SOME HARDY INEQUALITIES ON THE SPHERE

Yingxiong Xiao

(*Communicated by S. Stević*)

Abstract. In this paper we establish some Hardy inequalities related to the geodesic distance on the sphere and compute the corresponding sharp constants. Moreover, some Rellich inequalities have been also established and the constants are also sharp.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqslant 3$, and $0 \in \Omega$. The Hardy inequality reads, for all $f \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla f|^2 dx \geqslant \frac{(N-2)^2}{4} \int_\Omega \frac{f^2}{|x|^2} dx. \tag{1}$$

Furthermore, the constant $\frac{(N-2)^2}{4}$ in (1) is sharp and never archived. It is well-known that Hardy inequalities play an important role in the study of linear and nonlinear partial differential equations (see e.g. [1, 2, 3, 4, 5, 8, 9, 17]).

In the case of Riemannian manifold M, Carron [6] obtained some weighted L^2 Hardy inequalities. It reads that, for $f \in C_0^\infty(M - \rho^{-1}\{0\})$,

$$\int_M \frac{|\nabla f|^2}{\rho^\beta} \geqslant \frac{(C-\beta-1)^2}{4} \int_M \frac{f^2}{\rho^{2+\beta}}, \tag{2}$$

where $C - \beta - 1 > 0$ and the weight function ρ satisfies the Eikonal equation $|\nabla \rho| = 1$ and $\Delta \rho \geqslant C/\rho$ in the sense of distribution. Recently, Kome and Özaydin [12, 13] proved, among other results, that if $M = \mathbb{B}^n$, the Poincaré conformal disc model, and $\rho = \log \frac{1+|x|}{1-|x|}$, the corresponding geodesic distance, Hardy inequalities reads, for $n \geqslant 3$,

$$\int_{\mathbb{B}^n} |\nabla_{\mathbb{H}} f|^2 dV \geqslant \frac{(n-2)^2}{4} \int_{\mathbb{B}^n} \frac{f^2}{\rho^2} dV, \tag{3}$$

where $f \in C_0^\infty(\mathbb{B}^n)$. Furthermore, the constant $\frac{(n-2)^2}{4}$ is sharp. For more Hardy inequalities on Riemannian manifold, we refer to [7, 11, 16].

One of the aims of this paper is to look for the sharp constants of Hardy inequalities related to the geodesic distance on the sphere. For some basic properties on the sphere, we refer to [10], page 161 and page 167. The main result is the following theorem.

THEOREM 1. *Let $n \geqslant 3$ and $p \in \mathbb{S}^n$. Denote by $\nabla_\mathbb{S}$ the gradient on $\mathbb{S}^n$. There exists a positive constant C_1 such that for $f \in C^\infty(\mathbb{S}^n)$, there holds*

$$C_1 \int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla_\mathbb{S} f|^2 dV \geqslant \frac{(n-2)^2}{4} \int_{\mathbb{S}^n} \left(\frac{f^2}{d(x,p)^2} + \frac{f^2}{(\pi - d(x,p))^2} \right) dV,$$

where $d(x,p)$ is the geodesic distance from x to p on $\mathbb{S}^n$. Furthermore, the constant $\frac{(n-2)^2}{4}$ is sharp in the sense that

$$\frac{(n-2)^2}{4} = \inf_{f \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{C_1 \int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla_\mathbb{S} f|^2 dV}{\int_{\mathbb{S}^n} \frac{f^2}{d(x,p)^2} dV}$$

and

$$\frac{(n-2)^2}{4} = \inf_{f \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{C_1 \int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla_\mathbb{S} f|^2 dV}{\int_{\mathbb{S}^n} \frac{f^2}{(\pi - d(x,p))^2} dV}.$$

REMARK 1. *If $f(-x) = f(x)$ for all $x \in \mathbb{S}^n$, then $\int_{\mathbb{S}^n} \frac{f^2}{d(x,p)^2} dV = \int_{\mathbb{S}^n} \frac{f^2}{(\pi - d(x,p))^2} dV$. Therefore, by Theorem 1.1, we have*

$$C_1 \int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla_\mathbb{S} f|^2 dV \geqslant \frac{(n-2)^2}{2} \int_{\mathbb{S}^n} \frac{f^2}{d(x,p)^2} dV.$$

We note a similar phenomenon occurs in the Moser-Trudinger inequality. In fact, Moser ([14]) shows

$$\sup_{\int_{\mathbb{S}^2} f dV = 0, \int_{\mathbb{S}^2} |\nabla_\mathbb{S} f|^2 dV \leqslant 1} \int_{\mathbb{S}^2} e^{4\pi f^2} dV < \infty. \tag{4}$$

Furthermore, if $f(-x) = f(x)$ for all $x \in \mathbb{S}^n$, then the constant 4π in (4) can be replaced by 8π (see [15]).

We also obtain a sharp Rellich inequality on $\mathbb{S}^n$.

THEOREM 2. *Let $n \geqslant 5$ and $p \in \mathbb{S}^n$. Denote by $\Delta_\mathbb{S}$ the Laplace-Beltrami operator on $\mathbb{S}^n$. There exists a positive constant C_2 such that for $f \in C^\infty(\mathbb{S}^n)$, there holds*

$$C_2 \int_{\mathbb{S}^n} \frac{f^2}{\sin^2 d(x,p)} dV + \int_{\mathbb{S}^n} |\Delta_\mathbb{S} f|^2 dV \geqslant \left(\frac{n(n-4)}{4} \right)^2 \int_{\mathbb{S}^n} \left(\frac{f^2}{d(x,p)^4} + \frac{f^2}{(\pi - d(x,p))^4} \right) dV.$$

Furthermore, the constant $\left(\frac{n(n-4)}{4} \right)^2$ is sharp in the sense that

$$\left(\frac{n(n-4)}{4} \right)^2 = \inf_{f \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{C_2 \int_{\mathbb{S}^n} \frac{f^2}{\sin^2 d(x,p)} dV + \int_{\mathbb{S}^n} |\Delta_\mathbb{S} f|^2 dV}{\int_{\mathbb{S}^n} \frac{f^2}{d(x,p)^4} dV}$$

and

$$\left(\frac{n(n-4)}{4} \right)^2 = \inf_{f \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{C_2 \int_{\mathbb{S}^n} \frac{f^2}{\sin^2 d(x,p)} dV + \int_{\mathbb{S}^n} |\Delta_\mathbb{S} f|^2 dV}{\int_{\mathbb{S}^n} \frac{f^2}{(\pi - d(x,p))^4} dV}.$$

2. Notations and preliminaries

Denote by
$$\mathbb{S}^n = \{x = (x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : |x| = 1\}$$
the unit sphere of dimension n. Let $(\theta_1, \theta_2, \cdots, \theta_n)$ be the angular variables on $\mathbb{S}^n$ and θ_n satisfies $x_{n+1} = |x| \cos \theta_n$. To simplify notation we set $\theta = \theta_n$ and θ will be the only relevant angular variable in this paper. The polar coordinate associated with θ on $\mathbb{S}^n$ is

$$\int_{\mathbb{S}^n} f dV = \int_{\mathbb{S}^{n-1}} \int_0^\pi f \cdot (\sin\theta)^{n-1} d\sigma d\theta, \ f \in L^1(\mathbb{S}^n), \tag{5}$$

where $d\sigma$ denotes the canonical measure of the unit sphere $\mathbb{S}^{n-1}$.

A function on $\mathbb{S}^n$ is called radial if f depends only on θ. If f is radial, then

$$\Delta_{\mathbb{S}} f = \frac{d^2 f}{d\theta^2} + (n-1)\cot\theta \frac{df}{d\theta} = (\sin\theta)^{1-n} \frac{d}{d\theta}\left((\sin\theta)^{n-1} \frac{d}{d\theta}\right) \tag{6}$$

and

$$|\nabla_{\mathbb{S}} f| = |f'(\theta)|. \tag{7}$$

We end this section with the following lemma:

LEMMA 1. *Let $n \geqslant 3$ and $0 \leqslant \alpha < n - 2$. There exists a positive constant $A_{n,\alpha} > 0$ such that for all $f \in C^\infty(\mathbb{S}^n)$, there holds*

$$A_{n,\alpha} \int_{\mathbb{S}^n} \frac{f^2}{\sin^\alpha\theta} dV + \int_{\mathbb{S}^n} \frac{|\nabla_{\mathbb{S}} f|^2}{\sin^\alpha\theta} dV \geqslant \frac{(n-2-\alpha)^2}{4} \int_{\mathbb{S}^n} \frac{f^2}{\sin^{2+\alpha}\theta} dV. \tag{8}$$

Proof. The proof is similar to that in [12], page 6193. Using the substitution $f = (\sin\theta)^{-\frac{n-2-\alpha}{2}} \varphi$, we get

$$|\nabla_{\mathbb{S}} f|^2 = \frac{|\nabla_{\mathbb{S}} \varphi|^2}{(\sin\theta)^{n-2-\alpha}} + \varphi^2 |\nabla_{\mathbb{S}}(\sin\theta)^{-\frac{n-2-\alpha}{2}}|^2 + 2(\sin\theta)^{-\frac{n-2-\alpha}{2}} \varphi \langle \nabla_{\mathbb{S}}(\sin\theta)^{-\frac{n-2-\alpha}{2}}, \nabla_{\mathbb{S}} \varphi \rangle$$

$$= \frac{|\nabla_{\mathbb{S}} \varphi|^2}{(\sin\theta)^{n-2-\alpha}} + \varphi^2 |\nabla(\sin\theta)^{-\frac{n-2-\alpha}{2}}|^2 + \frac{1}{2} \langle \nabla_{\mathbb{S}}(\sin\theta)^{-(n-2-\alpha)}, \nabla_{\mathbb{S}} \varphi^2 \rangle.$$

Therefore,

$$\int_{\mathbb{S}^n} \frac{|\nabla_{\mathbb{S}} f|^2}{\sin^\alpha\theta} dV$$

$$= \int_{\mathbb{S}^n} \frac{1}{\sin^\alpha\theta} \left[\frac{|\nabla_{\mathbb{S}} \varphi|^2}{(\sin\theta)^{n-2-\alpha}} + |\nabla_{\mathbb{S}^n}(\sin\theta)^{-\frac{n-2-\alpha}{2}}|^2 \varphi^2 + \frac{1}{2} \langle \nabla_{\mathbb{S}}(\sin\theta)^{-(n-2-\alpha)}, \nabla_{\mathbb{S}} \varphi^2 \rangle \right] dV$$

$$\geqslant \int_{\mathbb{S}^n} \frac{\varphi^2}{\sin^\alpha\theta} |\nabla_{\mathbb{S}^n}(\sin\theta)^{-\frac{n-2-\alpha}{2}}|^2 dV + \frac{1}{2} \int_{\mathbb{S}^n} \frac{\langle \nabla_{\mathbb{S}}(\sin\theta)^{-(n-2-\alpha)}, \nabla_{\mathbb{S}} \varphi^2 \rangle}{\sin^\alpha\theta} dV$$

$$= \frac{(n-2-\alpha)^2}{4} \int_{\mathbb{S}^n} \frac{\varphi^2}{\sin^n\theta} \cos^2\theta dV + \frac{n-2-\alpha}{2(n-2)} \int_{\mathbb{S}^n} \langle \nabla_{\mathbb{S}}(\sin\theta)^{-(n-2)}, \nabla_{\mathbb{S}} \varphi^2 \rangle dV$$

$$= \frac{(n-2-\alpha)^2}{4} \int_{\mathbb{S}^n} \frac{\varphi^2}{\sin^n\theta} \cos^2\theta dV - \frac{n-2-\alpha}{2(n-2)} \int_{\mathbb{S}^n} \varphi^2 \Delta_{\mathbb{S}}(\sin\theta)^{-(n-2)} dV.$$

By (6),

$$\Delta_\mathbb{S}(\sin\theta)^{-(n-2)} = (\sin\theta)^{1-n}\frac{d}{d\theta}\left((\sin\theta)^{n-1}\frac{d}{d\theta}(\sin\theta)^{2-n}\right) = (n-2)(\sin\theta)^{2-n}.$$

Therefore, we have

$$\begin{aligned}
\int_{\mathbb{S}^n}\frac{|\nabla_\mathbb{S}f|^2}{\sin^\alpha\theta}dV &\geqslant \frac{(n-2-\alpha)^2}{4}\int_{\mathbb{S}^n}\frac{\varphi^2}{\sin^n\theta}\cos^2\theta dV - \frac{n-2-\alpha}{2}\int_{\mathbb{S}^n}\varphi^2(\sin\theta)^{2-n}dV \\
&= \frac{(n-2-\alpha)^2}{4}\int_{\mathbb{S}^n}\frac{\varphi^2}{\sin^n\theta}(1-\sin^2\theta)dV - \frac{n-2-\alpha}{2}\int_{\mathbb{S}^n}\varphi^2(\sin\theta)^{2-n}dV \\
&= \frac{(n-2-\alpha)^2}{4}\int_{\mathbb{S}^n}\frac{\varphi^2}{\sin^n\theta}dV - A_{n,\alpha}\int_{\mathbb{S}^n}\frac{\varphi^2}{(\sin\theta)^{n-2}}dV \\
&= \frac{(n-2-\alpha)^2}{4}\int_{\mathbb{S}^n}\frac{f^2}{\sin^{2+\alpha}\theta}dV - A_{n,\alpha}\int_{\mathbb{S}^n}\frac{f^2}{(\sin\theta)^\alpha}dV,
\end{aligned}$$

where $A_{n,\alpha} = \frac{(n-2-\alpha)^2}{4} + \frac{n-2-\alpha}{2}$. The desired result follows. $\square$

3. The proofs

We firstly prove Theorem 1.1.

Proof of Theorem 1.1. It is enough to show that

$$C_1\int_{\mathbb{S}^n}f^2 dV + \int_{\mathbb{S}^n}|\nabla_\mathbb{S}f|^2 dV \geqslant \frac{(n-2)^2}{4}\int_{\mathbb{S}^n}\left(\frac{f^2}{\theta^2} + \frac{f^2}{(\pi-\theta)^2}\right)dV \qquad (9)$$

and $\frac{(n-2)^2}{4}$ is sharp. Choosing $\alpha = 0$ in Lemma 2.1, we have

$$A_n\int_{\mathbb{S}^n}f^2 dV + \int_{\mathbb{S}^n}|\nabla_\mathbb{S}f|^2 dV \geqslant \frac{(n-2)^2}{4}\int_{\mathbb{S}^n}\frac{f^2}{\sin^2\theta}dV,$$

i.e.

$$A_n\int_{\mathbb{S}^n}f^2 dV + \frac{(n-2)^2}{4}\int_{\mathbb{S}^n}f^2\left(\frac{1}{\theta^2} + \frac{1}{(\pi-\theta)^2} - \frac{1}{\sin^2\theta}\right)dV + \int_{\mathbb{S}^n}|\nabla_\mathbb{S}f|^2 dV$$
$$\geqslant \frac{(n-2)^2}{4}\int_{\mathbb{S}^n}\left(\frac{f^2}{\theta^2} + \frac{f^2}{(\pi-\theta)^2}\right)dV, \qquad (10)$$

where $A_n = A_{n,0} = \frac{(n-2)^2}{4} + \frac{n-2}{2}$. Set $V_1(\theta) = \frac{1}{\theta^2} + \frac{1}{(\pi-\theta)^2} - \frac{1}{\sin^2\theta}$, $0 < \theta < \pi$. We claim $V_1(\theta)$ is bounded in $(0,\pi)$. In fact, since $\sin\theta$ has the asymptotic expansion

$$\sin\theta = \theta - \frac{1}{6}\theta^3 + o(\theta^3), \quad \theta \to 0, \qquad (11)$$

it is easy to check that

$$\lim_{\theta \to 0+} V_1(\theta) = \frac{1}{\pi^2} + \lim_{\theta \to 0+} \left(\frac{1}{\theta^2} - \frac{1}{\sin^2 \theta} \right)$$

$$\frac{1}{\pi^2} + \lim_{\theta \to 0+} \left(\frac{1}{\theta} - \frac{1}{\sin \theta} \right) \left(\frac{1}{\theta} + \frac{1}{\sin \theta} \right)$$

$$= \frac{1}{\pi^2} + \lim_{\theta \to 0+} \frac{\sin \theta - \theta}{\theta^2 \sin \theta} \left(1 + \frac{\theta}{\sin \theta} \right) = \frac{1}{\pi^2} - \frac{1}{3};$$

$$\lim_{\theta \to \pi-} V_1(\theta) = \lim_{t \to 0+} V_1(\pi - t) = \lim_{t \to 0+} V_1(t) = \frac{1}{\pi^2} - \frac{1}{3}.$$

Therefore, V_1 is bounded in $(0, \pi)$. Thus, if we set

$$C_1 = A_n + \frac{(n-2)^2}{4} \sup_{\theta \in (0,\pi)} |V_1(\theta)| < \infty,$$

then, by (10), inequality (9) holds. This proves the first part of Theorem 1.1.

Now we show the constant $\frac{(n-2)^2}{4}$ is sharp. It is enough to show

$$\inf_{f \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{C_1 \int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla_{\mathbb{S}} f|^2 dV}{\int_{\mathbb{S}^n} \frac{f^2}{\theta^2} dV} \leqslant \frac{(n-2)^2}{4}.$$

The proof is similar to that in [16], page 9-13. Let $\phi : \mathbb{R} \to [0,1]$ be a cut-off function which is equal to one in $[-1,1]$ and zero outside $[-2,2]$. Set $H(t) = 1 - \phi(t)$ and

$$f_\varepsilon(\theta) = \begin{cases} 0, & \theta = 0; \\ H(\frac{\theta}{\varepsilon}) \theta^{\frac{2-n}{2}}, & 0 < \theta \leqslant \pi. \end{cases}$$

Without loss of generality, we may assume $0 < \varepsilon < 1/2$ and $0 \leqslant \phi(t) \leqslant 1$, $t \in \mathbb{R}$. Then $f_\varepsilon(\theta)$ is a radial and smooth function on $\mathbb{S}^n$. By (5), we have

$$\int_{\mathbb{S}^n} f_\varepsilon^2 dV = |\mathbb{S}^{n-1}| \int_\varepsilon^\pi H^2 \left(\frac{\theta}{\varepsilon} \right) \theta^{2-n} (\sin \theta)^{n-1} d\theta$$

$$\leqslant |\mathbb{S}^{n-1}| \int_\varepsilon^\pi \theta^{2-n} \cdot \theta^{n-1} d\theta = |\mathbb{S}^{n-1}| \cdot \frac{\pi^2 - \varepsilon^2}{2};$$

$$\int_{\mathbb{S}^n} \frac{f_\varepsilon^2}{\theta^2} dV = |\mathbb{S}^{n-1}| \int_\varepsilon^\pi H^2 \left(\frac{\theta}{\varepsilon} \right) \theta^{-n} (\sin \theta)^{n-1} d\theta$$

$$\geqslant |\mathbb{S}^{n-1}| \int_{2\varepsilon}^\pi H^2 \left(\frac{\theta}{\varepsilon} \right) \theta^{-n} (\sin \theta)^{n-1} d\theta = |\mathbb{S}^{n-1}| \int_{2\varepsilon}^\pi \theta^{-n} (\sin \theta)^{n-1} d\theta,$$

where $|\mathbb{S}^{n-1}|$ is the volume of $\mathbb{S}^{n-1}$. Moreover, by Minkowski's inequality,

$$\left(\int_{\mathbb{S}^n} |\nabla_{\mathbb{S}} f_\varepsilon|^2 dV \right)^{\frac{1}{2}}$$

$$= |\mathbb{S}^{n-1}|^{\frac{1}{2}} \left(\int_\varepsilon^\pi \left| H'\left(\frac{\theta}{\varepsilon}\right) \frac{1}{\varepsilon} \theta^{\frac{2-n}{2}} + \frac{2-n}{2} H\left(\frac{\theta}{\varepsilon}\right) \theta^{-\frac{n}{2}} \right|^2 (\sin\theta)^{n-1} d\theta \right)^{\frac{1}{2}}$$

$$\leqslant \frac{|\mathbb{S}^{n-1}|^{\frac{1}{2}}}{\varepsilon} \left(\int_\varepsilon^\pi \left| H'\left(\frac{\theta}{\varepsilon}\right) \right|^2 \theta^{2-n} (\sin\theta)^{n-1} d\theta \right)^{\frac{1}{2}}$$

$$+ \frac{n-2}{2} |\mathbb{S}^{n-1}|^{\frac{1}{2}} \left(\int_\varepsilon^\pi H^2\left(\frac{\theta}{\varepsilon}\right) \theta^{-n} (\sin\theta)^{n-1} d\theta \right)^{\frac{1}{2}}$$

$$= \frac{|\mathbb{S}^{n-1}|^{\frac{1}{2}}}{\varepsilon} \left(\int_\varepsilon^{2\varepsilon} \left| H'\left(\frac{\theta}{\varepsilon}\right) \right|^2 \theta^{2-n} (\sin\theta)^{n-1} d\theta \right)^{\frac{1}{2}}$$

$$+ \frac{n-2}{2} |\mathbb{S}^{n-1}|^{\frac{1}{2}} \left(\int_\varepsilon^\pi H^2\left(\frac{\theta}{\varepsilon}\right) \theta^{-n} (\sin\theta)^{n-1} d\theta \right)^{\frac{1}{2}}$$

$$\leqslant \frac{|\mathbb{S}^{n-1}|^{\frac{1}{2}}}{\varepsilon} \cdot \max_{t\in[0,2]} |H'(t)| \cdot \left(\int_\varepsilon^{2\varepsilon} \theta^{2-n} \cdot \theta^{n-1} d\theta \right)^{\frac{1}{2}}$$

$$+ \frac{n-2}{2} |\mathbb{S}^{n-1}|^{\frac{1}{2}} \left(\int_\varepsilon^\pi \theta^{-n} (\sin\theta)^{n-1} d\theta \right)^{\frac{1}{2}}$$

$$= |\mathbb{S}^{n-1}|^{\frac{1}{2}} \max_{t\in[0,2]} |H'(t)| \cdot \sqrt{\frac{3}{2}} + \frac{n-2}{2} |\mathbb{S}^{n-1}|^{\frac{1}{2}} \left(\int_\varepsilon^\pi \theta^{-n} (\sin\theta)^{n-1} d\theta \right)^{\frac{1}{2}}.$$

Therefore,

$$\inf_{f\in C^\infty(\mathbb{S}^n)\setminus\{0\}} \frac{C_1 \int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla_{\mathbb{S}} f|^2 dV}{\int_{\mathbb{S}^n} \frac{f^2}{\theta^2} dV} \leqslant \frac{C_1 \int_{\mathbb{S}^n} f_\varepsilon^2 dV + \int_{\mathbb{S}^n} |\nabla_{\mathbb{S}} f_\varepsilon|^2 dV}{\int_{\mathbb{S}^n} \frac{f_\varepsilon^2}{\theta^2} dV}$$

$$\leqslant \frac{C_1}{2} \cdot \frac{\pi^2 - \varepsilon^2}{\int_{2\varepsilon}^\pi \theta^{-n} (\sin\theta)^{n-1} d\theta} \tag{12}$$

$$+ \left[\frac{\max_{t\in[0,2]} |H'(t)| \cdot \sqrt{\frac{3}{2}}}{(\int_{2\varepsilon}^\pi \theta^{-n} (\sin\theta)^{n-1} d\theta)^{1/2}} + \frac{n-2}{2} \cdot \frac{(\int_\varepsilon^\pi \theta^{-n} (\sin\theta)^{n-1} d\theta)^{\frac{1}{2}}}{(\int_{2\varepsilon}^\pi \theta^{-n} (\sin\theta)^{n-1} d\theta)^{\frac{1}{2}}} \right]^2.$$

Notice that

$$\lim_{\varepsilon\to 0+} \int_{2\varepsilon}^\pi \theta^{-n} (\sin\theta)^{n-1} d\theta = \infty.$$

We have

$$\lim_{\varepsilon\to 0+} \frac{\pi^2 - \varepsilon^2}{\int_{2\varepsilon}^\pi \theta^{-n} (\sin\theta)^{n-1} d\theta} = \lim_{\varepsilon\to 0+} \frac{\max_{t\in[0,2]} |H'(t)| \cdot \sqrt{\frac{3}{2}}}{(\int_{2\varepsilon}^\pi \theta^{-n} (\sin\theta)^{n-1} d\theta)^{1/2}} = 0.$$

Therefore, passing to the limit as $\varepsilon \to 0+$ in (12) yields

$$\inf_{f \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{C_1 \int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla_{\mathbb{S}} f|^2 dV}{\int_{\mathbb{S}^n} \frac{f^2}{\theta^2} dV} \leqslant \frac{(n-2)^2}{4} \left[\lim_{\varepsilon \to 0+} \frac{\left(\int_\varepsilon^\pi \theta^{-n} (\sin\theta)^{n-1} d\theta \right)^{\frac{1}{2}}}{\left(\int_{2\varepsilon}^\pi \theta^{-n} (\sin\theta)^{n-1} d\theta \right)^{\frac{1}{2}}} \right]^2$$

$$= \frac{(n-2)^2}{4} \lim_{\varepsilon \to 0+} \frac{\int_\varepsilon^\pi \theta^{-n} (\sin\theta)^{n-1} d\theta}{\int_{2\varepsilon}^\pi \theta^{-n} (\sin\theta)^{n-1} d\theta}$$

$$= \frac{(n-2)^2}{4}.$$

To get the last equality above, we use the L'Hospital's rule

$$\lim_{\varepsilon \to 0+} \frac{\int_\varepsilon^\pi \theta^{-n} (\sin\theta)^{n-1} d\theta}{\int_{2\varepsilon}^\pi \theta^{-n} (\sin\theta)^{n-1} d\theta} = \lim_{\varepsilon \to 0+} \frac{-\varepsilon^{-n} (\sin\varepsilon)^{n-1}}{-2(2\varepsilon)^{-n} (\sin(2\varepsilon))^{n-1}} = 1.$$

Therefore,

$$\inf_{f \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{C_1 \int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla_{\mathbb{S}} f|^2 dV}{\int_{\mathbb{S}^n} \frac{f^2}{\theta^2} dV} = \frac{(n-2)^2}{4}.$$

Similarly, taking $f_\varepsilon(\pi - \theta)$ as the test function and following the proof above, one can also check that $\frac{(n-2)^2}{4}$ is sharp in the sense that

$$\frac{(n-2)^2}{4} = \inf_{f \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{C_1 \int_{\mathbb{S}^n} f^2 dV + \int_{\mathbb{S}^n} |\nabla_{\mathbb{S}} f|^2 dV}{\int_{\mathbb{S}^n} \frac{f^2}{(\pi-\theta)^2} dV}.$$

The proof of Theorem 1.1 is thereby completed. $\quad\square$

Before the proof of Theorem 1.2, we need the following lemma.

LEMMA 2. *If $f \in C^\infty(\mathbb{S}^n)$, then*

$$-2f\Delta_{\mathbb{S}} f = 2|\nabla_{\mathbb{S}} f|^2 - \Delta_{\mathbb{S}} f^2.$$

Proof. It is enough to show that for all $\varphi \in C^\infty(\mathbb{S}^n)$, there holds

$$-2 \int_{\mathbb{S}^n} \varphi f \Delta_{\mathbb{S}} f dV = 2 \int_{\mathbb{S}^n} \varphi |\nabla_{\mathbb{S}} f|^2 dV - \int_{\mathbb{S}^n} \varphi \Delta_{\mathbb{S}} f^2 dV.$$

In fact, by integration by parts, we have

$$-2 \int_{\mathbb{S}^n} \varphi f \Delta_{\mathbb{S}} f dV = 2 \int_{\mathbb{S}^n} \langle \nabla_{\mathbb{S}} (\varphi f), \nabla_{\mathbb{S}} f \rangle dV$$

$$= 2 \int_{\mathbb{S}^n} \varphi \langle \nabla_{\mathbb{S}} f, \nabla_{\mathbb{S}} f \rangle dV + 2 \int_{\mathbb{S}^n} f \langle \nabla_{\mathbb{S}} \varphi, \nabla_{\mathbb{S}} f \rangle dV$$

$$= 2 \int_{\mathbb{S}^n} \varphi |\nabla_{\mathbb{S}} f|^2 dV + \int_{\mathbb{S}^n} \langle \nabla_{\mathbb{S}} \varphi, \nabla_{\mathbb{S}} f^2 \rangle dV$$

$$= 2 \int_{\mathbb{S}^n} \varphi |\nabla_{\mathbb{S}} f|^2 dV - \int_{\mathbb{S}^n} \varphi \Delta_{\mathbb{S}} f^2 dV.$$

The desired result follows. □

Proof of Theorem 1.2. It is enough to show that

$$C_2 \int_{\mathbb{S}^n} \frac{f^2}{\sin^2 d(x,p)} dV + \int_{\mathbb{S}^n} |\Delta_{\mathbb{S}} f|^2 dV \geqslant \left(\frac{n(n-4)}{4}\right)^2 \int_{\mathbb{S}^n} \left(\frac{f^2}{d(x,p)^4} + \frac{f^2}{(\pi - d(x,p))^4}\right) dV$$

and the constant $\left(\frac{n(n-4)}{4}\right)^2$ is sharp.

By Lemma 3.1,

$$-2f\Delta_{\mathbb{S}} f = 2|\nabla_{\mathbb{S}} f|^2 - \Delta_{\mathbb{S}} f^2.$$

Integrating this inequality over $\mathbb{S}^n$ with weight $(\sin\theta)^{-2}$ yields

$$\begin{aligned}
-2 \int_{\mathbb{S}^n} \frac{f\Delta_{\mathbb{S}} f}{\sin^2\theta} dV &= 2 \int_{\mathbb{S}^n} \frac{|\nabla_{\mathbb{S}} f|^2}{\sin^2\theta} dV - \int_{\mathbb{S}^n} \frac{\Delta_{\mathbb{S}} f^2}{\sin^2\theta} dV \\
&= 2 \int_{\mathbb{S}^n} \frac{|\nabla_{\mathbb{S}} f|^2}{\sin^2\theta} dV - \int_{\mathbb{S}^n} f^2 \Delta_{\mathbb{S}} (\sin\theta)^{-2} dV.
\end{aligned} \tag{13}$$

By (6),

$$\begin{aligned}
\Delta_{\mathbb{S}} (\sin\theta)^{-2} &= (\sin\theta)^{1-n} \frac{d}{d\theta} \left((\sin\theta)^{n-1} \frac{d}{d\theta} (\sin\theta)^{-2}\right) \\
&= -2(\sin\theta)^{1-n} \frac{d}{d\theta} \left((\sin\theta)^{n-4} \cos\theta\right) \\
&= \frac{2}{\sin^2\theta} - 2(n-4) \frac{\cos^2\theta}{\sin^4\theta}.
\end{aligned} \tag{14}$$

Combing (13),(14) and (8) yields

$$\begin{aligned}
-2 \int_{\mathbb{S}^n} \frac{f\Delta_{\mathbb{S}} f}{\sin^2\theta} dV &\geqslant \frac{(n-4)^2}{2} \int_{\mathbb{S}^n} \frac{f^2}{\sin^4\theta} dV - 2A_{n,2} \int_{\mathbb{S}^n} \frac{f^2}{\sin^2\theta} dV \\
&\quad - 2 \int_{\mathbb{S}^n} \frac{f^2}{\sin^2\theta} dV + 2(n-4) \int_{\mathbb{S}^n} \frac{f^2}{\sin^4\theta} \cos^2\theta dV \\
&= \frac{n(n-4)}{2} \int_{\mathbb{S}^n} \frac{f^2}{\sin^4\theta} dV - 2(A_{n,2} - n + 3) \int_{\mathbb{S}^n} \frac{f^2}{\sin^2\theta} dV.
\end{aligned} \tag{15}$$

On the other hand,

$$\begin{aligned}
-2 \int_{\mathbb{S}^n} \frac{f\Delta_{\mathbb{S}} f}{\sin^2\theta} dV &\leqslant 2 \int_{\mathbb{S}^n} \frac{|f\Delta_{\mathbb{S}} f|}{\sin^2\theta} dV \\
&\leqslant \frac{n(n-4)}{4} \int_{\mathbb{S}^n} \frac{f^2}{\sin^4\theta} dV + \frac{4}{n(n-4)} \int_{\mathbb{S}^n} |\nabla_{\mathbb{S}} f|^2 dV.
\end{aligned} \tag{16}$$

To get the second inequality in above, we use the following elementary inequality

$$2ab \leqslant \frac{n(n-4)}{4} a^2 + \frac{4}{n(n-4)} b^2, \quad a > 0, \, b > 0.$$

Combing (15) and (16) yields

$$\frac{4}{n(n-4)} \int_{\mathbb{S}^n} |\nabla_{\mathbb{S}} f|^2 dV \geqslant \frac{n(n-4)}{4} \int_{\mathbb{S}^n} \frac{f^2}{\sin^4 \theta} dV - 2(A_{n,2} - n + 3) \int_{\mathbb{S}^n} \frac{f^2}{\sin^2 \theta} dV,$$

i.e.

$$\frac{(A_{n,2} - n + 3)n(n-4)}{2} \int_{\mathbb{S}^n} \frac{f^2}{\sin^2 \theta} dV + \int_{\mathbb{S}^n} \frac{f^2}{\sin^2 \theta} V_2(\theta) dV + \int_{\mathbb{S}^n} |\nabla_{\mathbb{S}} f|^2 dV$$

$$\geqslant \left(\frac{n(n-4)}{4} \right)^2 \int_{\mathbb{S}^n} \left(\frac{f^2}{\theta^4} + \frac{f^2}{(\pi - \theta)^4} \right) dV,$$

(17)

where

$$V_2(\theta) = \left(\frac{n(n-4)}{4} \right)^2 \sin^2 \theta \cdot \left(\frac{1}{\theta^4} + \frac{1}{(\pi - \theta)^4} - \frac{1}{\sin^4 \theta} \right), \quad 0 < \theta < \pi.$$

We note V_2 is bounded in $(0, \pi)$ since

$$\lim_{\theta \to 0+} V_2(\theta) = \left(\frac{n(n-4)}{4} \right)^2 \lim_{\theta \to 0+} \left(\frac{\sin^2 \theta}{\theta^4} - \frac{1}{\sin^2 \theta} \right)$$

$$= \left(\frac{n(n-4)}{4} \right)^2 \lim_{\theta \to 0+} \left(\frac{1}{\theta^2} - \frac{1}{\sin^2 \theta} \right) \left(\frac{\sin^2 \theta}{\theta^2} + 1 \right)$$

$$= 2 \left(\frac{n(n-4)}{4} \right)^2 \lim_{\theta \to 0+} \left(\frac{1}{\theta^2} - \frac{1}{\sin^2 \theta} \right)$$

$$= 2 \left(\frac{n(n-4)}{4} \right)^2 \lim_{\theta \to 0+} \frac{\sin \theta - \theta}{\theta^2 \sin \theta} \left(1 + \frac{\theta}{\sin \theta} \right)$$

$$= -\frac{2}{3} \cdot \left(\frac{n(n-4)}{4} \right)^2;$$

$$\lim_{\theta \to \pi-} V_2(\theta) = \lim_{t \to 0+} V_2(\pi - t) = \lim_{t \to 0+} V_1(2) = -\frac{2}{3} \cdot \left(\frac{n(n-4)}{4} \right)^2.$$

Therefore, if we set

$$C_2 = \left| \frac{(A_{n,2} - n + 3)n(n-4)}{2} \right| + \sup_{\theta \in (0,\pi)} |V_2(\theta)|,$$

then, by (17),

$$C_2 \int_{\mathbb{S}^n} \frac{f^2}{\sin^2 d(x,p)} dV + \int_{\mathbb{S}^n} |\Delta_{\mathbb{S}} f|^2 dV \geqslant \left(\frac{n(n-4)}{4} \right)^2 \int_{\mathbb{S}^n} \left(\frac{f^2}{d(x,p)^4} + \frac{f^2}{(\pi - d(x,p))^4} \right) dV.$$

Now we show the constant $\left(\frac{n(n-4)}{4} \right)^2$ is sharp. It is enough to show

$$\inf_{f \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{C_2 \int_{\mathbb{S}^n} \frac{f^2}{\sin^2 \theta} dV + \int_{\mathbb{S}^n} |\Delta_{\mathbb{S}} f|^2 dV}{\int_{\mathbb{S}^n} \frac{f^2}{\theta^4} dV} \leqslant \left(\frac{n(n-4)}{4} \right)^2.$$

Let ϕ be the cut-off function defined in the proof of Theorem 1.1 and $H = 1 - \phi$. We take $g_\varepsilon(\theta) = H\left(\frac{\theta}{\varepsilon}\right)\theta^{-\frac{n-4}{2}}$ as the test function. By (5), we have

$$\int_{\mathbb{S}^n} \frac{g_\varepsilon^2}{\sin^2\theta} dV = |\mathbb{S}^{n-1}| \int_\varepsilon^\pi H^2\left(\frac{\theta}{\varepsilon}\right)\theta^{4-n}(\sin\theta)^{n-3}d\theta$$

$$\leqslant |\mathbb{S}^{n-1}| \int_\varepsilon^\pi \theta^{4-n}\cdot\theta^{n-3}d\theta = |\mathbb{S}^{n-1}|\cdot\frac{\pi^2-\varepsilon^2}{2};$$

$$\int_{\mathbb{S}^n} \frac{g_\varepsilon^2}{\theta^4} dV = |\mathbb{S}^{n-1}| \int_\varepsilon^\pi H^2\left(\frac{\theta}{\varepsilon}\right)\theta^{-n}(\sin\theta)^{n-1}d\theta$$

$$\geqslant |\mathbb{S}^{n-1}| \int_{2\varepsilon}^\pi H^2\left(\frac{\theta}{\varepsilon}\right)\theta^{-n}(\sin\theta)^{n-1}d\theta$$

$$= |\mathbb{S}^{n-1}| \int_{2\varepsilon}^\pi \theta^{-n}(\sin\theta)^{n-1}d\theta.$$

By (6), we have

$$\Delta_{\mathbb{S}}g_\varepsilon = \left(\frac{d^2}{d\theta^2} + (n-1)\cot\theta\frac{d}{d\theta}\right)H\left(\frac{\theta}{\varepsilon}\right)\theta^{-\frac{n-4}{2}}$$

$$= H\left(\frac{\theta}{\varepsilon}\right)\left(\theta^{-\frac{n-4}{2}}\right)'' + 2\frac{1}{\varepsilon}H'\left(\frac{\theta}{\varepsilon}\right)\left(\theta^{-\frac{n-4}{2}}\right)' + \frac{1}{\varepsilon^2}H''\left(\frac{\theta}{\varepsilon}\right)\theta^{-\frac{n-4}{2}}$$

$$+ (n-1)\cot\theta\left[\frac{1}{\varepsilon}H'\left(\frac{\theta}{\varepsilon}\right)\theta^{-\frac{n-4}{2}} + H\left(\frac{\theta}{\varepsilon}\right)\left(\theta^{-\frac{n-4}{2}}\right)'\right]$$

$$= H\left(\frac{\theta}{\varepsilon}\right)\left[\frac{(4-n)(2-n)}{4}\theta^{-\frac{n}{2}} + \frac{(4-n)(n-1)}{2}\theta^{\frac{2-n}{2}}\cot\theta\right]$$

$$+ \frac{1}{\varepsilon}H'\left(\frac{\theta}{\varepsilon}\right)\left[(4-n)\theta^{\frac{2-n}{2}} + (n-1)\theta^{\frac{4-n}{2}}\cot\theta\right] + \frac{1}{\varepsilon^2}H''\left(\frac{\theta}{\varepsilon}\right)\theta^{\frac{4-n}{2}}.$$

Therefore, by (5) and Minkowski's inequality,

$$\frac{1}{|\mathbb{S}^{n-1}|^{\frac{1}{2}}}\left(\int_{\mathbb{S}^n}|\Delta_{\mathbb{S}}g_\varepsilon|^2 dV\right)^{\frac{1}{2}}$$

$$\leqslant \left(\int_\varepsilon^\pi H^2\left(\frac{\theta}{\varepsilon}\right)\left|\frac{(4-n)(2-n)}{4}\theta^{-\frac{n}{2}} + \frac{(4-n)(n-1)}{2}\theta^{\frac{2-n}{2}}\cot\theta\right|^2 (\sin\theta)^{n-1}d\theta\right)^{\frac{1}{2}}$$

$$+ \frac{1}{\varepsilon}\left(\int_\varepsilon^{2\varepsilon}\left|H'\left(\frac{\theta}{\varepsilon}\right)\left[(4-n)\theta^{\frac{2-n}{2}} + (n-1)\theta^{\frac{4-n}{2}}\cot\theta\right]\right|^2 (\sin\theta)^{n-1}d\theta\right)^{\frac{1}{2}}$$

$$+ \frac{1}{\varepsilon^2}\left(\int_\varepsilon^{2\varepsilon}\left|H''\left(\frac{\theta}{\varepsilon}\right)\theta^{\frac{4-n}{2}}\right|^2 (\sin\theta)^{n-1}d\theta\right)^{\frac{1}{2}}$$

$$:= (I) + (II) + (III).$$

It is easy to check

$$(I) = \left(\int_\varepsilon^\pi H^2 \left(\frac{\theta}{\varepsilon} \right) \left| \frac{(4-n)(2-n)}{4} \theta^{-\frac{n}{2}} + \frac{(4-n)(n-1)}{2} \theta^{\frac{2-n}{2}} \cot\theta \right|^2 (\sin\theta)^{n-1} d\theta \right)^{\frac{1}{2}}$$

$$\leqslant \left(\int_\varepsilon^\pi \left| \frac{(4-n)(2-n)}{4} \theta^{-\frac{n}{2}} + \frac{(4-n)(n-1)}{2} \theta^{\frac{2-n}{2}} \cot\theta \right|^2 (\sin\theta)^{n-1} d\theta \right)^{\frac{1}{2}} ;$$

$$(II) = \frac{1}{\varepsilon} \left(\int_\varepsilon^{2\varepsilon} \left| H' \left(\frac{\theta}{\varepsilon} \right) \left[(4-n)\theta^{\frac{2-n}{2}} + (n-1)\theta^{\frac{4-n}{2}} \cot\theta \right] \right|^2 (\sin\theta)^{n-1} d\theta \right)^{\frac{1}{2}}$$

$$\leqslant \frac{1}{\varepsilon} \max_{t\in[0,2]} H'(t) \left(\int_\varepsilon^{2\varepsilon} |(4-n)\sin\theta + (n-1)\theta\cos\theta|^2 \theta^{2-n}(\sin\theta)^{n-3} d\theta \right)^{\frac{1}{2}}$$

$$\leqslant \frac{1}{\varepsilon} \max_{t\in[0,2]} H'(t) \left(\int_\varepsilon^{2\varepsilon} [(n-4)\theta + (n-1)\theta]^2 \theta^{2-n} \theta^{n-3} d\theta \right)^{\frac{1}{2}}$$

$$= \max_{t\in[0,2]} H'(t) \cdot \sqrt{\frac{6n-15}{2}} ;$$

$$(III) = \frac{1}{\varepsilon^2} \left(\int_\varepsilon^{2\varepsilon} \left| H'' \left(\frac{\theta}{\varepsilon} \right) \theta^{\frac{4-n}{2}} \right|^2 (\sin\theta)^{n-1} d\theta \right)^{\frac{1}{2}}$$

$$\leqslant \frac{1}{\varepsilon^2} \max_{t\in[0,2]} H''(t) \left(\int_\varepsilon^{2\varepsilon} \theta^{4-n} \theta^{n-1} d\theta \right)^{\frac{1}{2}}$$

$$= \frac{\sqrt{15}}{2} \max_{t\in[0,2]} H''(t).$$

Thus,

$$\inf_{f\in C^\infty(\mathbb{S}^n)\setminus\{0\}} \frac{C_2 \int_{\mathbb{S}^n} \frac{f^2}{\sin^2\theta} dV + \int_{\mathbb{S}^n} |\Delta_\mathbb{S} f|^2 dV}{\int_{\mathbb{S}^n} \frac{f^2}{\theta^4} dV} \leqslant \frac{C_2 \int_{\mathbb{S}^n} \frac{g_\varepsilon^2}{\sin^2\theta} dV + \int_{\mathbb{S}^n} |\Delta_\mathbb{S} g_\varepsilon|^2 dV}{\int_{\mathbb{S}^n} \frac{g_\varepsilon^2}{\theta^4} dV}$$

$$\leqslant C_2 \frac{\frac{\pi^2-\varepsilon^2}{2}}{\int_{2\varepsilon}^\pi \theta^{-n}(\sin\theta)^{n-1} d\theta} + \left[\frac{(I)+(II)+(III)}{\sqrt{\int_{2\varepsilon}^\pi \theta^{-n}(\sin\theta)^{n-1} d\theta}} \right]^2$$

$$\leqslant C_2 \frac{\frac{\pi^2-\varepsilon^2}{2}}{\int_{2\varepsilon}^\pi \theta^{-n}(\sin\theta)^{n-1} d\theta} + \left[\frac{\max_{t\in[0,2]} H'(t) \cdot \sqrt{\frac{6n-15}{2}} + \frac{\sqrt{15}}{2} \max_{t\in[0,2]} H''(t)}{\sqrt{\int_{2\varepsilon}^\pi \theta^{-n}(\sin\theta)^{n-1} d\theta}} \right.$$

$$\left. + \frac{\left(\int_\varepsilon^\pi \left| \frac{(4-n)(2-n)}{4} \theta^{-\frac{n}{2}} + \frac{(4-n)(n-1)}{2} \theta^{\frac{2-n}{2}} \cot\theta \right|^2 (\sin\theta)^{n-1} d\theta \right)^{\frac{1}{2}}}{\sqrt{\int_{2\varepsilon}^\pi \theta^{-n}(\sin\theta)^{n-1} d\theta}} \right]^2 .$$

Passing to the limit as $\varepsilon \to 0+$ yields

$$\inf_{f \in C^{\infty}(\mathbb{S}^n)\setminus\{0\}} \frac{C_2 \int_{\mathbb{S}^n} \frac{f^2}{\sin^2 \theta} dV + \int_{\mathbb{S}^n} |\Delta_{\mathbb{S}} f|^2 dV}{\int_{\mathbb{S}^n} \frac{f^2}{\theta^4} dV}$$

$$\leqslant \lim_{\varepsilon \to 0+} \frac{\int_{\varepsilon}^{\pi} \left| \frac{(4-n)(2-n)}{4} \theta^{-\frac{n}{2}} + \frac{(4-n)(n-1)}{2} \theta^{\frac{2-n}{2}} \cot \theta \right|^2 (\sin \theta)^{n-1} d\theta}{\int_{2\varepsilon}^{\pi} \theta^{-n}(\sin \theta)^{n-1} d\theta}$$

$$= \left(\frac{n(n-4)}{4} \right)^2.$$

To get the last equality above, we use the L'Hospital's rule

$$\lim_{\varepsilon \to 0+} \frac{\int_{\varepsilon}^{\pi} \left| \frac{(4-n)(2-n)}{4} \theta^{-\frac{n}{2}} + \frac{(4-n)(n-1)}{2} \theta^{\frac{2-n}{2}} \cot \theta \right|^2 (\sin \theta)^{n-1} d\theta}{\int_{2\varepsilon}^{\pi} \theta^{-n}(\sin \theta)^{n-1} d\theta}$$

$$= \lim_{\varepsilon \to 0+} \frac{- \left| \frac{(4-n)(2-n)}{4} \varepsilon^{-\frac{n}{2}} + \frac{(4-n)(n-1)}{2} \varepsilon^{\frac{2-n}{2}} \cot \varepsilon \right|^2 (\sin \varepsilon)^{n-1}}{-2(2\varepsilon)^{-n}(\sin 2\varepsilon)^{n-1}}$$

$$= \left(\frac{n(n-4)}{4} \right)^2.$$

Therefore,

$$\inf_{f \in C^{\infty}(\mathbb{S}^n)\setminus\{0\}} \frac{C_2 \int_{\mathbb{S}^n} \frac{f^2}{\sin^2 \theta} dV + \int_{\mathbb{S}^n} |\Delta_{\mathbb{S}} f|^2 dV}{\int_{\mathbb{S}^n} \frac{f^2}{\theta^4} dV} = \left(\frac{n(n-4)}{4} \right)^2.$$

Similarly, taking $g_{\varepsilon}(\pi - \theta)$ as the test function and following the proof above, one can also check that $\left(\frac{n(n-4)}{4} \right)^2$ is sharp in the sense that

$$\left(\frac{n(n-4)}{4} \right)^2 = \inf_{f \in C^{\infty}(\mathbb{S}^n)\setminus\{0\}} \frac{C_2 \int_{\mathbb{S}^n} \frac{f^2}{\sin^2 \theta} dV + \int_{\mathbb{S}^n} |\Delta_{\mathbb{S}} f|^2 dV}{\int_{\mathbb{S}^n} \frac{f^2}{(\pi-\theta)^4} dV}.$$

The proof of Theorem 1.2 is thereby completed.   □

REFERENCES

[1] B. ABDELLAOUI, D. COLORADO, I. PERAL, *Some improved Caffarelli-Kohn-Nirenberg inequalities*, Calc. Var. Partial Differential Equations, **23**, (2005), 327–345.

[2] ADIMURTHI, S. FILIPPAS, A. TERTIKAS, *On the best constant of Hardy-Sobolev inequalities*, Nonlinear Analysis, **70**, (2009), 2826–2833.

[3] G. BARBATIS, S. FILIPPAS AND A. TERTIKAS, *A unified approach to improved L^p Hardy inequalities with best constants*, Trans. Amer. Math. Soc., **356**, (2004), 2169–2196.

[4] P. BARAS, J. A. GOLDSTEIN, *The heat equation with a singular potential*, Trans. Amer. Math. Soc. **284**, (1984), 121–139.

[5] H. BREZIS, J. L. VÁZQUEZ, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Comp. Madrid, **10** (1997), 443–469.

[6] G. CARRON, *Inégalités de Hardy sur les variétés riemanniennes non-compactes*, J. Math. Pures Appl. **76**, (1997), 883–891.

[7] L. D'AMBROSIO, S. DIPIERRO, *Hardy inequalities on Riemannian manifolds and applications*, Annales de l'Institut Henri Poincare (C) Non Linear Analysis, **31**, 3 (2014), 449–475.

[8] E. B. DAVIES, A. M. HINZ, *Explicit constants for Rellich inequalities in $L_p(\Omega)$*, Math. Z., **227** (1998), 511–523.

[9] S. FILIPPAS, A. TERTIKAS, *Optimizing improved Hardy inequalities*, J. Funct. Anal. **192**, 1 (2002), 186–233.

[10] S. GALLOT, D. HULIN, J. LAFONTAINE, *Riemannian Geometry*, 3rd edn. Springer-Verlag, Berlin, 2004.

[11] H. JABER, *Optimal Hardy-Sobolev Inequalities on Compact Riemannain Manifolds*, J. Math. Anal. Appl. **421** (2015), 1869–1888.

[12] I. KOMBE, M. ÖZAYDIN, *Improved Hardy and Rellich inequalities on Riemannian manifolds*, Trans. Amer. Math. Soc. **361**, (2009), 6191–6203.

[13] I. KOMBE, M. ÖZAYDIN, *Hardy-Poincaré, Rellich and Uncertainty principle inequalities on Riemannian manifolds*, Trans. Amer. Math. Soc. **365** (2013), 5035–5050.

[14] J. MOSER, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970), 1077–1092.

[15] J. MOSER, *On a non-linear problem in differential geometry and dynamical systems*, Academic Press, N.Y. (ed. M. Peixoto) 1973.

[16] Q. YANG, D. SU, Y. KONG, *Hardy inequalities on Riemannian manifolds with negative curvature*, comm. contemp. Math. **16**, 1350043 (2014) [24 pages].

[17] J. L. VAZQUEZ, E. ZUAZUA, *The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential*, J. Funct. Anal. **173** (2000), 103–153.

Yingxiong Xiao
School of Mathematics and Statistics
Hubei Engineering University
Xiaogan, Hubei, P. C. 432000, People's Republic of China
e-mail: yxxiao.math@gmail.com

Journal of Mathematical Inequalities
www.ele-math.com
jmi@ele-math.com