6]. See [2, 3, 6, 7] for some characterizations of the boundedness from below of composition operator on $\mathscr{B}$. The boundedness from below of multiplication operator on $\mathscr{B}$ was studied in [4].

In this paper, we investigate the condition for the boundedness from below of composition operator on $\mathscr{L}\mathscr{B}^{\alpha}$. We extend the results in [3] to the case of $\mathscr{L}\mathscr{B}^{\alpha}$.

Throughout the paper, we denote by C a positive constant which may differ from one occurrence to the next.

2. Main results and proofs

In this section, we give our main results and their proofs. Before stating these results, we need some auxiliary results, which are incorporated in the lemmas which follow.

LEMMA 1. *Let* $\alpha > 0$ *and* $0 \leqslant x, y < 1$. *Let* $f(x) = (1-x)^\alpha \ln \frac{2}{1-x}$. *Then*

$$\frac{1}{\frac{1}{\alpha e \ln 2} + \left(\frac{1-y}{1-x}\right)^\alpha} \leqslant \frac{f(x)}{f(y)} \leqslant \frac{1}{\alpha e \ln 2} + \left(\frac{1-x}{1-y}\right)^\alpha.$$

Proof. A short calculation shows that

$$\left(\frac{1-x}{1-y}\right)^\alpha \ln \frac{1-y}{1-x} \leqslant \frac{1}{\alpha e}.$$

Hence we get

$$\begin{aligned} \frac{f(x)}{f(y)} &= \frac{(1-x)^\alpha \ln \frac{2}{1-x}}{(1-y)^\alpha \ln \frac{2}{1-y}} \\ &= \left(\frac{1-x}{1-y}\right)^\alpha \frac{\ln \frac{2}{1-x} - \ln \frac{2}{1-y}}{\ln \frac{2}{1-y}} + \left(\frac{1-x}{1-y}\right)^\alpha \\ &\leqslant \frac{1}{\alpha e \ln 2} + \left(\frac{1-x}{1-y}\right)^\alpha. \end{aligned}$$

The other inequality follows in a similar way. □

LEMMA 2. [3, Lemma 4.1] *Let* $h \subset S(\mathbb{D})$ *such that* $h(0) = 0$. *If* $|h'(0)| \geqslant \varepsilon > 0$, *then there exist* $\delta_1, \delta_2 > 0$, *depending only on* ε, *such that*

(a) $|h'(z)| \geqslant \frac{\varepsilon}{2}$ *for* $z \in D(0, \delta_1)$;

(b) $\overline{D}(0, \delta_2) \subset h(D(0, \delta_1))$.

To state our main results, we need a definition and some notations. For $\alpha > 0$, a subset F of $\mathbb{D}$ is called a sampling set for $\mathscr{L}\mathscr{B}^\alpha$, if there exists a $k > 0$ such that

$$\|f\|_{\mathscr{L}\mathscr{B}^\alpha} \leqslant k \sup_{z \in F} (1-|z|^2)^\alpha \ln \frac{2}{1-|z|^2} |f'(z)|$$

hold for $f \in \mathscr{L}\mathscr{B}^\alpha$. For $\varepsilon > 0$, let

$$\Omega_\varepsilon^\alpha = \{z \in \mathbb{D} : \tau_\varphi^\alpha(z) \geqslant \varepsilon\}, \quad G_\varepsilon^\alpha = \varphi(\Omega_\varepsilon^\alpha).$$

THEOREM 1. *Let* $\alpha > 0$ *and* $\varphi \in S(\mathbb{D})$ *such that* C_φ *is bounded on* $\mathscr{L}\mathscr{B}^\alpha$. *Then* C_φ *is bounded below on* $\mathscr{L}\mathscr{B}^\alpha$ *if and only if there exists an* $\varepsilon > 0$ *such that* G_ε^α *is a sampling set for* $\mathscr{L}\mathscr{B}^\alpha$.

Proof. Suppose that G_ε^α with an $\varepsilon > 0$ is a sampling set for $\mathscr{L}\mathscr{B}^\alpha$. Then for any $f \in \mathscr{L}\mathscr{B}^\alpha$, there is a $z_f \in \mathbb{D}$ such that

$$\varphi(z_f) \in G_\varepsilon^\alpha, \quad \tau_\varepsilon^\alpha(z_f) \geqslant \varepsilon$$

and

$$\begin{aligned}\|f\|_{\mathscr{L}\mathscr{B}^\alpha} &\leqslant k \sup_{w \in G_\varepsilon^\alpha} (1-|w|^2)^\alpha \ln \frac{2}{1-|w|^2} |f'(w)| \\ &\leqslant 2k(1-|\varphi(z_f)|^2)^\alpha \ln \frac{2}{1-|\varphi(z_f)|^2} |f'(\varphi(z_f))|,\end{aligned}$$

where $k > 0$ is independent of f. So,

$$\begin{aligned}\|C_\varphi(f)\|_{\mathscr{L}\mathscr{B}^\alpha} &\geqslant (1-|z_f|^2)^\alpha \ln \frac{2}{1-|z_f|^2} |(C_\varphi(f))'(z_f)| \\ &= \tau_\varphi^\alpha(z_f)(1-|\varphi(z_f)|^2)^\alpha \ln \frac{2}{1-|\varphi(z_f)|^2} |f'(\varphi(z_f))| \\ &\geqslant \frac{\varepsilon \|f\|_{\mathscr{L}\mathscr{B}^\alpha}}{2k}.\end{aligned}$$

Thus, C_φ is bounded below on $\mathscr{L}\mathscr{B}^\alpha$.

Conversely, we assume that C_φ is bounded below on $\mathscr{L}\mathscr{B}^\alpha$, i.e., $\|C_\varphi(f)\|_{\mathscr{L}\mathscr{B}^\alpha} \geqslant \delta \|f\|_{\mathscr{L}\mathscr{B}^\alpha}$ for any $f \in \mathscr{L}\mathscr{B}^\alpha$ with $\delta > 0$ independent of f. Then, for any $f \in \mathscr{L}\mathscr{B}^\alpha$ with $\|f\|_{\mathscr{L}\mathscr{B}^\alpha} > 0$, there is a $z_f \in \mathbb{D}$ such that

$$\begin{aligned}&\tau_\varphi^\alpha(z_f)(1-|\varphi(z_f)|^2)^\alpha \ln \frac{2}{1-|\varphi(z_f)|^2} |f'(\varphi(z_f))| \\ &= \frac{(1-|z_f|^2)^\alpha \ln \frac{2}{1-|z_f|^2} |\varphi'(z_f)|}{(1-|\varphi(z_f)|^2)^\alpha \ln \frac{2}{1-|\varphi(z_f)|^2}} (1-|\varphi(z_f)|^2)^\alpha \ln \frac{2}{1-|\varphi(z_f)|^2} |f'(\varphi(z_f))| \\ &= (1-|z_f|^2)^\alpha \ln \frac{2}{1-|z_f|^2} |(C_\varphi(f))'(z_f)| \\ &\geqslant \frac{\delta}{2} \|f\|_{\mathscr{L}\mathscr{B}^\alpha}.\end{aligned}$$

In addition,

$$(1-|\varphi(z_f)|^2)^\alpha \ln \frac{2}{1-|\varphi(z_f)|^2} |f'(\varphi(z_f))| \leqslant \|f\|_{\mathscr{L}\mathscr{B}^\alpha}.$$

Thus,

$$\tau_\varphi^\alpha(z_f) \geqslant \frac{\delta}{2} \text{ and } (1-|\varphi(z_f)|^2)^\alpha \ln \frac{2}{1-|\varphi(z_f)|^2} |f'(\varphi(z_f))| \geqslant \frac{\delta}{2M} \|f\|_{\mathscr{L}\mathscr{B}^\alpha}.$$

Here $M := \sup_{z \in \mathbb{D}} \tau_\varphi^\alpha(z) < \infty$. Taking $\varepsilon = \frac{\delta}{2M}$, we see that G_ε^α contains all $\varphi(z_f)$ and is a sampling set for $\mathscr{L}\mathscr{B}^\alpha$. $\square$

THEOREM 2. *Let $\alpha > 1$. A sampling set for $\mathscr{L}\mathscr{B}^\alpha$ is a pseudo r-net. Conversely, if E is a pseudo r-net, then for any $\delta > 0$, the set $E_\delta = \bigcup_{z\in E} \overline{\Delta}(z,\delta)$ is a sampling set for $\mathscr{L}\mathscr{B}^\alpha$.*

Proof. Let $w \in \mathbb{D}$. Take

$$f_w(z) = 2\int_0^z \frac{1-|w|^2}{\ln \frac{4}{1-\bar{w}u}} \frac{1}{(1-\bar{w}u)^{\alpha+1}} du, \ \ z \in \mathbb{D}.$$

It is easy to check that $f_w \in \mathscr{L}\mathscr{B}^\alpha$ and $\|f_w\|_{\mathscr{L}\mathscr{B}^\alpha} \geqslant 1$. Suppose that F is a sampling set for $\mathscr{L}\mathscr{B}^\alpha$, then there exists a $z \in F$ such that

$$\|f_w\|_{\mathscr{L}\mathscr{B}^\alpha} \leqslant k(1-|z|^2)^\alpha \ln \frac{2}{1-|z|^2} |f_w'(z)|. \tag{4}$$

Similarly to the proof of Lemma 2.2 in [9], we get

$$\begin{aligned}
1 &\leqslant k(1-|z|^2)^\alpha \ln \frac{2}{1-|z|^2} |f_w'(z)| \\
&= \frac{2k(1-|z|^2)^\alpha \ln \frac{2}{1-|z|^2}(1-|w|^2)}{|\ln \frac{4}{1-\bar{w}z}||1-\bar{w}z|^{\alpha+1}} \\
&= \frac{2k \ln \frac{2}{1-|z|^2}(1-|z|^2)^{\alpha-1}}{|\ln \frac{4}{1-\bar{w}z}||1-\bar{w}z|^{\alpha-1}} \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{w}z|^2} \\
&\leqslant \frac{2k \ln \frac{2}{1-|z|^2}(1-|z|^2)^{\alpha-1}}{\ln \frac{4}{|1-\bar{w}z|}(|1-\bar{w}z|^{\alpha-1})} \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{w}z|^2} \\
&\leqslant 2k \max\left\{1, \frac{2^{\alpha-1} e^{\frac{1}{\alpha-1}}}{(\alpha-1)e\ln 4}\right\}(1-|\sigma_w(z)|^2).
\end{aligned}$$

Thus,

$$\rho(z,w) = |\sigma_w(z)| \leqslant r = \sqrt{1-1/C_\alpha},$$

where

$$C_\alpha = 2k \max\left\{1, \frac{2^{\alpha-1} e^{\frac{1}{\alpha-1}}}{(\alpha-1)e\ln 4}\right\}.$$

Therefore, F is a pseudo r-net.

Now, suppose that E is a pseudo r-net. We next to prove that E_δ is a sampling set for $\mathscr{L}\mathscr{B}^\alpha$ for any $\delta > 0$. Suppose on the contrary that there are a δ and a sequence $f_n \in \mathscr{L}\mathscr{B}^\alpha$ such that

$$\|f_n\|_{\mathscr{L}\mathscr{B}^\alpha} = 1 \ \text{ for } n = 1,2,3.\cdots \tag{5}$$

and

$$\sup_{z\in E_\delta}(1-|z|^2)^\alpha \ln \frac{2}{1-|z|^2}|f_n'(z)| = \varepsilon_n \to 0 \ \text{ as } n \to \infty. \tag{6}$$

For $n=1,2,3,\cdots$, let $z_n\in\mathbb{D}$ satisfy

$$(1-|z_n|^2)^{\alpha}\ln\frac{2}{1-|z_n|^2}|f_n'(z_n)|\geqslant\frac{1}{2}. \tag{7}$$

Since E is a pseudo r-net, we have a sequence $z_n'\in E$ such that $\rho(z_n,z_n')\leqslant r$ for $n=1,2,3,\cdots$. Let $t_n=\sigma_{z_n'}(z_n)$ and $g_n=f_n\circ\sigma_{z_n'}$ for $n=1,2,3,\cdots$.

Let $n\geqslant 1$ be fixed. We have

$$|t_n|=|\sigma_{z_n'}(z_n)|=\rho(z_n,z_n')\leqslant r. \tag{8}$$

For $w\in\mathbb{D}$, we obtain

$$\begin{aligned}
&(1-|w|^2)^{\alpha}\ln\frac{2}{1-|w|^2}|g_n'(w)| \qquad (9)\\
&=(1-|w|^2)^{\alpha-1}(1-|w|^2)\ln\frac{2}{1-|w|^2}|g_n'(w)|\\
&=(1-|w|^2)^{\alpha-1}(1-|\sigma_{z_n'}(w)|^2)\ln\frac{2}{1-|w|^2}|f_n'(\sigma_{z_n'}(w))|\\
&=\frac{(1-|w|^2)^{\alpha-1}\ln\frac{2}{1-|w|^2}}{(1-|\sigma_{z_n'}(w)|^2)^{\alpha-1}\ln\frac{2}{1-|\sigma_{z_n'}(w)|^2}}\times(1-|\sigma_{z_n'}(w)|^2)^{\alpha}\ln\frac{2}{1-|\sigma_{z_n'}(w)|^2}|f_n'(\sigma_{z_n'}(w))|.
\end{aligned}$$

By (3), (5) and Lemma 1, we have

$$\begin{aligned}
(1-|w|^2)^{\alpha}\ln\frac{2}{1-|w|^2}|g_n'(w)| &\leqslant\frac{1}{(\alpha-1)e\ln 2}+\Big(\frac{1-|w|^2}{1-|\sigma_{z_n'}(w)|^2}\Big)^{\alpha-1}\\
&\leqslant\frac{1}{(\alpha-1)e\ln 2}+\frac{(1+|z_n'|)^{\alpha-1}}{(1-|z_n'|)^{\alpha-1}}\\
&\leqslant\frac{1}{(\alpha-1)e\ln 2}+\frac{4^{\alpha-1}}{(1-|z_n'|^2)^{\alpha-1}}.
\end{aligned} \tag{10}$$

Set $w=t_n$ in (9). By (1), (7) and (8), we get

$$\begin{aligned}
&(1-|t_n|^2)^{\alpha}\ln\frac{2}{1-|t_n|^2}|g_n'(t_n)| \qquad (11)\\
&=\frac{(1-|t_n|^2)^{\alpha-1}\ln\frac{2}{1-|t_n|^2}}{(1-|\sigma_{z_n'}(t_n)|^2)^{\alpha-1}\ln\frac{2}{1-|\sigma_{z_n'}(t_n)|^2}}(1-|\sigma_{z_n'}(t_n)|^2)^{\alpha}\ln\frac{2}{1-|\sigma_{z_n'}(t_n)|^2}|f_n'(\sigma_{z_n'}(t_n))|\\
&=\frac{(1-|t_n|^2)^{\alpha-1}\ln\frac{2}{1-|t_n|^2}}{(1-|z_n|^2)^{\alpha-1}\ln\frac{2}{1-|z_n|^2}}(1-|z_n|)^{\alpha}\ln\frac{2}{1-|z_n|^2}|f_n'(z_n)|\\
&\geqslant\frac{1}{2\big(\frac{1}{(\alpha-1)e\ln 2}+\frac{(1-|z_n|^2)^{\alpha-1}}{(1-|t_n|^2)^{\alpha-1}}\big)}=\frac{1}{2\big(\frac{1}{(\alpha-1)e\ln 2}+|\sigma_{z_n'}'(t_n)|^{\alpha-1}\big)}\\
&\geqslant\frac{1}{2\big(\frac{1}{(\alpha-1)e\ln 2}+\frac{(1-|z_n'|^2)^{\alpha-1}}{(1-r)^{2(\alpha-1)}}\big)}\geqslant\frac{1}{2\big(\frac{1}{(\alpha-1)e\ln 2}+\frac{1}{(1-r)^{2(\alpha-1)}}\big)}.
\end{aligned}$$

If $|w| \leqslant \delta$, then

$$\rho(\sigma_{z'_n}(w), z'_n) = \rho(\sigma_{z'_n}(w), \varphi_{z'_n}(0)) = |w| \leqslant \delta.$$

Hence $\sigma_{z'_n}(w) \in E_\delta$. Thus, by (6),

$$(1-|\sigma_{z'_n}(w)|^2)^\alpha \ln \frac{2}{1-|\sigma_{z'_n}(w)|} |f'_n(\sigma_{z'_n}(w))| \leqslant \varepsilon_n,$$

which combing with (10) and (3) imply

$$\begin{aligned} &(1-|w|^2)^\alpha \ln \frac{2}{1-|w|^2} |g'_n(w)| \qquad (12)\\ = &\frac{(1-|w|^2)^{\alpha-1} \ln \frac{2}{1-|w|^2}}{(1-|\sigma_{z'_n}(w)|^2)^{\alpha-1} \ln \frac{2}{1-|\sigma_{z'_n}(w)|^2}} (1-|\sigma_{z'_n}(w)|^2)^\alpha \ln \frac{2}{1-|\sigma_{z'_n}(w)|^2} |f'_n(\sigma_{z'_n}(w))| \\ \leqslant &\left(\frac{1}{(\alpha-1)e\ln 2} + \frac{4^{\alpha-1}}{(1-|z'_n|^2)^{\alpha-1}}\right)\varepsilon_n, \quad |w| \leqslant \delta. \end{aligned}$$

For $n = 1, 2, \cdots$, let

$$h_n(w) = \left(\frac{1}{(\alpha-1)e\ln 2} + \frac{4^{\alpha-1}}{(1-|z'_n|^2)^{\alpha-1}}\right)^{-1} g'_n(w), \quad w \in \mathbb{D}.$$

By (10), h_n is bounded locally uniformly in $\mathbb{D}$. Using Montel's theorem, we may assume that h_n converges to a $h \in H(\mathbb{D})$ locally uniformly in $\mathbb{D}$, and $t_n \to t_0$ with $|t_0| \leqslant r$ by (10). Letting $n \to \infty$ in (11) and (12), we have

$$h(t_0) \geqslant \frac{1}{2\big((\alpha-1)e\ln 2 + \frac{1}{(1-r)^{2(\alpha-1)}}\big)} \frac{1}{(1-|t_0|^2) \ln \frac{2}{1-|t_0|^2}}$$

and $h(w) = 0$. Thus, we arrive at a contradiction. Therefore E_δ is a sampling set for any δ. $\square$

LEMMA 3. *Let $\alpha > 1$. For $\varepsilon > 0$, there exist δ, $\varepsilon' > 0$, which only depend on $\varphi(0)$, ε and α only, such that*

$$(G_\varepsilon^\alpha)_\delta = \bigcup_{w' \in G_\varepsilon^\alpha} \overline{\Delta}(w', \delta) \subset G_{\varepsilon'}^\alpha.$$

Proof. Let $w' = \varphi(z') \in G_\varepsilon^\alpha$, $\tau_\varphi^\alpha(z') \geqslant \varepsilon$ and $h = \sigma_{w'} \circ \varphi \circ \sigma_{z'}$. By Lemma 1 and (3), we get

$$\begin{aligned}|h'(0)| &= \frac{(1-|z'|^2)|\varphi'(z')|}{1-|w'|^2} \\ &= \frac{(1-|w'|)^{\alpha-1}\ln\frac{2}{1-|w'|^2}}{(1-|z'|)^{\alpha-1}\ln\frac{2}{1-|z'|^2}}\tau_\varphi^\alpha(z') \\ &\geqslant \frac{1}{\frac{1}{(\alpha-1)e\ln 2}+\left(\frac{1-|z'|}{1-|w'|}\right)^{\alpha-1}}\tau_\varphi^\alpha(z') \\ &\geqslant \left(\frac{1}{\frac{1}{(\alpha-1)e\ln 2}+\left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\alpha-1}}\right)\varepsilon = \varepsilon_1.\end{aligned}$$

By Lemma 2, there exists $t_1, t_2 > 0$ satisfying (a) and (b) of Lemma 2 with ε replaced by ε_1.

For $w \in \overline{\triangle}(w', t_2)$, let $\lambda = \sigma_{w'}(w) \in \overline{D}(0, t_2)$. By Lemma 2, there exists a $\xi \in D(0, t_1)$ such that $h(\xi) = \lambda$ and $h'(\xi) \geqslant \varepsilon_1/2$. Let $z = \sigma_{z'}(\xi)$. Then $\varphi(z) = w$. By Lemma 1, (1) and (3), we get

$$\begin{aligned}\tau_\varphi^\alpha(z) &= \frac{(1-|z|^2)^\alpha\ln\frac{2}{1-|z|^2}}{(1-|w|^2)^\alpha\ln\frac{2}{1-|w|^2}}|\varphi'(z)| \\ &= \frac{1-|z|^2}{1-|\xi|^2}\frac{1-|\xi|^2}{1-|\lambda|^2}\frac{1-|\lambda|^2}{1-|w|^2}|\sigma_{w'}'(\lambda)h'(\xi)\sigma_{z'}'(z)|\frac{(1-|z|^2)^{\alpha-1}\ln\frac{2}{1-|z|^2}}{(1-|w|^2)^{\alpha-1}\ln\frac{2}{1-|w|^2}} \\ &\geqslant \frac{\varepsilon_1}{2}\left(\frac{1-|z|^2}{1-|\sigma_{z'}(z)|^2}|\sigma_{z'}'(z)|\frac{1-|\lambda|^2}{1-|\sigma_{w'}(\lambda)|^2}|\sigma_{w'}'(\lambda)|\right)\frac{1-|\xi|^2}{1-|\lambda|^2}\frac{1}{\frac{1}{(\alpha-1)e\ln 2}+\left(\frac{1-|w|^2}{1-|z|^2}\right)^{\alpha-1}} \\ &\geqslant \frac{\varepsilon_1}{2}(1-t_1^2)\frac{1}{\frac{1}{(\alpha-1)e\ln 2}+\left(\frac{1+|\varphi(0)|^2}{1-|\varphi(0)|^2}\right)^{\alpha-1}},\end{aligned}$$

which implies that $\overline{\Delta}(w', t_2) \subset G_{\varepsilon'}^\alpha$, for $w' \in G_\varepsilon^\alpha$. The proof of the lemma is finished. $\square$

THEOREM 3. *Let $\alpha > 1$ and $\varphi \in S(\mathbb{D})$ such that C_φ is bounded on $\mathscr{LB}^\alpha$. Then, C_φ is bounded below on $\mathscr{LB}^\alpha$ if and only if there exist an $\varepsilon > 0$ and an r with $0 < r < 1$ such that G_ε^α is a pseudo r-net.*

Proof. First we assume that there exist an $\varepsilon > 0$ and an $r \in (0,1)$ such that G_ε^α is a pseudo r-net. By Lemma 3, there exist $\delta, \varepsilon' > 0$ such that $(G_\varepsilon^\alpha)_\delta \subset G_{\varepsilon'}^\alpha$. From Theorem 2, we see that $(G_\varepsilon^\alpha)_\delta$ is a sampling set for $\mathscr{LB}^\alpha$. Since $(G_\varepsilon^\alpha)_\delta \subset G_{\varepsilon'}^\alpha$, we see that $G_{\varepsilon'}^\alpha$ is a sampling set for $\mathscr{LB}^\alpha$.

Conversely, assume that C_φ is bounded below. Using Theorem 1, there exists an $\varepsilon > 0$ such that G_ε^α is a sampling set for $\mathscr{LB}^\alpha$. Therefore G_ε^α is a pseudo r-net with $r \in (0,1)$ by Theorem 2, completing the proof. $\square$

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