Journal of
Mathematical
Inequalities
Volume 10, Number 3 (2016), 713–723 doi:10.7153/jmi-10-58

REVERSES AND VARIATIONS OF YOUNG'S INEQUALITIES WITH KANTOROVICH CONSTANT

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(*Communicated by J. Pečarić*)

Abstract. In this paper, we obtain some improved Young and Heinz inequalities and the reverse versions for scalars and matrices with Kantorovich constant, equipped with the Hilbert-Schmidt norm, and then we present the corresponding interpolations of recent refinements in the literature.

1. Introduction

Let $B(H)$ be the C^*-algebra of all bounded linear operators on a Hilbert space H equipped with the operator norm and $S(H)$ the set of all bounded self-adjoint operators. For $X,Y \in S(H)$, we write $X \leqslant Y$ if $Y-X$ is positive, and $X<Y$ if $Y-X$ is positive invertible. The set of all positive operators of $S(H)$ will be denoted by $P(H)$.

Let $\mathbb{M}_n$ be the set of all $n\times n$ matrices with entries in the complex field $\mathbb{C}$. For $A=(a_{ij})\in\mathbb{M}_n$, unitarily invariant norms $|||\cdot|||$ are defined on the matrix algebra $\mathbb{M}_n$ so that $|||UAV|||=|||A|||$ for any unitary matrices U,V. The Hilbert-Schmidt norm of A is defined by $||A||_2=(\sum\limits_{j=1}^{n} s_j^2(A))^{1/2}$, where $s_1(A),s_2(A),\cdots,s_n(A)$ are the singular values of A, i.e. the eigenvalues of the positive matrix $|A|=(A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. It is known that the Hilbert-Schmidt norm is unitarily invariant.

The classical Young inequality says that if $a,b\geqslant 0$ and $0\leqslant v\leqslant 1$, then

$$a^v b^{1-v}\leqslant va+(1-v)b \tag{1.1}$$

with equality if and only if $a=b$.

This inequality has been studied, generalized and refined in different directions. It is worth to mention that in [7], J. Wu and J. Zhao obtained an improved version which can be stated as follows:

$$K(\sqrt{h},2)^{r'}a^v b^{1-v}+r(\sqrt{a}-\sqrt{b})^2\leqslant va+(1-v)b, \tag{1.2}$$

Mathematics subject classification (2010): 15A48, 15A60, 47A63.
Keywords and phrases: Young's inequality, Hilbert-Schmidt norm, Positive semi-definite matrix, Kantorovich constant.

where $h = \frac{b}{a}$, $r = \min\{v, 1-v\}$, $r' = \min\{2r, 1-2r\}$ and $K(\cdot, 2)$ is Kantorovich constant, defined by $K(t,2) = \frac{(t+1)^2}{4t}$ for $t > 0$.

On the other hand, they [7] also presented a reverse of the scalar Young type inequality with $a, b \in \mathbb{R}^+$ and $v \in [0,1] - \{\frac{1}{2}\}$:

$$K(\sqrt{h}, 2)^{-r'} a^v b^{1-v} + s(\sqrt{a} - \sqrt{b})^2 \geqslant va + (1-v)b, \tag{1.3}$$

where $h = \frac{b}{a}$, $s = \max\{v, 1-v\}$, $r = \min\{v, 1-v\}$ and $r' = \min\{2r, 1-2r\}$.

In [6], M. Sababheh, A. Yousef and R. Khalil presented a generalization of the Young's inequality as follow:

$$a^p b^q \leqslant \frac{p-q+r}{p-q+2r} a^{p+r} b^{q-r} + \frac{r}{p-q+2r} a^{q-r} b^{p+r}, \tag{1.4}$$

where $a, b \in \mathbb{R}^+$ and $p \geqslant q \geqslant r \geqslant 0$.

Then, they proved a series of interpolated inequalities, reverse inequalities and their matrix versions.

Since then, many researchers have tried to give new refinements and generalizations of these inequalities and have obtained a series of improvements. One can refer to the references of [2, 3, 4].

These inequalities are extended to matrices in various contexts. The original Young's inequality was first extended to $\mathbb{M}_n$ in [1] as follows: For $A, B, X \in \mathbb{M}_n$ and $A, B \in P(H)$, we have

$$|||A^p X B^q||| \leqslant \frac{p}{p+q} |||A^{p+q} X||| + \frac{q}{p+q} |||X B^{p+q}|||,$$

for all $p, q > 0$.

In [5], M. Sababheh interpolated the above inequality as follow:

$$|||A^p X B^q||| \leqslant \frac{p-q+r}{p-q+2r} |||A^{p+r} X B^{q-r}||| + \frac{r}{p-q+2r} |||A^{q-r} X B^{p+r}|||,$$

for all $p \geqslant q \geqslant r \geqslant 0$.

Then, each refinement of the scalar Young's inequality accompanies a corresponding refinement of the matrix inequality. For example, let $A, B \in P(H)$. Then the matrix versions of (1.2) and (1.3) are

$$K(\sqrt{h}, 2)^{r'} ||A^v X B^{1-v}||_2 + r||A^{\frac{1}{2}} X - X B^{\frac{1}{2}}||_2^2 \leqslant ||vAX + (1-v)XB||_2, \quad 0 \leqslant v \leqslant 1$$

and

$$K(\sqrt{h}, 2)^{-r'} ||A^v X B^{1-v}||_2 + s||A^{\frac{1}{2}} X - X B^{\frac{1}{2}}||_2^2 \geqslant ||vAX + (1-v)XB||_2, \quad 0 \leqslant v \leqslant 1$$

respectively, where $h = \frac{||B||_2}{||A||_2}$, $s = \max\{v, 1-v\}$, $r = \min\{v, 1-v\}$ and $r' = \min\{2r, 1-2r\}$.

Getting the matrix version from the scalars version is somehow easy in the case of the Hilbert-Schmidt norm, however, it is not always valid for general norms. In [6], M.

Sababheh, A. Yousef and R. Khalil gave a series of generalizations of the scalar Young type interpolated inequalities and the corresponding matrix versions of [2, 3, 4] for the Hilbert-Schmidt norm, furthermore, some reverse inequalities were obtained. However, interpolated inequalities for unitarily invariant norms have appeared recently in [5].

In this paper, we obtain 3-term refinements of Young's inequality, different from most results in the literature that treat 2-term refinements.

2. Refinements of the Young's inequality for scalars

We begin this section with an improvement of the Young type inequality with Kantorovich constant.

THEOREM 2.1. *Let* $a,b \in \mathbb{R}^+$ *and let* $p \geqslant q \geqslant r \geqslant 0$. *Then*

$$K(\sqrt{h},2)^{r'} a^p b^q + \frac{r}{p-q+2r}\left(a^{\frac{p+r}{2}} b^{\frac{q-r}{2}} - a^{\frac{q-r}{2}} b^{\frac{p+r}{2}}\right)^2 \leqslant \frac{p-q+r}{p-q+2r} a^{p+r} b^{q-r} + \frac{r}{p-q+2r} a^{q-r} b^{p+r}, \tag{2.1}$$

where $h = (\frac{b}{a})^{p-q+2r}$ *and* $r' = \min\left\{\frac{2r}{p-q+2r}, \frac{p-q}{p-q+2r}\right\}$.

Proof. Let $\frac{p-q+r}{p-q+2r} = v$. Then $\frac{r}{p-q+2r} = 1-v$, and by the inequality (1.2),we have

$$\begin{aligned}
&\frac{p-q+r}{p-q+2r} a^{p+r} b^{q-r} + \frac{r}{p-q+2r} a^{q-r} b^{p+r} \\
&= v(a^{p+r} b^{q-r}) + (1-v)(a^{q-r} b^{p+r}) \\
&\geqslant K(\sqrt{h},2)^{r'} (a^{p+r} b^{q-r})^v (a^{q-r} b^{p+r})^{1-v} + \frac{r}{p-q+2r}\left(a^{\frac{p+r}{2}} b^{\frac{q-r}{2}} - a^{\frac{q-r}{2}} b^{\frac{p+r}{2}}\right)^2 \\
&= K(\sqrt{h},2)^{r'} a^p b^q + \frac{r}{p-q+2r}\left(a^{\frac{p+r}{2}} b^{\frac{q-r}{2}} - a^{\frac{q-r}{2}} b^{\frac{p+r}{2}}\right)^2.
\end{aligned}$$

This completes the proof. □

In the following, we present some refinements of this inequality together with their reverse inequalities by using computations similar to those in [2, 3, 4].

To facilitate our statements, let $\alpha = \frac{p-q+r}{p-q+2r}$ and $\beta = \frac{r}{p-q+2r}$, for $p \geqslant q \geqslant r \geqslant 0$.

THEOREM 2.2. *Let* $a,b \in \mathbb{R}^+$ *and let* $p > q \geqslant r \geqslant 0$. *Then*

$$\begin{aligned}
&K(\sqrt{(1-2\beta)h},2)^{r'} (\alpha-\beta)^{2\beta} a^{2p} b^{2q} + \beta^2 (a^{p+r} b^{q-r} + a^{q-r} b^{p+r})^2 \\
&\quad + \gamma_0 (\alpha-\beta) a^{p+r} b^{q-r}\left(\frac{1}{\sqrt{1-2\beta}} a^{\frac{p+r}{2}} b^{\frac{q-r}{2}} - a^{\frac{q-r}{2}} b^{\frac{p+r}{2}}\right)^2 \\
&\leqslant (\alpha a^{p+r} b^{q-r} + \beta a^{q-r} b^{p+r})^2,
\end{aligned} \tag{2.2}$$

where $h=(\frac{b}{a})^{p-q+2r}$, $\gamma_0=\min\{2\beta,1-2\beta\}$ *and* $r'=\min\{2\gamma_0,1-2\gamma_0\}$.

Proof. Observe that

$$\begin{aligned}&(\alpha a^{p+r}b^{q-r}+\beta a^{q-r}b^{p+r})^2-\beta^2(a^{p+r}b^{q-r}+a^{q-r}b^{p+r})^2\\&\quad-\gamma_0(\alpha-\beta)a^{p+r}b^{q-r}\left(\frac{1}{\sqrt{1-2\beta}}a^{\frac{p+r}{2}}b^{\frac{q-r}{2}}-a^{\frac{q-r}{2}}b^{\frac{p+r}{2}}\right)^2\\&=(\alpha-\beta)a^{p+r}b^{q-r}\left[a^{p+r}b^{q-r}+2\beta a^{q-r}b^{p+r}-\gamma_0\left(\frac{1}{\sqrt{1-2\beta}}a^{\frac{p+r}{2}}b^{\frac{q-r}{2}}-a^{\frac{q-r}{2}}b^{\frac{p+r}{2}}\right)^2\right]\\&=(\alpha-\beta)a^{p+r}b^{q-r}\left[(1-2\beta)\frac{a^{p+r}b^{q-r}}{1-2\beta}+2\beta a^{q-r}b^{p+r}-\gamma_0\left(\frac{1}{\sqrt{1-2\beta}}a^{\frac{p+r}{2}}b^{\frac{q-r}{2}}-a^{\frac{q-r}{2}}b^{\frac{p+r}{2}}\right)^2\right]\\&\geqslant(\alpha-\beta)a^{p+r}b^{q-r}\left[K(\sqrt{(1-2\beta)h},2)^{r'}\left(\frac{a^{p+r}b^{q-r}}{1-2\beta}\right)^{1-2\beta}(a^{q-r}b^{p+r})^{2\beta}\right]\\&=K(\sqrt{(1-2\beta)h},2)^{r'}(\alpha-\beta)^{2\beta}a^{2p}b^{2q}.\end{aligned}$$

This completes the proof. □

Now we present the v-version of the inequality (2.2) as an application of the Theorem 2.2.

COROLLARY 2.3. *Let* $a,b\in\mathbb{R}^+$. *Then for* $0<v<\frac{1}{2}$,

$$K(\sqrt{(1-2v)h^{-1}},2)^{r'}(1-2v)^{2v}(a^vb^{1-v})^2+v^2(a+b)^2+\gamma_1(1-2v)b\left(\sqrt{a}-\sqrt{\frac{b}{1-2v}}\right)^2\leqslant(va+(1-v)b)^2,$$

for $\frac{1}{2}<v\leqslant 1$,

$$\begin{aligned}&K(\sqrt{(2v-1)h},2)^{r''}(2v-1)^{2(1-v)}(a^vb^{1-v})^2+(1-v)^2(a+b)^2\\&\quad+\gamma_2(2v-1)a\left(\sqrt{\frac{a}{2v-1}}-\sqrt{b}\right)^2\\&\leqslant(va+(1-v)b)^2,\end{aligned}$$

where $h=\frac{b}{a}$, $\gamma_1=\min\{2v,1-2v\}$, $\gamma_2=\min\{2v-1,2-2v\}$, $r'=\min\{2\gamma_1,1-2\gamma_1\}$ *and* $r''=\min\{2\gamma_2,1-2\gamma_2\}$.

Proof. Suppose first that $1-v>v$. Then, by replacing p by $1-v$, q by v and r by v in (2.2), we get

$$K(\sqrt{(1-2v)h^{-1}},2)^{r'}(1-2v)^{2v}(a^vb^{1-v})^2+v^2(a+b)^2+\gamma_1(1-2v)b\left(\sqrt{a}-\sqrt{\frac{b}{1-2v}}\right)^2\leqslant(va+(1-v)b)^2.$$

A similar argument works if $v>1-v$. □

THEOREM 2.4. *Let* $a,b \in \mathbb{R}^+$ *and let* $p \geqslant q \geqslant r \geqslant 0$. *Then*

$$\begin{aligned}&K(\sqrt{h},2)^{r'} a^{2p} b^{2q} + \beta^2 (a^{p+r} b^{q-r} - a^{q-r} b^{p+r})^2 \\ &\quad + \gamma_0 a^{p+r} b^{q-r} \left(a^{\frac{p+r}{2}} b^{\frac{q-r}{2}} - a^{\frac{q-r}{2}} b^{\frac{p+r}{2}} \right)^2 \\ &\leqslant (\alpha a^{p+r} b^{q-r} + \beta a^{q-r} b^{p+r})^2,\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&K(\sqrt{h},2)^{-r'} a^{2p} b^{2q} + \beta^2 (a^{p+r} b^{q-r} - a^{q-r} b^{p+r})^2 \\ &\quad + s_0 a^{p+r} b^{q-r} \left(a^{\frac{p+r}{2}} b^{\frac{q-r}{2}} - a^{\frac{q-r}{2}} b^{\frac{p+r}{2}} \right)^2 \\ &\geqslant (\alpha a^{p+r} b^{q-r} + \beta a^{q-r} b^{p+r})^2,\end{aligned} \tag{2.4}$$

where $h = (\frac{b}{a})^{p-q+2r}$, $\gamma_0 = \min\{2\beta, 1-2\beta\}$, $s_0 = \max\{2\beta, 1-2\beta\}$ *and* $r' = \min\{2\gamma_0, 1-2\gamma_0\}$.

Proof. For (2.3), observe that

$$\begin{aligned}&(\alpha a^{p+r} b^{q-r} + \beta a^{q-r} b^{p+r})^2 - \beta^2 (a^{p+r} b^{q-r} - a^{q-r} b^{p+r})^2 \\ &\quad - \gamma_0 a^{p+r} b^{q-r} \left(a^{\frac{p+r}{2}} b^{\frac{q-r}{2}} - a^{\frac{q-r}{2}} b^{\frac{p+r}{2}} \right)^2 \\ &= a^{p+r} b^{q-r} \left[(\alpha-\beta) a^{p+r} b^{q-r} + 2\beta a^{q-r} b^{p+r} - \gamma_0 \left(a^{\frac{p+r}{2}} b^{\frac{q-r}{2}} - a^{\frac{q-r}{2}} b^{\frac{p+r}{2}} \right)^2 \right] \\ &\geqslant a^{p+r} b^{q-r} \left[K(\sqrt{h},2)^{r'} (a^{p+r} b^{q-r})^{\alpha-\beta} (a^{q-r} b^{p+r})^{2\beta} \right] \\ &= K(\sqrt{h},2)^{r'} a^{2p} b^{2q}.\end{aligned}$$

As for (2.4), using the inequality (1.3), observe that

$$\begin{aligned}&(\alpha a^{p+r} b^{q-r} + \beta a^{q-r} b^{p+r})^2 - \beta^2 (a^{p+r} b^{q-r} - a^{q-r} b^{p+r})^2 \\ &\quad - s a^{p+r} b^{q-r} \left(a^{\frac{p+r}{2}} b^{\frac{q-r}{2}} - a^{\frac{q-r}{2}} b^{\frac{p+r}{2}} \right)^2 \\ &= a^{p+r} b^{q-r} \left[(\alpha-\beta) a^{p+r} b^{q-r} + 2\beta a^{q-r} b^{p+r} - s \left(a^{\frac{p+r}{2}} b^{\frac{q-r}{2}} - a^{\frac{q-r}{2}} b^{\frac{p+r}{2}} \right)^2 \right] \\ &\leqslant a^{q-r} b^{p+r} \left[K(\sqrt{h},2)^{-r'} (a^{p+r} b^{q-r})^{\alpha-\beta} (a^{q-r} b^{p+r})^{2\beta} \right] \\ &= K(\sqrt{h},2)^{-r'} a^{2p} b^{2q}. \quad \square\end{aligned}$$

By the same processing methods of Theorem 2.4, we can obtain the following Theorems.

THEOREM 2.5. *Let* $a,b \in \mathbb{R}^+$ *and let* $p \geqslant q \geqslant r \geqslant 0$. *Then*

$$\begin{aligned} &K(\sqrt{h},2)^{r'}\beta^{2\beta}a^p b^q + \beta^2\left(a^{\frac{p+r}{2}}b^{\frac{q-r}{2}} - a^{\frac{q-r}{2}}b^{\frac{p+r}{2}}\right)^2 \\ &\quad + \gamma_0 a^{\frac{p+r}{2}}b^{\frac{q-r}{2}}\left(a^{\frac{p+r}{4}}b^{\frac{q-r}{4}} - \sqrt{\beta}a^{\frac{q-r}{4}}b^{\frac{p+r}{4}}\right)^2 \\ &\leqslant \alpha^2 a^{p+r}b^{q-r} + \beta^2 a^{q-r}b^{p+r}, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &K(\sqrt{h},2)^{-r'}\beta^{2\beta}a^p b^q + \beta^2\left(a^{\frac{p+r}{2}}b^{\frac{q-r}{2}} - a^{\frac{q-r}{2}}b^{\frac{p+r}{2}}\right)^2 \\ &\quad + s_0 a^{\frac{p+r}{2}}b^{\frac{q-r}{2}}\left(a^{\frac{p+r}{4}}b^{\frac{q-r}{4}} - \sqrt{\beta}a^{\frac{q-r}{4}}b^{\frac{p+r}{4}}\right)^2 \\ &\geqslant \alpha^2 a^{p+r}b^{q-r} + \beta^2 a^{q-r}b^{p+r}, \end{aligned} \tag{2.6}$$

where $h = (\frac{b}{a})^{p-q+2r}$, $\gamma_0 = \min\{2\beta, 1-2\beta\}$, $s_0 = \max\{2\beta, 1-2\beta\}$ *and* $r' = \min\{2\gamma_0, 1-2\gamma_0\}$.

THEOREM 2.6. *Let* $a,b \in \mathbb{R}^+$ *and let* $p \geqslant q \geqslant r \geqslant 0$. *Then*

$$\begin{aligned} &K(\sqrt{h},2)^{r'}a^p b^q + \beta\left(a^{\frac{p+r}{2}}b^{\frac{q-r}{2}} - a^{\frac{q-r}{2}}b^{\frac{p+r}{2}}\right)^2 \\ &\quad + \gamma_0 a^{\frac{p+r}{2}}b^{\frac{q-r}{2}}\left(a^{\frac{p+r}{4}}b^{\frac{q-r}{4}} - a^{\frac{q-r}{4}}b^{\frac{p+r}{4}}\right)^2 \\ &\leqslant \alpha a^{p+r}b^{q-r} + \beta a^{q-r}b^{p+r}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &K(\sqrt{h},2)^{-r'}a^p b^q + \beta\left(a^{\frac{p+r}{2}}b^{\frac{q-r}{2}} - a^{\frac{q-r}{2}}b^{\frac{p+r}{2}}\right)^2 \\ &\quad + s_0 a^{\frac{p+r}{2}}b^{\frac{q-r}{2}}\left(a^{\frac{p+r}{4}}b^{\frac{q-r}{4}} - a^{\frac{q-r}{4}}b^{\frac{p+r}{4}}\right)^2 \\ &\geqslant \alpha a^{p+r}b^{q-r} + \beta a^{q-r}b^{p+r}, \end{aligned} \tag{2.8}$$

where $h = (\frac{b}{a})^{p-q+2r}$, $\gamma_0 = \min\{2\beta, 1-2\beta\}$, $s_0 = \max\{2\beta, 1-2\beta\}$ *and* $r' = \min\{2\gamma_0, 1-2\gamma_0\}$.

The following Corollary can be easily obtained by applying Theorem 2.6 twice and will be used to prove the refined interpolated Heinz inequality.

COROLLARY 2.7. *Let* $a,b \in \mathbb{R}^+$ *and let* $p \geqslant q \geqslant r \geqslant 0$. *Then*

$$\begin{aligned} &K(\sqrt{h},2)^{r'}(a^p b^q + a^q b^p)^2 + 2\beta(a^{p+r}b^{q-r} - a^{q-r}b^{p+r})^2 \\ &\quad + \gamma_0(a^{p+r}b^{q-r} + a^{q-r}b^{p+r})\left(a^{\frac{p+r}{2}}b^{\frac{q-r}{2}} - a^{\frac{q-r}{2}}b^{\frac{p+r}{2}}\right)^2 - \left(K(\sqrt{h},2)^{r'} - 1\right)a^{p+q}b^{p+q} \\ &\leqslant (a^{p+r}b^{q-r} + a^{q-r}b^{p+r})^2, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned}&K(\sqrt{h},2)^{-r'}(a^pb^q+a^qb^p)^2+2\beta(a^{p+r}b^{q-r}-a^{q-r}b^{p+r})^2 \\ &+s_0(a^{p+r}b^{q-r}+a^{q-r}b^{p+r})\left(a^{\frac{p+r}{2}}b^{\frac{q-r}{2}}-a^{\frac{q-r}{2}}b^{\frac{p+r}{2}}\right)^2-\left(K(\sqrt{h},2)^{-r'}-1\right)a^{p+q}b^{p+q} \\ &\geqslant(a^{p+r}b^{q-r}+a^{q-r}b^{p+r})^2,\end{aligned} \tag{2.10}$$

where $h=(\frac{b}{a})^{p-q+2r}$, $\gamma_0=\min\{2\beta,1-2\beta\}$, $s_0=\max\{2\beta,1-2\beta\}$ *and* $r'=\min\{2\gamma_0, 1-2\gamma_0\}$.

REMARK 1. Since $K(t,2)=\frac{(t+1)^2}{4t}\geqslant 1$ for all $t>0$, the inequalities (2.1)–(2.10) except the reverse inequalities, are the improvements of the scalar Young type inequalities of [6].

REMARK 2. Obviously, the inequalities (2.1)–(2.10) are 3-term refinements of Young's inequality, different from most results in the literature that treat 2-term refinements.

3. Refinements of the Young's inequality for matrices

Based on the improvements of the scalar Young type inequalities (2.1)–(2.10), we present matrix versions of these inequalities.

We first prove the matrix version of Theorem 2.2.

THEOREM 3.1. *Let* $A,B,X\in\mathbb{M}_n$ *such that* $A,B\in P(H)$ *and let* $p>q\geqslant r\geqslant 0$. *Then*

$$\begin{aligned}&K(\sqrt{(1-2\beta)h},2)^{r'}(\alpha-\beta)^{2\beta}||A^pXB^q||_2^2+\beta^2||A^{p+r}XB^{q-r}\pm A^{q-r}XB^{p+r}||_2^2 \\ &\quad+\gamma_0(\alpha-\beta)\left\|\frac{1}{\sqrt{1-2\beta}}A^{p+r}XB^{q-r}-A^{\frac{p+q}{2}}XB^{\frac{p+q}{2}}\right\|_2^2 \\ &\leqslant||\alpha A^{p+r}XB^{q-r}+\beta A^{q-r}XB^{p+r}||_2^2,\end{aligned} \tag{3.1}$$

where $h=\left(\frac{||B||_2}{||A||_2}\right)^{p-q+2r}$, $\gamma_0=\min\{2\beta,1-2\beta\}$ *and* $r'=\min\{2\gamma_0,1-2\gamma_0\}$.

Proof. Since $A,B\geqslant 0$, then by the spectral decomposition, there are unitary matrices $U,V\in\mathbb{M}_n$ such that $A=U\Lambda U^*$ and $B=V\Gamma V^*$, where $\Lambda=diag(\lambda_1,\lambda_2,\cdots,\lambda_n)$, $\Gamma=diag(\mu_1,\mu_2,\cdots,\mu_n)$, and λ_j, μ_j $(j=1,\cdots,n)$ are the eigenvalues of A and B, respectively. Let $Y=U^*XV=[y_{ij}]$. Then

$$A^pXB^q=U\Lambda^pY\Gamma^qV^*=U\left[\lambda_i^p\mu_j^qy_{ij}\right]V^*, \tag{3.2}$$

$$A^{p+r}XB^{q-r}\pm A^{q-r}XB^{p+r}=U\left[\left(\lambda_i^{p+r}\mu_j^{q-r}\pm\lambda_i^{q-r}\mu_j^{p+r}\right)y_{ij}\right]V^*, \tag{3.3}$$

$$\alpha A^{p+r}XB^{q-r}+\beta A^{q-r}XB^{p+r}=U\left[\left(\alpha\lambda_i^{p+r}\mu_j^{q-r}+\beta\lambda_i^{q-r}\mu_j^{p+r}\right)y_{ij}\right]V^*, \tag{3.4}$$

$$\frac{1}{\sqrt{1-2\beta}}A^{p+r}XB^{q-r}-A^{\frac{p+q}{2}}XB^{\frac{p+q}{2}}=U\left[\left(\frac{1}{\sqrt{1-2\beta}}\lambda_i^{p+r}\mu_j^{q-r}-\lambda_i^{\frac{p+q}{2}}\mu_j^{\frac{p+q}{2}}\right)y_{ij}\right]V^*, \tag{3.5}$$

It follows from (3.2), (3.3), (3.4), (3.5) and Theorem 2.2 that

$$\begin{aligned}
&K(\sqrt{(1-2\beta)h},2)^{r'}(\alpha-\beta)^{2\beta}||A^pXB^q||_2^2+\beta^2||A^{p+r}XB^{q-r}\pm A^{q-r}XB^{p+r}||_2^2\\
&\quad+\gamma_0(\alpha-\beta)||\frac{1}{\sqrt{1-2\beta}}A^{p+r}XB^{q-r}-A^{\frac{p+q}{2}}XB^{\frac{p+q}{2}}||_2^2\\
=&K(\sqrt{(1-2\beta)h},2)^{r'}(\alpha-\beta)^{2\beta}\sum_{i,j=1}^{n}\left(\lambda_i^{2p}\mu_j^{2q}\right)|y_{ij}|^2\\
&\quad+\beta^2\sum_{i,j=1}^{n}\left(\lambda_i^{p+r}\mu_j^{q-r}\pm\lambda_i^{q-r}\mu_j^{p+r}\right)^2|y_{ij}|^2\\
&\quad+\gamma_0(\alpha-\beta)\sum_{i,j=1}^{n}\left(\frac{1}{\sqrt{1-2\beta}}\lambda_i^{p+r}\mu_j^{q-r}-\lambda_i^{\frac{p+q}{2}}\mu_j^{\frac{p+q}{2}}\right)^2|y_{ij}|^2\\
\leqslant&\sum_{i,j=1}^{n}\left(\alpha\lambda_i^{p+r}\mu_j^{q-r}+\beta\lambda_i^{q-r}\mu_j^{p+r}\right)^2|y_{ij}|^2\\
=&||\alpha A^{p+r}XB^{q-r}+\beta A^{q-r}XB^{p+r}||_2^2.
\end{aligned}$$

This completes the proof. □

By similar computations to Theorem 3.1, one can prove the matrix version of Theorem 2.4.

THEOREM 3.2. *Let $A,B,X\in\mathbb{M}_n$ such that $A,B\in P(H)$ and let $p\geqslant q\geqslant r\geqslant 0$. Then*

$$\begin{aligned}
&K(\sqrt{h},2)^{r'}||A^pXB^q||_2^2+\beta^2||A^{p+r}XB^{q-r}-A^{q-r}XB^{p+r}||_2^2\\
&\quad+\gamma_0||A^{p+r}XB^{q-r}-A^{\frac{p+q}{2}}XB^{\frac{p+q}{2}}||_2^2\\
\leqslant&||\alpha A^{p+r}XB^{q-r}+\beta A^{q-r}XB^{p+r}||_2^2,
\end{aligned}$$

and

$$\begin{aligned}
&K(\sqrt{h},2)^{-r'}||A^pXB^q||_2^2+\beta^2||A^{p+r}XB^{q-r}-A^{q-r}XB^{p+r}||_2^2\\
&\quad+s_0||A^{p+r}XB^{q-r}-A^{\frac{p+q}{2}}XB^{\frac{p+q}{2}}||_2^2\\
\geqslant&||\alpha A^{p+r}XB^{q-r}+\beta A^{q-r}XB^{p+r}||_2^2,
\end{aligned}$$

where $h=\left(\frac{||B||_2}{||A||_2}\right)^{p-q+2r}$, $\gamma_0=\min\{2\beta,1-2\beta\}$, $s_0=\max\{2\beta,1-2\beta\}$ and $r'=\min\{2\gamma_0,1-2\gamma_0\}$.

In the following, we give the matrix version of Theorem 2.5.

THEOREM 3.3. *Let $A,B,X \in \mathbb{M}_n$ such that $A,B \in P(H)$ and let $p \geqslant q \geqslant r \geqslant 0$. Then*

$$\begin{aligned} &K(\sqrt{h},2)^{r'}\beta^{2\beta}||A^pXB^q||_2^2+\beta^2||A^{p+r}XB^{q-r}-A^{q-r}XB^{p+r}||_2^2 \\ &\quad +2\alpha\beta||A^{\frac{p+q}{2}}XB^{\frac{p+q}{2}}||_2^2+\gamma_0||A^{\frac{p+r}{2}}XB^{\frac{q-r}{2}}-\sqrt{\beta}A^{\frac{p+q}{4}}XB^{\frac{p+q}{4}}||_2^2 \\ &\leqslant ||\alpha A^{p+r}XB^{q-r}+\beta A^{q-r}XB^{p+r}||_2^2, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &K(\sqrt{h},2)^{-r'}\beta^{2\beta}||A^pXB^q||_2^2+\beta^2||A^{p+r}XB^{q-r}-A^{q-r}XB^{p+r}||_2^2 \\ &\quad +2\alpha\beta||A^{\frac{p+q}{2}}XB^{\frac{p+q}{2}}||_2^2+s_0||A^{\frac{p+r}{2}}XB^{\frac{q-r}{2}}-\sqrt{\beta}A^{\frac{p+q}{4}}XB^{\frac{p+q}{4}}||_2^2 \\ &\geqslant ||\alpha A^{p+r}XB^{q-r}+\beta A^{q-r}XB^{p+r}||_2^2, \end{aligned} \tag{3.7}$$

where $h=\left(\frac{||B||_2}{||A||_2}\right)^{p-q+2r}$, $\gamma_0=\min\{2\beta,1-2\beta\}$, $s_0=\max\{2\beta,1-2\beta\}$ and $r'=\min\{2\gamma_0,1-2\gamma_0\}$.

Proof. For (3.6), following the same notations of the Theorem 3.1, we have

$$A^{\frac{p+q}{2}}XB^{\frac{p+q}{2}}=U\left[\left(\lambda_i^{\frac{p+q}{2}}\mu_j^{\frac{p+q}{2}}\right)y_{ij}\right]V^*, \tag{3.8}$$

$$A^{p+r}XB^{q-r}-A^{q-r}XB^{p+r}=U\left[\left(\lambda_i^{p+r}\mu_j^{q-r}-\lambda_i^{q-r}\mu_j^{p+r}\right)y_{ij}\right]V^*, \tag{3.9}$$

$$A^{\frac{p+r}{2}}XB^{\frac{q-r}{2}}-\sqrt{\beta}A^{\frac{p+q}{4}}XB^{\frac{p+q}{4}}=U\left[\left(\lambda_i^{\frac{p+r}{2}}\mu_j^{\frac{q-r}{2}}-\sqrt{\beta}\lambda_i^{\frac{p+q}{4}}\mu_j^{\frac{p+q}{4}}\right)y_{ij}\right]V^*, \tag{3.10}$$

It follows from (3.2), (3.4), (3.8)–(3.10) and Theorem 2.5 that

$$\begin{aligned} &K(\sqrt{h},2)^{r'}\beta^{2\beta}||A^pXB^q||_2^2+\beta^2||A^{p+r}XB^{q-r}-A^{q-r}XB^{p+r}||_2^2+2\alpha\beta||A^{\frac{p+q}{2}}XB^{\frac{p+q}{2}}||_2^2 \\ &\quad +\gamma_0||A^{\frac{p+r}{2}}XB^{\frac{q-r}{2}}-\sqrt{\beta}A^{\frac{p+q}{4}}XB^{\frac{p+q}{4}}||_2^2 \\ &=K(\sqrt{h},2)^{r'}\beta^{2\beta}\sum_{i,j=1}^n\left(\lambda_i^{2p}\mu_j^{2q}\right)|y_{ij}|^2+\beta^2\sum_{i,j=1}^n\left(\lambda_i^{p+r}\mu_j^{q-r}-\lambda_i^{q-r}\mu_j^{p+r}\right)^2|y_{ij}|^2 \\ &\quad +2\alpha\beta\sum_{i,j=1}^n\left(\lambda_i^{\frac{p+q}{2}}\mu_j^{\frac{p+q}{2}}\right)^2|y_{ij}|^2+\gamma_0\sum_{i,j=1}^n\left(\lambda_i^{\frac{p+r}{2}}\mu_j^{\frac{q-r}{2}}-\sqrt{\beta}\lambda_i^{\frac{p+q}{4}}\mu_j^{\frac{p+q}{4}}\right)^2|y_{ij}|^2 \\ &\leqslant\sum_{i,j=1}^n\left(\alpha^2\lambda_i^{p+r}\mu_j^{q-r}+\beta^2\lambda_i^{q-r}\mu_j^{p+r}\right)|y_{ij}|^2+2\alpha\beta\sum_{i,j=1}^n\left(\lambda_i^{\frac{p+q}{2}}\mu_j^{\frac{p+q}{2}}\right)^2|y_{ij}|^2 \\ &=\sum_{i,j=1}^n\left(\alpha\lambda_i^{p+r}\mu_j^{q-r}+\beta\lambda_i^{q-r}\mu_j^{p+r}\right)^2|y_{ij}|^2 \\ &=||\alpha A^{p+r}XB^{q-r}+\beta A^{q-r}XB^{p+r}||_2^2. \end{aligned}$$

For (3.7), the proceeding is similar to that of the above.

This completes the proof. □

Next, we present the matrix version of Corollary 2.7.

THEOREM 3.4. *Let $A,B,X \in \mathbb{M}_n$ such that $A,B \in P(H)$ and let $p \geqslant q \geqslant r \geqslant 0$. Then*

$$\begin{aligned}
&K(\sqrt{h},2)^{r'}||A^pXB^q + A^qXB^p||_2^2 + 2\beta||A^{p+r}XB^{q-r} - A^{q-r}XB^{p+r})^2 \\
&\quad + \gamma_0\left(||A^{p+r}XB^{q-r} - A^{\frac{p+q}{2}}XB^{\frac{p+q}{2}}||_2^2 + ||A^{q-r}XB^{p+r} - A^{\frac{p+q}{2}}XB^{\frac{p+q}{2}}||_2^2\right) \\
&\quad - \left(K(\sqrt{h},2)^{r'} - 1\right)||A^{\frac{p+q}{2}}XB^{\frac{p+q}{2}}||_2^2 \\
&\leqslant ||A^{p+r}XB^{q-r} + A^{q-r}XBb^{p+r}||_2^2,
\end{aligned}$$

and

$$\begin{aligned}
&K(\sqrt{h},2)^{-r'}||A^pXB^q + A^qXB^p||_2^2 + 2\beta||A^{p+r}XB^{q-r} - A^{q-r}XB^{p+r})^2 \\
&\quad + s_0\left(||A^{p+r}XB^{q-r} - A^{\frac{p+q}{2}}XB^{\frac{p+q}{2}}||_2^2 + ||A^{q-r}XB^{p+r} - A^{\frac{p+q}{2}}XB^{\frac{p+q}{2}}||_2^2\right) \\
&\quad - \left(K(\sqrt{h},2)^{-r'} - 1\right)||A^{\frac{p+q}{2}}XB^{\frac{p+q}{2}}||_2^2 \\
&\geqslant ||A^{p+r}XB^{q-r} + A^{q-r}XBb^{p+r}||_2^2,
\end{aligned}$$

where $h = \left(\frac{||B||_2}{||A||_2}\right)^{p-q+2r}$, $\gamma_0 = \min\{2\beta, 1-2\beta\}$, $s_0 = \max\{2\beta, 1-2\beta\}$ *and* $r' = \min\{2\gamma_0, 1-2\gamma_0\}$.

REMARK 3. Obviously, the inequalities of section 3 are the improvements of the matrix version Young type inequalities of [6]. And the reverse inequalities are the refinements of the Young type inequality which are different from those in [6].

Acknowledgement. The authors thank the handling editor and referees for careful reading of the manuscript and useful comments.

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(Received June 8, 2015)

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