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SOME NEW RESULTS OF TWO OPEN PROBLEMS RELATED TO INTEGRAL INEQUALITIES

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Abstract. In this paper, we have solved two open problems, and as consequence some interesting integral inequalities are obtained.

1. Introduction

More recently, Liu et al. (see [1]) obtained the following theorem.

THEOREM 1.1. *Let $f(x) \geqslant 0$ be a continuous function on $[a,b]$ satisfying*

$$\int_x^b f^{\min\{1,\beta\}}(t)dt \geqslant \int_x^b (t-a)^{\min\{1,\beta\}}dt, \quad \forall x \in [a,b] \tag{1.1}$$

Then the inequality

$$\int_a^b f^{\alpha+\beta}(x)dx \geqslant \int_a^b (x-a)^{\alpha} f^{\beta}(x)dx \tag{1.2}$$

holds for every positive real number $\alpha > 0$ and $\beta > 0$.

THEOREM 1.2. *Let $f(x), g(x), h(x) > 0$ be continuous functions on $[a,b]$ with $f(x) \leqslant h(x)$ for all x and such that $\frac{f(x)}{h(x)}$ is decreasing and $f(x), g(x)$ are increasing. Assume that $\varphi(x)$ is a convex function with $\varphi(0) = 0$.*

Then the inequality

$$\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \geqslant \frac{\int_a^b \varphi(f(x))g(x)dx}{\int_a^b \varphi(h(x))g(x)dx} \tag{1.3}$$

holds.

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Liu et al. (see [2]) presented the following two open problems.

Open Problem 1. Under what conditions does the inequality

$$\int_a^b f^{\alpha+\beta}(x)dx \geqslant \left(\int_a^b (x-a)^{\alpha} f^{\beta}(x)dx\right)^{\lambda} \tag{1.4}$$

hold for α, β and λ?

Open Problem 2. Assume that $\varphi(x)$ is a convex function with $\varphi(0)=0$. Under what conditions does the inequality

$$\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \geqslant \frac{\left(\int_a^b \varphi(f(x))g(x)dx\right)^{\delta}}{\left(\int_a^b \varphi(h(x))g(x)dx\right)^{\lambda}} \tag{1.5}$$

hold for δ and λ?

2. Main results

THEOREM 2.1. *Let $f(x) \geqslant 0$ be a continuous function on $[a,b]$ satisfying*

$$\int_x^b (t-a)^{\min\{1,\beta\}} dt \leqslant \int_x^b f^{\min\{1,\beta\}}(t)dt, \quad \forall x \in [a,b] \tag{2.1}$$

Then the inequality

$$\int_a^b f^{\alpha+\beta}(x)dx \geqslant \left(\int_a^b (x-a)^{\alpha} f^{\beta}(x)dx\right)^{\lambda}, \quad \forall \lambda \geqslant 1 \tag{2.2}$$

holds under each of the following conditions:

1. *For all $\beta > 1$ and $\alpha > 0$ such that*

$$\frac{(b-a)^{\alpha+2}}{\alpha+2} \leqslant 1$$

2. *For $\beta \in (0,1]$ and $\alpha > 0$ such that*

$$\frac{(b-a)^{\alpha+\beta+1}}{\alpha+\beta+1} \leqslant 1$$

Proof. If $\lambda = 1$ then (2.2) holds for every positive real number $\alpha > 0$ and $\beta > 0$ by theorem 1.1. Let $\lambda > 1$.

Then

$$\left(\int_a^b (x-a)^{\alpha} f^{\beta}(x)dx\right)^{\lambda} = \left(\int_a^b (x-a)^{\alpha} f^{\beta}(x)dx\right) \cdot \left(\int_a^b (x-a)^{\alpha} f^{\beta}(x)dx\right)^{\lambda-1}$$

$$\Longrightarrow \left(\int_a^b (x-a)^\alpha f^\beta(x)dx\right)^\lambda \leqslant \int_a^b f^{\alpha+\beta}(x)dx \Longleftrightarrow \left(\int_a^b (x-a)^\alpha f^\beta(x)dx\right)^{\lambda-1} \leqslant 1$$

$$\Longrightarrow \left(\int_a^b (x-a)^\alpha f^\beta(x)dx\right)^{\lambda-1} \leqslant 1 \Longleftrightarrow 0 \leqslant \int_a^b (x-a)^\alpha f^\beta(x)dx \leqslant 1. \tag{2.3}$$

By using integration by parts, we obtain the following relation

$$\int_a^b (x-a)^\alpha f^\beta(x)dx = \alpha \int_a^b (x-a)^{\alpha-1}\left(\int_x^b f^\beta(t)dt\right)dx \tag{2.4}$$

But, by the hypothesis of theorem 2.1

$$\int_x^b (t-a)^{\min\{1,\beta\}}dt \leqslant \int_x^b f^{\min\{1,\beta\}}(t)dt, \quad \forall x \in [a,b]$$

We have the following two cases:

(1) For all $\beta > 1$ and $\alpha > 0$ such that

$$\frac{(b-a)^{\alpha+2}}{\alpha+2} \leqslant 1$$

by simple calculations inequality (2.3) follows.

(2) For $\beta \in (0,1]$ and $\alpha > 0$ such that

$$\frac{(b-a)^{\alpha+\beta+1}}{\alpha+\beta+1} \leqslant 1$$

by simple calculations inequality (2.3) holds. □

THEOREM 2.2. *Let $f(x), g(x), h(x) > 0$ be continuous functions on $[a,b]$ with $f(x) \leqslant h(x)$ for all x and such that $\frac{f(x)}{h(x)}$ is decreasing and $f(x), g(x)$ are increasing. Assume that $\varphi(x)$ is positive and convex function with $\varphi(0) = 0$.*

Then the inequality

$$\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \geqslant \frac{\left(\int_a^b \varphi(f(x))g(x)dx\right)^\delta}{\left(\int_a^b \varphi(h(x))g(x)dx\right)^\lambda} \tag{2.5}$$

holds under each of the following conditions:

1. $\lambda = \delta = 0$ *and* $f(x) = h(x)$*, for all* $x \in [a,b]$*;*
2. $\lambda = \delta \in [1,+\infty)$*, for all* $x \in [a,b]$*;*
3. $\varphi(f(a)) \geqslant \frac{1}{(b-a)g(a)}$ *for* $1 \leqslant \delta < \lambda$*;*

4. $\varphi(f(b)) \leqslant \frac{1}{(b-a)g(b)}$ *for* $1 \leqslant \lambda < \delta$.

Proof.

1. If $\lambda = \delta = 0$ and $f(x) = h(x)$, for all $x \in [a,b]$ then inequality (2.5) turns into an equality.

2. If $\lambda = \delta = 1$ inequality (2.5) coincides with theorem 1.2.

   Now let $\lambda = \delta > 1$ and denote by $d = \frac{\int_a^b f(x)dx}{\int_a^b h(x)dx}$. Since $0 < f(x) \leqslant h(x)$, for all $x \in [a,b]$ then $d \in [0,1]$. By theorem 1.2 and the fact that $\varphi(x)$ is positive and convex function with $\varphi(0) = 0$, we have the following inequalities

$$\left(\frac{\int_a^b \varphi(f(x))g(x)dx}{\int_a^b \varphi(h(x))g(x)dx}\right)^{\delta} \leqslant \left(\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx}\right)^{\delta} \leqslant \frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \tag{2.6}$$

   since $d \in [0,1]$, for all $\delta > 1$. So inequality (2.5) follows.

3. For $1 \leqslant \delta < \lambda$ there exists a real positive number r such that $\lambda = \delta + r$. Using case (2) for $\lambda = \delta \in [1,+\infty)$ we have

$$\frac{\left(\int_a^b \varphi(f(x))g(x)dx\right)^{\delta}}{\left(\int_a^b \varphi(h(x))g(x)dx\right)^{\lambda}} = \left(\frac{\int_a^b \varphi(f(x))g(x)dx}{\int_a^b \varphi(h(x))g(x)dx}\right)^{\delta} \cdot \frac{1}{\left(\int_a^b \varphi(h(x))g(x)dx\right)^{r}}$$
$$\leqslant \left(\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx}\right) \cdot \frac{1}{\left(\int_a^b \varphi(h(x))g(x)dx\right)^{r}} \leqslant \frac{\int_a^b f(x)dx}{\int_a^b h(x)dx}$$

   The last inequality above follows by the fact that $\left(\int_a^b \varphi(h(x))g(x)dx\right)^{r} \geqslant 1$ for $r > 0$, since we have assumed that $\varphi(f(a)) \geqslant \frac{1}{(b-a)g(a)}$. So inequality (2.5) holds.

4. For $1 \leqslant \lambda < \delta$ there exists a real positive number r_1 such that $\delta = \lambda + r_1$. Using case (2) for $\lambda = \delta \in [1,+\infty)$ we have

$$\frac{\left(\int_a^b \varphi(f(x))g(x)dx\right)^{\delta}}{\left(\int_a^b \varphi(h(x))g(x)dx\right)^{\lambda}} = \left(\frac{\int_a^b \varphi(f(x))g(x)dx}{\int_a^b \varphi(h(x))g(x)dx}\right)^{\lambda} \cdot \left(\int_a^b \varphi(f(x))g(x)dx\right)^{r_1}$$
$$\leqslant \left(\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx}\right) \cdot \left(\int_a^b \varphi(f(x))g(x)dx\right)^{r_1} \leqslant \frac{\int_a^b f(x)dx}{\int_a^b h(x)dx}$$

In the last inequality we have used the fact that $\left(\int_a^b \varphi(f(x))g(x)dx\right)^{r_1} \leqslant 1$ for $r_1 > 0$, since we have assumed that $\varphi(f(b)) \leqslant \dfrac{1}{(b-a)g(b)}$. So inequality (2.5) follows. □

3. Applications

COROLLARY 3.1. *Let $f(x) \geqslant 0$ be a continuous function on $[0,1]$ satisfying*

$$\int_x^1 t^{\min\{1,\beta\}}dt \leqslant \int_x^1 f^{\min\{1,\beta\}}(t)dt, \quad \forall x \in [0,1] \tag{3.1}$$

Then the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geqslant \left(\int_0^1 x^\alpha f^\beta(x)dx\right)^\lambda, \quad \forall \lambda \geqslant 1 \tag{3.2}$$

holds for $\alpha, \beta > 0$.

COROLLARY 3.2. *Let $f(x) \geqslant 0$ be a continuous function on $[a,b]$ satisfying*

$$\int_x^b (t-a)^{\min\{1,\alpha\}}dt \leqslant \int_x^b f^{\min\{1,\alpha\}}(t)dt, \quad \forall x \in [a,b] \tag{3.3}$$

Then the inequality

$$\int_a^b f^{2\alpha}(x)dx \geqslant \left(\int_a^b ((x-a)\cdot f(x))^\alpha dx\right)^\lambda, \quad \forall \lambda \geqslant 1 \tag{3.4}$$

holds under each of the following conditions:

1. *For $\alpha > 1$ such that*
$$\frac{(b-a)^{\alpha+2}}{\alpha+2} \leqslant 1$$
2. *For $\alpha \in (0,1]$ such that*
$$\frac{(b-a)^{2\alpha+1}}{2\alpha+1} \leqslant 1$$

Proof. Let $\alpha = \beta$ and applying theorem 2.1. □

COROLLARY 3.3. *Let $f(x), g(x), h(x) > 0$ be continuous functions on $[a,b]$ with $f(x) \leqslant h(x)$ for all x and such that $\dfrac{f(x)}{h(x)}$ is decreasing and $f(x), g(x)$ are increasing. Assume that $\varphi(x)$ is positive and convex function with $\varphi(0) = 0$. Then the inequality*

$$\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \geqslant \frac{\left(\int_a^b \varphi(f(x))g^p(x)dx\right)^\delta}{\left(\int_a^b \varphi(h(x))g^p(x)dx\right)^\lambda}, \quad \forall p \geqslant 0 \tag{3.5}$$

holds under each of the following conditions:

1. $\lambda = \delta = 0$ *and* $f(x) = h(x)$, *for all* $x \in [a,b]$;

2. $\lambda = \delta \in [1,+\infty)$, *for all* $x \in [a,b]$;

3. $\varphi(f(a)) \geqslant \dfrac{1}{(b-a)g^p(a)}$ *for* $1 \leqslant \delta < \lambda$;

4. $\varphi(f(b)) \leqslant \dfrac{1}{(b-a)g^p(b)}$ *for* $1 \leqslant \lambda < \delta$.

Proof. Let $g_p(x) = g^p(x)$, for all $x \in [a,b]$ and for all $p \geqslant 0$. Since $g(x)$ is increasing function and $g(x) > 0$, then $g_p(x)$ are increasing functions for all $p \geqslant 0$. By applying theorem 2.2, inequality (3.5) follows. □

COROLLARY 3.4. *Let* $f(x), g(x), h(x) > 0$ *be continuous functions on* $[a,b]$ *with* $f(x) \leqslant h(x)$ *for all* x *and such that* $\dfrac{f(x)}{h(x)}$ *is decreasing and* $f(x), g(x)$ *are increasing. Then the inequality*

$$\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \geqslant \frac{\left(\int_a^b f^k(x)g^p(x)dx\right)^{\delta}}{\left(\int_a^b h^k(x)g^p(x)dx\right)^{\lambda}}, \quad \forall k \geqslant 1 \text{ and } \forall p \geqslant 0 \tag{3.6}$$

holds under each of the following conditions:

1. $\lambda = \delta = 0$ *and* $f(x) = h(x)$, *for all* $x \in [a,b]$;

2. $\lambda = \delta \in [1,+\infty)$, *for all* $x \in [a,b]$;

3. $f^k(a) \geqslant \dfrac{1}{(b-a)g^p(a)}$ *for* $1 \leqslant \delta < \lambda$;

4. $f^k(b) \leqslant \dfrac{1}{(b-a)g^p(b)}$ *for* $1 \leqslant \lambda < \delta$.

Proof. Let $\varphi(x) = x^k$ where $k \geqslant 1$. φ is a convex function and $\varphi(0) = 0$. By corollary 3.3, inequality (3.6) follows. □

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