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NUMERICAL RADIUS INEQUALITIES FOR HILBERT SPACE OPERATORS

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Abstract. In this article, we give several inequalities involving powers numerical radii and the usual operator norms of Hilbert space operators. In particular, if A_i, B_i and X_i are bounded linear operators $(i = 1,2,\cdots,n \in \mathbb{N})$, then we estimate the norm as well as the numerical radius to $\sum_{i=1}^{n} X_i A_i^m B_i$ for some $m \in \mathbb{N}$.

1. Introduction

Let H be a complex Hilbert with inner product $\langle .,. \rangle$, and let $B(H)$ be the space of C^*-algebra of all bounded linear operators on H. For $T \in B(H)$, let T^* denote the adjoint of T. Also, let $w(T)$ denote the the numerical radius of T given by

$$w(T) = \sup\{|\langle Tx,x\rangle| : x \in H, \|x\| = 1\}.$$

It is well known that $w(.)$ is a norm on $B(H)$, which is equivalent to the usual operator norm $\| . \|$ defined, for $T \in B(H)$, by

$$\|T\| = \sup\{\|Tx\| : x \in H, \|x\| = 1\},$$

where $\|Tx\| = \langle Tx,Tx\rangle^{\frac{1}{2}}$. More precisely, for $T \in B(H)$, [7] showed that

$$\frac{1}{2}\|T\| \leqslant w(T) \leqslant \|T\|. \tag{1.1}$$

The inequalities in (1.1) are sharp: if $T^2 = 0$, then the first inequality becomes an equality, while the second inequality becomes equality if T is normal.

A fundamental inequality for the numerical radius is the power inequality, which says that for $T \in B(H)$, we have

$$w(T^n) \leqslant w^n(T),$$

for $n = 1,2,3,...$ (see, e.g., [1], p. 118).

Several numerical radius inequalities that provide alternative upper bounds for $w(.)$ have received much attention from many authors. We refer the readers to [1], [6],

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and [8] for their history and significance, and [2], [9], and [11] for recent developments in this area. For example, [9] proved that for $T \in H$,

$$w(T) \leqslant \frac{1}{2}\left\| |T| + |T^*| \right\| \leqslant \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{\frac{1}{2}} \right), \tag{1.2}$$

where $|T| = (T^*T)^{\frac{1}{2}}$ is the absolute value of T. [11] determine that

$$\frac{1}{4}\|T^*T + TT^*\| \leqslant w^2(T) \leqslant \frac{1}{2}\|T^*T + TT^*\|. \tag{1.3}$$

Considerable generalizations of the first inequality in (1.2) and the second inequality in (1.3) have been established in [4] for the numerical radius for one operator and for sum of two operators. It has been shown that if $A, B \in B(H)$, then

$$w^r(A) \leqslant \frac{1}{2}\left\| |A|^{2r\alpha} + |A^*|^{2r(1-\alpha)} \right\| \tag{1.4}$$

$$w^r(A+B) \leqslant 2^{r-2}\left\| |A|^{2r\alpha} + |A^*|^{2r(1-\alpha)} + |B|^{2r\alpha} + |B^*|^{2r(1-\alpha)} \right\|, \tag{1.5}$$

for $\alpha \in (0,1)$ and $r \geqslant 1$.

A general numerical radius inequality has been proved by Kittaneh, it has been showed in [11] that if $A, B, C, D, S, T \in B(H)$, then

$$w(ATB + CSD) \leqslant \frac{1}{2}\left\| A|T^*|^{2(1-\alpha)}A^* + B^*|T|^{2\alpha}B + C|S^*|^{2(1-\alpha)}C^* + D^*|S|^{2\alpha}D \right\| \tag{1.6}$$

for all $\alpha \in (0,1)$.

Although some open problems related to the numerical radius inequalities for bounded linear operator still remain open, the investigation to establish numerical radius inequalities for several bounded linear operators has been started, (see for instance [3] and [6]). For example, If A, B $\in B(H)$, [6] evidenced that

$$w(AB) \leqslant 4w(A)w(B).$$

Moreover, in the case $AB = BA$, [6] verified that

$$w(AB) \leqslant 2w(A)w(B).$$

However, the sharp inequality

$$w(AB) \leqslant w(A)w(B).$$

still has not been reached. A useful result in the direction, which can be found in [5], says that for any A, B $\in B(H)$,

$$w(AB \pm BA^*) \leqslant 2\|A\| w(B)$$

Very recently, for A, $B \in B(H)$ and $r \geqslant 1$, [3] showed that

$$w^r(B^*A) \leqslant \frac{1}{2}\|(A^*A)^r + (BB^*)^r\|. \tag{1.7}$$

Moreover, for A, $B \in B(H)$, $\alpha \in (0,1)$, and $r \geqslant 1$, [3] applied a different approach to obtain

$$w^{2r}(B^*A) \leqslant \left\|\alpha(A^*A)^{\frac{r}{\alpha}} + (1-\alpha)(BB^*)^{\frac{r}{1-\alpha}}\right\|. \tag{1.8}$$

The purpose of this work is to establish some new inequalities for the numerical radius of sums of bounded linear operators in Hilbert spaces. Also, we generalize the inequalities (1.4)–(1.8).

2. The main results

In this section, we establish a general numerical radius inequalities for Hilbert space operators which yields well known and new numerical radius inequalities as special cases. To prove our result we need the following basic lemmas. The first lemma is a generalized form of the mixed Schwarz inequality, which has been proved by Kittaneh [10].

LEMMA 2.1. ([10]) *Let $A,B \in B(H)$, and let f and g be non-negative functions on $[0,\infty)$ which are continuous such that $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then*

$$|\langle Ax,y\rangle| \leqslant \|f(|A|)x\|\,\|g(|A^*|)y\|,$$

for all $x,y \in H$.

The second lemma, which is called Hölder-McCarthy inequality is a well known result that follows from the spectral theorem for the positive operators and Jensen's inequality (see [10]).

LEMMA 2.2. ([10]) *Let $A \in B(H)$ be positive operator and let $x \in H$ be any unit vector. Then*

$$\langle Ax,x\rangle^r \leqslant \langle A^r x,x\rangle \ \ for\ r \geqslant 1,$$

and

$$\langle A^r x,x\rangle \leqslant \langle Ax,x\rangle^r \ \ for\ 0 < r \leqslant 1.$$

The third lemma is a simple consequence of the classical Jensen's inequality concerning the convexity or the concavity of certain power functions. It is a special case of Schlömilch's inequality for the weighted means of non-negative real numbers (see, e.g., [7].)

LEMMA 2.3. ([7]) *For $a,b \geqslant 0$, $0 < \alpha < 1$, and $r \neq 0$, let $M_r(a,b,\alpha) = (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}}$ and let $M_0(a,b,\alpha) = a^\alpha b^{1-\alpha}$. Then*

$$M_r(a,b,\alpha) \leqslant M_s(a,b,\alpha) \ \ for\ r \leqslant s.$$

The fourth lemma concerned with positive real numbers, and it is a consequence of the convexity of the function $f(t) = t^r,\ r \geqslant 1$.

LEMMA 2.4. *let* $a_i > 0,\ (i = 1,2,...n)$. *Then*

$$\left(\sum_{i=1}^{n} a_i\right)^r \leqslant n^{r-1}\sum_{i=1}^{n} a_i^r \ \textit{for } r \geqslant 1.$$

The first result in this paper, which leads to a generalization of (1.4), (1.5), (1.6) and (1.7) can be stated as follows.

THEOREM 2.5. *Let* $A_i, B_i, X_i \in B(H),\ (i = 1,2,...,n),\ m \in \mathbb{N}$ *and let* f *and* g *be as in Lemma 2.1. Then*

$$w^r\left(\sum_{i=1}^{n} X_i A_i^m B_i\right) \leqslant \frac{n^{r-1}}{2m}\sum_{j=1}^{m}\left\|\sum_{i=1}^{n} (M_{ij})^r + (N_{ij})^r\right\| \ \textit{for all } r \geqslant 1, \tag{2.1}$$

where $M_{ij} = X_i f^2\left(\left|A_i^{*^j}\right|\right) X_i^*$ *and* $N_{ij} = (A_i^{m-j}B_i)^* g^2\left(\left|A_i^j\right|\right) A_i^{m-j}B_i$.

Proof. Let $x \in H$ be any unit vector. Then by Lemma 2.1, Lemma 2.2 and Lemma 2.4 we obtain that

$$\begin{aligned}
\left|\left\langle \sum_{i=1}^{n} X_i A_i^m B_i x, x\right\rangle\right|^r &= \frac{1}{m}\sum_{j=1}^{m}\left|\left\langle \sum_{i=1}^{n} X_i A_i^m B_i x, x\right\rangle\right|^r \\
&\leqslant \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{i=1}^{n} |\langle X_i A_i^m B_i x, x\rangle|\right)^r \\
&\leqslant \frac{n^{r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n} |\langle X_i A_i^m B_i x, x\rangle|^r \\
&\leqslant \frac{n^{r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left|\left\langle A_i^{*^j} X_i^* x, A_i^{m-j}B_i x\right\rangle\right|^r \\
&\leqslant \frac{n^{r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\|f\left(\left|A_i^{*^j}\right|\right) X_i^* x\right\|^r \left\|g\left(\left|A_i^j\right|\right) A_i^{m-j}B_i x\right\|^r \\
&\leqslant \frac{n^{r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle M_{ij}x, x\right\rangle^{\frac{r}{2}}\left\langle N_{ij}x, x\right\rangle^{\frac{r}{2}} \\
&\leqslant \frac{n^{r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle (M_{ij})^r x, x\right\rangle^{\frac{1}{2}}\left\langle (N_{ij})^r x, x\right\rangle^{\frac{1}{2}} \\
&\leqslant \frac{n^{r-1}}{2m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle ((M_{ij})^r + (N_{ij})^r)x, x\right\rangle.
\end{aligned}$$

We finish the proof by taking the supremum over all unit vectors $x \in H$. □

Inequality (2.1) includes several numerical radius inequalities as special cases. Samples of inequalities are demonstrated in what follows, for $f(t) = t^{\alpha}$ and $g(t) = t^{1-\alpha}$, $\alpha \in (0,1)$ in inequality (2.1) we get the following inequality that t generalizes (1.6).

COROLLARY 2.6. *Let $A_i, B_i, X_i \in B(H)$, $(i = 1,2,...,n)$, $m \in \mathbb{N}$ and $\alpha \in (0,1)$. Then*

$$w^r\left(\sum_{i=1}^{n} X_i A_i^m B_i\right) \leqslant \frac{n^{r-1}}{2m} \sum_{j=1}^{m} \left\| \sum_{i=1}^{n} (M_{ij})^r + (N_{ij})^r \right\| \; for\ r \geqslant 1 \tag{2.2}$$

where $M_{ij} = X_i \left| A_i^{*^j} \right|^{2\alpha} X_i^*$ *and* $N_{ij} = (A_i^{m-j} B_i)^* \left| A_i^j \right|^{2(1-\alpha)} A_i^{m-j} B_i$.

For $X_i = B_i = I$ in inequality (2.1) we get the following numerical radius inequality that generalizes (1.6).

COROLLARY 2.7. *Let $A_i \in B(H)$ $(i = 1,2,...,n)$, $m \in \mathbb{N}$ and let f and g as in Lemma 2.1. Then*

$$w^r\left(\sum_{i=1}^{n} A_i^m\right) \leqslant \frac{n^{r-1}}{2m} \sum_{j=1}^{m} \left\| \sum_{i=1}^{n} f^{2r}\left(\left| A_i^{*^j} \right|\right) + \left(A_i^{*^{m-j}} g^2 \left| A_i^j \right| A_i^{m-j} \right)^r \right\| \; for\ r \geqslant 1.$$

An application of Corollary 2.7 can be seen in the following result. It involves a numerical radius inequality for the powers of operator.

COROLLARY 2.8. *Let $A \in B(H)$, $m \in \mathbb{N}$, $r \geqslant 1$ and $\alpha \in (0,1)$. Then*

$$w^r(A^m) \leqslant \frac{1}{2m} \sum_{j=1}^{m} \left\| \left| A^{*^j} \right|^{2\alpha r} + \left(A^{*^{m-j}} \left| A^j \right|^{2(1-\alpha)} A^{m-j} \right)^r \right\|$$

In particular

$$w(A^m) \leqslant \frac{1}{2m} \sum_{j=1}^{m} \left\| \left| A^{*^j} \right| + \left(A^{*^{m-j}} \left| A^j \right| A^{m-j} \right) \right\|$$

Following the same arguments used in the proof of Theorem 2.5, we achieve the following theorem.

THEOREM 2.9. *Let $A_i, B_i, X_i \in B(H)$, $m \in \mathbb{N}$, $\alpha \in (0,1)$ and let f and g be as in Lemma 2.1. Then*

$$w\left(\sum_{i=1}^{n} X_i A_i^m B_i\right) \leqslant \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \left\| \alpha (M_{ij})^{\frac{r}{\alpha}} + (1-\alpha)(N_{ij})^{\frac{r}{1-\alpha}} \right\|^{\frac{1}{2r}} \; for\ r \geqslant 1,$$

where $M_{ij} = X_i f^2\left(\left| A_i^{*^j} \right|\right) X_i^*$ *and* $N_{ij} = (A_i^{m-j} B_i)^* g^2\left(\left| A_i^j \right|\right) A_i^{m-j} B_i$.

Proof. Let $x \in H$ be a unit vector. Lemma 2.2, Lemma 2.3 and approaches used in the proof of Theorem 2.5 give that

$$\begin{aligned}
\left|\left\langle \sum_{i=1}^{n} X_i A_i^m B_i x, x \right\rangle\right| &\leqslant \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \left\langle M_{ij} x, x \right\rangle^{\frac{1}{2}} \left\langle N_{ij} x, x \right\rangle^{\frac{1}{2}} \\
&\leqslant \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \left\langle (M_{ij})^{\frac{1}{\alpha}} x, x \right\rangle^{\alpha} \left\langle (N_{ij})^{\frac{1}{1-\alpha}} x, x \right\rangle^{1-\alpha} \\
&\leqslant \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\alpha \left\langle (M_{ij})^{\frac{1}{\alpha}} x, x \right\rangle^{r} + (1-\alpha) \left\langle (N_{ij})^{\frac{1}{1-\alpha}} x, x \right\rangle^{r} \right)^{\frac{1}{2r}} \\
&\leqslant \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \left\langle \left(\alpha (M_{ij})^{\frac{r}{\alpha}} + (1-\alpha)(N_{ij})^{\frac{r}{1-\alpha}} \right) x, x \right\rangle^{\frac{1}{2r}}.
\end{aligned}$$

The proof is finished by taking the supremum over all unit vectors $x \in H$. □

The following applications of Theorem 2.9 present numerical radius inequalities for sums of operators and for the power operator.

COROLLARY 2.10. *Let* $A_i, B_i, X_i \in B(H)$, $(i = 1, 2, \ldots, n)$, $m \in \mathbb{N}$. *Then*

$$w\left(\sum_{i=1}^{n} X_i A_i^m B_i\right) \leqslant \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \left\| \alpha (O_{ij})^{\frac{r}{\alpha}} + (1-\alpha)(R_{ij})^{\frac{r}{1-\alpha}} \right\|^{\frac{1}{2r}} \text{ for } r \geqslant 1$$

where $O_{ij} = X_i \left| A_i^{*^{j}} \right|^{2\beta} X_i^*$, $R_{ij} = \left(A_i^{m-j} B_i\right)^* \left|A_i^j\right|^{2(1-\beta)} \left(A_i^{m-j} B_i\right)$ *and* $\alpha, \beta \in (0,1)$.

In the above corollary, take $n = m = 1$, $A_1 = I$, $X_1 = A^*$ and $B_1 = B$, we get inequality (1.8).

COROLLARY 2.11. *Let* $A \in B(H)$, $m \in \mathbb{N}$ *and* $r \geqslant 1$. *Then*

$$w(A^m) \leqslant \frac{1}{m} \sum_{j=1}^{m} \left\| \frac{\left|A^{*^{j}}\right|^{2r} + \left(A^{*^{m-j}} \left|A^j\right| A^{m-j}\right)^{2r}}{2} \right\|^{\frac{1}{2r}}$$

Another relation for the numerical radius of the operator $\sum\limits_{i=1}^{m} A_i^m$ is as follows

THEOREM 2.12. *Let* $A_i \in B(H)$, $(i = 1, 2, \ldots, n)$ *and* $m \in \mathbb{N}$. *Then*

$$w\left(\sum_{i=1}^{n} A_i^m\right) \leqslant \frac{1}{2m} \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} \left|A_i^j\right|^2 + \left|A_i^{*^{j-1}}\right|^2 \right\|$$

Proof. Let $x \in H$ be any unit vector. Then

$$\begin{aligned}\left|\left\langle \sum_{i=1}^{n} A_i^m x, x\right\rangle\right| &\leqslant \frac{1}{m}\sum_{i=1}^{n}\sum_{j=1}^{m}\left|\left\langle A_i^j x, A_i^{*^{m-j}} x\right\rangle\right| \\ &\leqslant \frac{1}{m}\sum_{i=1}^{n}\sum_{j=1}^{m}\left\|A_i^j x\right\|\left\|A_i^{*^{m-j}} x\right\| \\ &\leqslant \frac{1}{2m}\sum_{i=1}^{n}\left(\sum_{j=1}^{m}\left\|A_i^j x\right\|^2 + \sum_{j=1}^{m}\left\|A_i^{*^{m-j}}\right\|^2\right) \\ &= \frac{1}{2m}\left\langle\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left(A_i^{*^j} A_i^j + A_i^{j-1} A_i^{*^{j-1}}\right)\right) x, x\right\rangle\end{aligned}$$

By taking the supremum over all unit vectors $x \in H$, we complete the proof. □

As a direct consequence of Theorem 2.12 we obtain the following corollary

COROLLARY 2.13. *Let $A \in B(H)$ and $m \in \mathbb{N}$. Then*

$$w(A^m) \leqslant \frac{1}{2m}\left(\left\|\sum_{j=1}^{m}\left|A^j\right|^2 + \left|A^{*^{j-1}}\right|^2\right\|\right)$$

The numerical radius inequalities in Corollary 2.11, Corollary 2.13, and Theorem 2.12 are sharp. For example in Corollary 2.11 and Corollary 2.13 choose $A = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$. In Theorem 2.12, equality holds if we choose $A_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ for $i = 1, 2, ..., n$.

Using the fact $\| |A| \| = \| |A^*| \| = \|A\|$ for $A \in B(H)$, a particular case of Corollary 2.13 can be stated:

COROLLARY 2.14. *Let $A \in B(H)$ such that $\|A\| \neq 1$ and $m \in N$. Then*

$$w(A^m) \leqslant \frac{\left(\|A\|^{2m} - 1\right)\left(\|A\|^2 + 1\right)}{2m\left(\|A\|^2 - 1\right)}.$$

3. General norm inequality

In this section we introduce a general norm inequalities for Hilbert space operators. The proof of the first inequality is similar to that of Theorem 2.5 under slight modification.

THEOREM 3.1. *Let $A_i, B_i, X_i \in B(H)$, $(i = 1, 2, ..., n)$, $m \in \mathbb{N}$ and let f and g be as in Lemma 2.1. Then*

$$\left\|\sum_{i=1}^{n} X_i A_i^m B_i\right\|^r \leqslant \frac{n^{r-1}}{2m}\sum_{j=1}^{m}\left(\left\|\sum_{i=1}^{n}(M_{ij})^r\right\| + \left\|\sum_{i=1}^{n}(N_{ij})^r\right\|\right) \; for\ r \geqslant 1 \qquad (3.1)$$

where $M_{ij} = X_i f^2\left(\left|A_i^{*^j}\right|\right) X_i^*$ *and* $N_{ij} = (A_i^{m-j}B_i)^* g^2\left(\left|A_i^j\right|\right) A_i^{m-j}B_i$.

An application of Theorem 3.1 can be seen in the following result.

COROLLARY 3.2. *Let* $A_i, B_i, X_i \in B(H)$ $(i = 1, 2, ..., n)$, $r \geqslant 1$, *and* $\alpha \in (0,1)$. *Then*

$$\left\|\sum_{i=1}^{n} X_i A_i^m B_i\right\|^r \leqslant \frac{n^{r-1}}{2m}\sum_{j=1}^{m}\left(\left\|\sum_{i=1}^{n}(M_{ij})^r\right\| + \left\|\sum_{i=1}^{n}(N_{ij})^r\right\|\right) \tag{3.2}$$

where $M_{ij} = X_i\left|A_i^{*^j}\right|^{2\alpha} X_i^*$ *and* $N_{ij} = (A_i^{m-j}B_i)^*\left|A_i^j\right|^{2(1-\alpha)} A_i^{m-j}B_i$.

For $X_i = B_i = I$ $(i = 1, 2, ..., n)$ in inequality (3.2), we get the following inequality for sums of operators.

COROLLARY 3.3. *Let* $A_i \in B(H)$, $(i = 1, 2, ..., n)$, $r \geqslant 1$ *and* $\alpha \in (0,1)$. *Then*

$$\left\|\sum_{i=1}^{n} A_i^m\right\|^r \leqslant \frac{n^{r-1}}{2m}\sum_{j=1}^{m}\left(\left\|\sum_{i=1}^{n}\left|A_i^{*^j}\right|^{2\alpha r}\right\| + \left\|\sum_{i=1}^{n}\left(A_i^{*^{m-j}}\left|A_i^j\right|^{2(1-\alpha)} A_i^{m-j}\right)^r\right\|\right)$$

Following the same arguments used in the proof of Theorem 2.12, we achieve the following theorem.

THEOREM 3.4. *Let* $A_i \in B(H)$, $(i = 1, 2, ..., n)$ *and* $m \in \mathbb{N}$. *Then*

$$\left\|\sum_{i=1}^{n} A_i^m\right\| \leqslant \frac{1}{2m}\left(\left\|\sum_{i=1}^{n}\sum_{j=1}^{m}\left|A_i^j\right|^2\right\| + \left\|\sum_{i=1}^{n}\sum_{j=1}^{m}\left|A_i^{*^{j-1}}\right|^2\right\|\right)$$

Finally, we end this paper by the following direct consequence of Theorem 3.4.

COROLLARY 3.5. *Let* $A \in B(H)$ *and* $m \in \mathbb{N}$. *Then*

$$\|A^m\| \leqslant \frac{1}{2m}\left(\left\|\sum_{j=1}^{m}\left|A^j\right|^2\right\| + \left\|\sum_{j=1}^{m}\left|A^{*^{j-1}}\right|^2\right\|\right)$$

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