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TOEPLITZ TYPE OPERATORS ASSOCIATED TO SINGULAR INTEGRAL WITH VARIABLE KERNEL ON WEIGHTED MORREY SPACES

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Abstract. Let $T^{k,1}$ be singular integrals with variable Calderón-Zygmund kernels or $\pm I$ (the identity operator), let $T^{k,2}$ and $T^{k,4}$ be the linear operators, and let $T^{k,3} = \pm I$. Denote the Toeplitz type operator by

$$T^b = \sum_{k=1}^{Q} (T^{k,1} M^b I_\alpha T^{k,2} + T^{k,3} I_\alpha M^b T^{k,4}),$$

where $M^b f = bf$, and I_α is the fractional integral operator. In this paper, we investigate the boundedness of the operator T^b on weighted Morrey space when b belongs to weighted BMO space.

1. Introduction and results

Let $K(x,\xi): \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a variable Calderón-Zygmund kernel, which depends on some parameter x and possesses good properties with respect to the second variable ξ. The singular integral with variable Calderón-Zygmund kernel is defined by

$$Tf(x) = p.v. \int_{\mathbb{R}^n} K(x, x-y) f(y) dy, \tag{1.1}$$

which was firstly studied by Calderón and Zygmund in [1].

Let b be a locally integrable function on $\mathbb{R}^n$. The Toeplitz type operator associated to the singular integral with variable Calderón-Zygmund kernel and fractional integral operator I_α is defined by

$$T^b = \sum_{k=1}^{Q} (T^{k,1} M^b I_\alpha T^{k,2} + T^{k,3} I_\alpha M^b T^{k,4}), \tag{1.2}$$

where $M^b f = bf$, and $T^{k,1}$ are the singular integrals with variable Calderón-Zygmund kernel or $\pm I$ (the identity operator), $T^{k,2}$ and $T^{k,4}$ are the linear operators, $T^{k,3} = \pm I$, $k = 1, \cdots, Q$.

Note that the commutators $[b, I_\alpha](f) = bI_\alpha(f) - I_\alpha(bf)$ are the particular cases of the Toeplitz type operators T^b. The Toeplitz type operators T^b are the non-trivial generalizations of these commutators.

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DEFINITION 1.1. ([2]) Let $1 \leqslant p < \infty$, $0 < \kappa < 1$ and ω be a weight function. Then the weighted Morrey space is defined by

$$L^{p,\kappa}(\omega) = \left\{ f \in L^p_{loc}(\omega) : \|f\|_{L^{p,\kappa}(\omega)} < \infty \right\},$$

where

$$\|f\|_{L^{p,\kappa}(\omega)} = \sup_B \left(\frac{1}{\omega(B)^\kappa} \int_B |f(x)|^p \omega(x)dx \right)^{1/p},$$

and the supremum is taken over all balls $B \subset \mathbb{R}^n$.

In order to deal with the fractional order case, we need to consider the weighted Morrey space with two weights.

DEFINITION 1.2. ([2]) Let $1 \leqslant p < \infty$ and $0 < \kappa < 1$. Then for two weights μ and ν, the weighted Morrey space is defined by

$$L^{p,\kappa}(\mu,\nu) = \left\{ f \in L^p_{loc}(\mu) : \|f\|_{L^{p,\kappa}(\mu,\nu)} < \infty \right\},$$

where

$$\|f\|_{L^{p,\kappa}(\mu,\nu)} = \sup_B \left(\frac{1}{\nu(B)^\kappa} \int_B |f(x)|^p \mu(x)dx \right)^{1/p},$$

and the supremum is taken over all balls $B \subset \mathbb{R}^n$.

In [3–5], some Toeplitz type operators associated to the singular integral operators are introduced, and the boundedness for the operators generated by BMO and Lipschitz functions are obtained. Motivated by these, in this paper, we investigate the boundedness of T^b on the weighted Morrey space when b belongs to weighted BMO space and have the following result.

THEOREM 1.1. *Suppose that T^b is defined as (1.2), and $b \in BMO(\omega)$(weighted BMO). Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$, $\omega^{q/p} \in A_1$, and the critical index of ω for the reverse Hölder condition $r_\omega > \frac{1-\kappa}{p/q-\kappa}$. If $T^1(f) = 0$ for any $f \in L^{p,\kappa}(\omega)$, $T^{k,2}$ and $T^{k,4}$ are the bounded operators on $L^{p,\kappa}(\omega)$, $k = 1, \cdots, Q$, then there exists a constant $C > 0$ such that,*

$$\|T^b(f)\|_{L^{q,\kappa q/p}(\omega^{1-(1-\alpha/n)q},\omega)} \leqslant C\|b\|_{BMO(\omega)}\|f\|_{L^{p,\kappa}(\omega)}.$$

The following results are immediately obtained from Theorem 1.1.

COROLLARY 1.1. ([6]) *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$ and $\omega^{q/p} \in A_1$. Suppose that $b \in BMO(\omega)$ and the critical index of ω for the reverse Hölder condition $r_\omega > \frac{1-\kappa}{p/q-\kappa}$, then $[b, I_\alpha]$ is bounded from $L^{p,\kappa}(\omega)$ to $L^{q,\kappa q/p}(\omega^{1-(1-\alpha/n)q},\omega)$.*

COROLLARY 1.2. *Suppose that T^b is defined as (1.2), and $b \in BMO$. Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$. If $T^1(f) = 0$ for any $f \in L^{p,\kappa}(\mathbb{R}^n)$, $T^{k,2}$ and $T^{k,4}$ are the bounded operators on $L^{p,\kappa}(\mathbb{R}^n)$, $k = 1,\cdots,Q$, then there exists a constant $C > 0$ such that,*

$$\|T^b(f)\|_{L^{q,\kappa q/p}(\mathbb{R}^n)} \leqslant C\|b\|_{BMO}\|f\|_{L^{p,\kappa}(\mathbb{R}^n)}.$$

2. Some preliminaries

First let us recall definitions and notation of weight classes.

A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B = B_r(x_0)$ denote the ball with the center x_0 and radius r, and $\lambda B = B_{\lambda r}(x_0)$ for any $\lambda > 0$. For a given weight function ω and a measurable set E, we also denote the Lebesgue measure of E by $|E|$ and set weighted measure $\omega(E) = \int_E \omega(x)dx$. For any given weight function ω on $\mathbb{R}^n$, $0 < p < \infty$, denote by $L^p(\omega)$ the space of all function f satisfying

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx\right)^{1/p} < \infty.$$

DEFINITION 2.1. ([7]) A weight ω is said to belong to the Muckenhoupt class A_p for $1 < p < \infty$, if there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \leqslant C$$

for every ball B. The class A_1 is defined by replacing the above inequality with

$$\frac{1}{|B|}\int_B \omega(y)dy \leqslant C \cdot \operatorname*{ess\,inf}_{x\in B} w(x). \tag{2.1}$$

When $p = \infty$, we defined $A_\infty = \cup_{1\leqslant p<\infty} A_p$.

DEFINITION 2.2. ([8]) A weight function ω belongs to $A_{p,q}$ for $1 < p < q < \infty$, if for every ball B in $\mathbb{R}^n$, there exists a positive constant C which is independent of B such that

$$\left(\frac{1}{|B|}\int_B \omega(y)^{-p'}dy\right)^{1/p'}\left(\frac{1}{|B|}\int_B \omega(y)^q dy\right)^{1/q} \leqslant C,$$

where p' denotes the conjugate exponent of $p > 1$; that is, $1/p + 1/p' = 1$.

From the definition of $A_{p,q}$, we can get that

$$\omega \in A_{p,q}, \text{ iff } \omega^q \in A_{1+q/p'}.$$

DEFINITION 2.3. ([9]) A weight function ω belongs to the reverse Hölder class RH_r if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B \omega(x)^r dx\right)^{1/r} \leqslant C\left(\frac{1}{|B|}\int_B \omega(x)dx\right)$$

holds for every ball $B \subset \mathbb{R}^n$.

It is well known that if $\omega \in A_p$ with $1 < p < \infty$, then $\omega \in A_r$ for all $r > p$, and $\omega \in A_q$ for some $1 < q < p$. If $\omega \in A_p$ with $1 \leqslant p < \infty$, then there exists $r > 1$ such that $\omega \in RH_r$. It follows directly from Hölder's inequality that $\omega \in RH_r$ implies $\omega \in RH_s$ for all $1 < s < r$. Moreover, if $\omega \in RH_r$, $r > 1$, then we have $\omega \in RH_{r+\varepsilon}$ for some $\varepsilon > 0$. We write $r_\omega = \sup\{r > 1 :\ \omega \in RH_r\}$ to denote the critical index of ω for the reverse Hölder condition.

LEMMA 2.1. ([9]) *Suppose* $\omega \in A_1$. *Then there exist two constant* C_1 *and* C_2, *such that*

$$C_1\omega(B) \leqslant |B| \inf_{x\in B}\omega(x) \leqslant C_2\omega(B). \tag{2.2}$$

LEMMA 2.2. ([9]) *Let* $\omega \in A_p, p \geqslant 1$. *Then, for any ball* B *and any* $\lambda > 1$, *there exists an absolute constant* $C > 0$ *such that*

$$\omega(\lambda B) \leqslant C\lambda^{pn}\omega(B), \tag{2.3}$$

where C *does not depend on* B *nor on* λ.

Next we shall introduce the Hardy-Littlewood maximal operator and several variants, the fractional integral operator and some function spaces.

DEFINITION 2.4. The Hardy-Littlewood maximal operator Mf is defined by

$$M(f)(x) = \sup_{x\in B}\frac{1}{|B|}\int_B |f(y)|dy.$$

For $0 < \delta < 1$, the sharp maximal operator $M^\sharp_\delta f$ is defined by

$$M^\sharp_\delta(f)(x) = \sup_{x\in B}\inf_{c}\left(\frac{1}{|B|}\int_B \left||f(y)|^\delta - |c|^\delta\right| dy\right)^{1/\delta}.$$

For $0 \leqslant \alpha < n$, $r \geqslant 1$, we define the fractional maximal operator $M_{\alpha,r}f$ by

$$M_{\alpha,r}(f)(x) = \sup_{x\in B}\left(\frac{1}{|B|^{1-\alpha r/n}}\int_B |f(y)|^r dy\right)^{1/r},$$

and define the fractional weighted maximal operator $M_{\alpha,r,\omega}f$ by

$$M_{\alpha,r,\omega}f(x)=\sup_{x\in B}\left(\frac{1}{\omega(B)^{1-\alpha r/n}}\int_B|f(y)|^r\omega(y)dy\right)^{1/r},$$

where the above supremum is taken over all balls B containing x. In order to simplify the notation, we set $M_{r,\omega}=M_{0,r,\omega}$ and $M_\omega=M_{1,\omega}$.

DEFINITION 2.5. For $0<\alpha<n$, the fractional integral operator I_α is defined by

$$I_\alpha(f)(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

DEFINITION 2.6. ([10]) Let $1\leqslant p<\infty$ and ω be a weighted function. A locally integrble function b is said $BMO_p(\omega)$, if

$$\sup_B\left[\frac{1}{\omega(B)}\int_B|b(x)-b_B|^p\omega(x)^{1-p}dx\right]^{1/p}\leqslant C<\infty,$$

where $b_B=\frac{1}{|B|}\int_B b(y)dy$ and the supremum is taken over all balls $B\subset\mathbb{R}^n$.

LEMMA 2.3. ([10]) *Let $\omega\in A_1$. Then for ang $1\leqslant p<\infty$, there exists an absolute constant $C>0$ such that $\|b\|_{BMO_p(\omega)}\leqslant C\|b\|_{BMO(\omega)}$.*

The following estimates will play a key role in the proof of our main result.

LEMMA 2.4. ([6]) *Let $0<\delta<1$, $1<p<\infty$, and $0<\kappa<1$. If μ, $\nu\in A_\infty$, then we have*

$$\|M_\delta f\|_{L^{p,\kappa}(\mu,\nu)}\leqslant C\|M_\delta^\sharp f\|_{L^{p,\kappa}(\mu,\nu)}$$

for all functions f such that the left hand side is finite. In particular, when $\mu=\nu=\omega$ and $\omega\in A_\infty$, then we have

$$\|M_\delta f\|_{L^{p,\kappa}(\omega)}\leqslant C\|M_\delta^\sharp f\|_{L^{p,\kappa}(\omega)}$$

for all functions f such that the left hand side is finite.

We list a series of lemmas which will be used in the proof of our theorem.

LEMMA 2.5. ([6]) *Let $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, $0<\kappa<p/q$ and $\omega\in A_\infty$. Then for any $1\leqslant r<p$, we have*

$$\|M_{\alpha,r,\omega}(f)\|_{L^{q,\kappa q/p}(\omega)}\leqslant C\|f\|_{L^{p,\kappa}(\omega)}.$$

LEMMA 2.6. ([6]) *Let* $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, $\omega^{p/q}\in A_1$ *and* $r_\omega>\frac{1-\kappa}{p/q-\kappa}$. *Then for every* $0<\kappa<p/q$, *and* $1\leqslant r<p$ *we have*

$$\|M_{r,\omega}(f)\|_{L^{q,\kappa q/p}(\omega^{q/p},\omega)}\leqslant C\|f\|_{L^{q,\kappa q/p}(\omega^{q/p},\omega)}.$$

LEMMA 2.7. ([6]) *Let* $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, $\omega^{p/q}\in A_1$. *Then if* $0<\kappa<p/q$ *and* $r_\omega>\frac{1-\kappa}{p/q-\kappa}$, *we have*

$$\|M_{\alpha,1}(f)\|_{L^{q,\kappa q/p}(\omega^{q/p},\omega)}\leqslant C\|f\|_{L^{p,\kappa}(\omega)}.$$

LEMMA 2.8. ([2]) *Let* $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, $\omega^{p/q}\in A_1$. *Then if* $0<\kappa<p/q$, *we have*

$$\|I_\alpha(f)\|_{L^{q,\kappa q/p}(\omega^{q/p},\omega)}\leqslant C\|f\|_{L^{p,\kappa}(\omega)}.$$

LEMMA 2.9. ([11]) *Let* I_α *be a fractional integral operator, and let* E *be a measurable set in* $\mathbb{R}^n$. *Then for any* $f\in L^1(\mathbb{R}^n)$, *there exists a constant* C *such that*

$$\int_E |I_\alpha f(x)|\,dx\leqslant C\|f\|_{L^1}|E|^{\alpha/n}.$$

Finally, we recall the definition of variable Calderón-Zygmund kernel and its properties.

DEFINITION 2.7. ([1]) The function $K(x,\xi):\ \mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ is called a variable Calderón-Zygmund kernel if:

(i) for every fixed x, the function $K(x,\cdot)$ is a constant kernel satisfying

  (1) $K(x,\cdot)\in C^\infty(\mathbb{R}^n\setminus\{0\})$;
  (2) for any $\mu>0$, $K(x,\mu\xi)=\mu^{-n}K(x,\xi)$;
  (3) $\int_{\mathbb{S}^{n-1}}K(x,\xi)d\xi=0$ and $\int_{\mathbb{S}^{n-1}}|K(x,\xi)|d\xi<\infty$;

(ii) for every multiindex β, $\sup_{\xi\in\mathbb{S}^{n-1}}|D_\xi^\beta K(x,\xi)|\leqslant C(\beta)$ is independent of x.

We need the spherical harmonics and their properties (see more detail in [1, 12]). Recall that any homogeneous polynomial $P:\mathbb{R}^n\to\mathbb{R}$ of degree m that satisfies $\Delta P=0$ is called an n-dimensional solid harmonic of degree m. Its restriction to the unit sphere $\mathbb{S}^{n-1}$ will be called an n-dimensional spherical harmonic of degree m Denote by $\mathbb{H}_m$ the space of all n-dimensional spherical harmonics of degree m. In general it results in a finite dimensional linear space with $g_m=dim\mathbb{H}_m$ such that $g_0=1$, $g_1=n$ and

$$g_m=C_{m+n-1}^{n-1}-C_{m+n-3}^{n-1}\leqslant C(n)m^{n-2},\qquad m\geqslant 2. \tag{2.4}$$

Furthermore, let $\{Y_{sm}\}_{s=1}^{g_m}$ be an orthonormal base of $\mathbb{H}_m$, then $\{Y_{sm}\}_{s=1m=0}^{g_m\infty}$ is a complete orthonormal system in $L^2(\mathbb{S}^{n-1})$ and

$$\sup_{x\in\mathbb{S}^{n-1}}|D_x^\beta Y_{sm}(x)|\leqslant C(n)m^{|\beta|+(n-2)/2},\qquad m=1,2,\cdots. \tag{2.5}$$

If, for instance, $\phi \in C^\infty(\mathbb{S}^{n-1})$, then $\sum_{s,m} a_{sm}Y_{sm}$ is the Fourier series expansion of $\phi(x)$ with respect to $\{Y_{sm}\}_{sm}$ then

$$a_{sm} = \int_{\mathbb{S}^{n-1}} \phi(y)Y_{sm}(y)d\sigma, \quad |a_{sm}| \leqslant C(n,l)m^{-2l} \sup_{|\beta|=2l} \sup_{y\in\mathbb{S}^{n-1}} |D_y^\beta \phi(y)|. \tag{2.6}$$

for any integer l. In particular, the expansion of ϕ into spherical harmonics converges uniformly to ϕ. For the proof of the above results see [13].

Let $x,y \in \mathbb{R}^n$, and $\overline{y} = y/|y| \in \mathbb{S}^{n-1}$. In view of the properties of the kernel K with respect to the second variable and the complete of $\{Y_{sm}(x)\}$ in $L^2(\mathbb{S}^{n-1})$, we get

$$\begin{aligned} K(x,x-y) &= |x-y|^{-n}K(x,\overline{x-y}) \\ &= |x-y|^{-n} \sum_{m=1}^{\infty} \sum_{s=1}^{g_m} a_{sm}(x)Y_{sm}(\overline{x-y}). \end{aligned}$$

Replacing the kernel with its series expansion, (1.1) can be written as

$$\begin{aligned} T(f)(x) &= \lim_{\varepsilon\to 0} T_\varepsilon(f)(x) \\ &= \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \sum_{m=1}^{\infty} \sum_{s=1}^{g_m} a_{sm}(x)|x-y|^{-n}Y_{sm}(\overline{x-y})f(y)dy. \end{aligned}$$

From the properties of (2.4)-(2.6), the series expansion

$$\sum_{m=1}^{N} \sum_{s=1}^{g_m} |a_{sm}(x)|x-y|^{-n}Y_{sm}(\overline{x-y})f(y)| \leqslant C\frac{|f(y)|}{|x-y|^n} \sum_{m=1}^{\infty} m^{3(n-2)/2-2l},$$

where the integer l is preliminarily chosen greater than $(3n-4)/4$. Along with the $|x-y|^{-n}f(y) \in L^1(\mathbb{R}^n)$ for almost everywhere $x \in \mathbb{R}^n$, by the Fubini dominated convergence theorem, we have

$$\begin{aligned} T(f)(x) &= \sum_{m=1}^{\infty} \sum_{s=1}^{g_m} a_{sm}(x) \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} H_{sm}(x-y)f(y)dy \\ &= \sum_{m=1}^{\infty} \sum_{s=1}^{g_m} a_{sm}(x)T_{sm}f(x). \end{aligned} \tag{2.7}$$

where

$$H_{sm}(x-y) = |x-y|^{-n}Y_{sm}(\overline{x-y}),$$

and H_{sm} satisfies pointwise Hörmander condition as follows:

$$|H_{sm}(x-y) - H_{sm}(x_0-y)| \leqslant Cm^{n/2}\frac{|x_0-x|}{|x-y|^{n+1}} \tag{2.8}$$

for each $x \in B$ and $y \notin 2B$. (see Lemma 3.2 in [13]). Then

$$\begin{aligned} T_{sm}f(x) &= \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} H_{sm}(x-y)f(y)dy \\ &= p.v. \int_{\mathbb{R}^n} H_{sm}(x-y)f(y)dy \end{aligned}$$

is a classical Calderón-Zygmund operator with a constant kernel.

3. Proof of Theorem 1.1

Let

$$T^b(f)(x)=\sum_{k=1}^{Q}(T^{k,1}M^bI_\alpha T^{k,2}(f)(x)+T^{k,3}I_\alpha M^bT^{k,4}(f)(x)).$$

Without loss generality, we may assume $T^{k,1}$ are singular integral operators with variable Calderón-Zygmund kernel ($k=1,\cdots,Q$). By (2.7),

$$\begin{aligned}T^b(f)(x)=&\sum_{k=1}^{Q}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}a_{sm}^{k,1}(x)T_{sm}^{k,1}M^bI_\alpha T^{k,2}(f)(x)\\&+\sum_{k=1}^{Q}T^{k,3}I_\alpha M^bT^{k,4}(f)(x)),\end{aligned}$$

where,

$$T_{sm}^{k,1}(f)(x)=p.v.\int_{\mathbb{R}^n}H_{sm}^{k,1}(x-y)f(y)dy$$

are classical Calderón-Zygmund operators with constant kernels. For arbitrary $x\in\mathbb{R}^n$, set B for the ball centered at x_0 and of radius r, and $B\ni x$. Since $T^1(f)=0$ for any $f\in L^{p,\kappa}(\omega)$, then

$$\begin{aligned}T^b(f)(x)&=T^{b-b_{2B}}(f)(x)\\&=\sum_{k=1}^{Q}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}a_{sm}^{k,1}(x)T_{sm}^{k,1}M^{b-b_{2B}}I_\alpha T^{k,2}(f)(x)+\sum_{k=1}^{Q}T^{k,3}I_\alpha M^{b-b_{2B}}T^{k,4}(f)(x).\end{aligned}\tag{3.1}$$

Let us first prove the following inequality:

$$\begin{aligned}&M_\delta^\sharp T_{sm}^{k,1}M^{b-b_{2B}}I_\alpha T^{k,2}(f)(x)\\&\leqslant Cm^{n/2}\|b\|_{BMO(\omega)}\omega(x)\left(M_{\omega,r}(I_\alpha T^{k,2}f)(x)+M_\omega(I_\alpha T^{k,2}(f))(x)\right),\end{aligned}\tag{3.2}$$

where $1<r<\infty$.

Write $T_{sm}^{k,1}M^{b-b_{2B}}I_\alpha T^{k,2}(f)(x)$ as

$$\begin{aligned}T_{sm}^{k,1}M^{b-b_{2B}}I_\alpha T^{k,2}(f)(y)&=T_{sm}^{k,1}M^{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(y)+T_{sm}^{k,1}M^{(b-b_{2B})\chi_{(2B)^c}}I_\alpha T^{k,2}(f)(y)\\&=\mathrm{U}_1(y)+\mathrm{U}_2(y).\end{aligned}$$

Taking $c=\mathrm{U}_2(x_0)$, then

$$\begin{aligned}&\left(\frac{1}{|B|}\int_B\left||T_{sm}^{k,1}M^{b-b_{2B}}I_\alpha T^{k,2}(f)(y)|^\delta-|c|^\delta\right|dy\right)^{1/\delta}\\&\leqslant\left(\frac{1}{|B|}\int_B|\mathrm{U}_1(y)|^\delta dy\right)^{1/\delta}+\left(\frac{1}{|B|}\int_B|\mathrm{U}_2(y)-\mathrm{U}_2(x_0)|^\delta dy\right)^{1/\delta}\\&=M_1+M_2.\end{aligned}$$

Since $T_{sm}^{k,1}$ are bounded from L^1 to WL^1, by Kolmogorov's inequality, Hölder's inequality and (2.1)-(2.3), we get

$$
\begin{aligned}
M_1 \leqslant & \frac{C}{|B|}\int_B |M^{(b-b_{2B})\chi_{2B}} I_\alpha T^{k,2}(f)(y)|dy \\
\leqslant & C\left(\frac{1}{|B|}\int_{2B}|b(y)-b_{2B}|^{r'}\omega(y)^{1-r'}dy\right)^{1/r'}\left(\frac{1}{|B|}\int_{2B}|I_\alpha T^{k,2}(f)(y)|^r\omega(y)dy\right)^{1/r} \\
\leqslant & C\|b\|_{BMO(\omega)}\frac{\omega(B)}{|B|}\left(\frac{1}{\omega(2B)}\int_{2B}|I_\alpha T^{k,2}(f)(y)|^r\omega(y)dy\right)^{1/r} \\
\leqslant & C\|b\|_{BMO(\omega)}\omega(x)M_{\omega,r}(I_\alpha T^{k,2}(f))(x).
\end{aligned}
$$

For any $y \in B$, and $z \in (2B)^c$, we have $|y-z| \sim |x_0-z|$. Then by Hölder's inequality and (2.8) we get,

$$
\begin{aligned}
M_2 \leqslant & \frac{1}{|B|}\int_B |T_{sm}^{k,1}M^{(b-b_{2B})\chi_{(2B)^c}} I_\alpha T^{k,2}(f)(y) - T_{sm}^{k,1}M^{(b-b_{2B})\chi_{(2B)^c}} I_\alpha T^{k,2}(f)(x_0)|dy \\
\leqslant & \frac{C}{|B|}\int_B\int_{(2B)^c}|b(z)-b_{2B}||H_{sm}^{k,1}(y-z)-H_{sm}^{k,1}(x_0-z)||I_\alpha T^{k,2}(f)(z)|dzdy \\
\leqslant & Cm^{n/2}\frac{1}{|B|}\int_B\int_{(2B)^c}|b(z)-b_{2B}|\frac{|x_0-y|}{|y-z|^{n+1}}|I_\alpha T^{k,2}(f)(z)|dzdy \\
\leqslant & Cm^{n/2}\sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^jB}|b(z)-b_{2B}|\frac{r_B}{|x_0-z|^{n+1}}|I_\alpha T^{k,2}(f)(z)|dz \\
\leqslant & Cm^{n/2}\sum_{j=1}^{\infty}\frac{r_B}{(2^jr_B)^{n+1}}\int_{2^{j+1}B}|b(z)-b_{2B}||I_\alpha T^{k,2}(f)(z)|dz \\
\leqslant & Cm^{n/2}\sum_{j=1}^{\infty}2^{-j}|b_{2^{j+1}B}-b_{2B}|\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|I_\alpha T^{k,2}(f)(z)|dz \\
& + Cm^{n/2}\sum_{j=1}^{\infty}2^{-j}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}||I_\alpha T^{k,2}(f)(z)|dz \\
= & M_{21}+M_{22}.
\end{aligned}
$$

By the defination of $BMO(\omega)$ and $\omega \in A_1$, we have

$$
\begin{aligned}
|b_{2^{j+1}B}-b_{2B}| \leqslant & \sum_{k=1}^{j}\frac{1}{|2^kB|}\int_{2^{k+1}B}|b(z)-b_{2^{k+1}B}|dz \\
\leqslant & C\|b\|_*\sum_{k=1}^{j}\frac{\omega(2^{k+1}B)}{|2^kB|} \\
\leqslant & Cj\omega(x)\|b\|_{BMO(\omega)}.
\end{aligned}
$$

Thus, by Lemma 2.1 we get

$$
\begin{aligned}
M_{21} &= Cm^{n/2}\sum_{j=1}^{\infty}2^{-j}|b_{2^{j+1}B}-b_{2B}|\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|I_\alpha T^{k,2}(f)(z)|dz\\
&\leqslant Cm^{n/2}\sum_{j=1}^{\infty}2^{-j}|b_{2^{j+1}B}-b_{2B}|\frac{1}{\omega(2^{j+1}B)}\int_{2^{j+1}B}|I_\alpha T^{k,2}(f)(z)|\omega(z)dz\\
&\leqslant Cm^{n/2}\|b\|_{BMO(\omega)}\omega(x)M_\omega(I_\alpha T^{k,2}(f))(x).
\end{aligned}
$$

By Hölder's inequality and (2.2), we have

$$
\begin{aligned}
M_{22} &= Cm^{n/2}\sum_{j=1}^{\infty}2^{-j}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}||I_\alpha T^{k,2}(f)(z)|dz\\
&\leqslant Cm^{n/2}\sum_{j=1}^{\infty}2^{-j}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}|^{r'}\omega(z)^{1-r'}dz\right)^{1/r'}\\
&\quad\times\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|I_\alpha T^{k,2}(f)(z)|^{r}\omega(z)dz\right)^{1/r}\\
&\leqslant Cm^{n/2}\|b\|_{BMO(\omega)}\sum_{j=1}^{\infty}2^{-j}\frac{\omega(2^{j+1}B)}{|2^{j+1}B|}\left(\frac{1}{\omega(2^{j+1}B)}\int_{2^{j+1}B}|I_\alpha T^{k,2}(f)(z)|^{r}\omega(z)dz\right)^{1/r}\\
&\leqslant\ Cm^{n/2}\|b\|_{BMO(\omega)}\omega(x)M_{\omega,r}(I_\alpha T^{k,2}(f))(x)\sum_{j=1}^{\infty}2^{-j}\\
&\leqslant\ Cm^{n/2}\|b\|_{BMO(\omega)}\omega(x)M_{\omega,r}(I_\alpha T^{k,2}(f))(x).
\end{aligned}
$$

Combining the estimates for $M_1, M_2,$ it finishes the proof of (3.2).

Since $\omega^{q/p}\in A_1$, by Lemma 2.4, Lemma 2.6, Lemma 2.8 and the boundedness of $T^{k,2}, T^{k,4}$ on $L^{p,\kappa}(\omega)$, we have

$$
\begin{aligned}
&\left\|T^{k,1}_{sm}M^{b-b_{2B}}I_\alpha T^{k,2}(f)\right\|_{L^{q,\kappa q/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&\leqslant C\left\|M^\sharp_\delta T^{k,1}_{sm}M^{b-b_{2B}}I_\alpha T^{k,2}(f)\right\|_{L^{q,\kappa q/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&\leqslant Cm^{n/2}\|b\|_{BMO(\omega)}\left(\left\|M_{r,\omega}(I_\alpha T^{k,2}f)\right\|_{L^{q,\kappa q/p}(\omega^{q/p},\omega)}+\left\|M_\omega(I_\alpha T^{k,2}f)\right\|_{L^{q,\kappa q/p}(\omega^{q/p},\omega)}\right)\\
&\leqslant Cm^{n/2}\|b\|_{BMO(\omega)}\left\|I_\alpha T^{k,2}f\right\|_{L^{q,\kappa q/p}(\omega^{q/p},\omega)}\\
&\leqslant Cm^{n/2}\|b\|_{BMO(\omega)}\left\|T^{k,2}f\right\|_{L^{p,\kappa}(\omega)}\\
&\leqslant Cm^{n/2}\|b\|_{BMO(\omega)}\|f\|_{L^{p,\kappa}(\omega)}.
\end{aligned}
\tag{3.3}
$$

Choosing $l > (3n-2)/4$, then by (2.4), (2.6) and (3.3) we get

$$\begin{aligned}
&\left\|\sum_{k=1}^{Q}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m} a_{sm}^{k,1}T_{sm}^{k,1}M^{b-b_{2B}}I_\alpha T^{k,2}(f)\right\|_{L^{q,\kappa q/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&\leqslant \sum_{k=1}^{Q}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}\|a_{sm}^{k,1}\|_{L^\infty}\|T_{sm}^{k,1}M^{b-b_{2B}}I_\alpha T^{k,2}(f)\|_{L^{q,\kappa q/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&\leqslant C\|b\|_{BMO(\omega)}\|f\|_{L^{p,\kappa}(\omega)}\sum_{k=1}^{Q}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m} m^{-2l+n/2}\\
&\leqslant C\|b\|_{BMO(\omega)}\|f\|_{L^{p,\kappa}(\omega)}\sum_{m=1}^{\infty} m^{-2l+n/2+n-2}\\
&\leqslant C\|b\|_{BMO(\omega)}\|f\|_{L^{p,\kappa}(\omega)}.
\end{aligned}\tag{3.4}$$

Next, we prove

$$\begin{aligned}
&M_\delta^\sharp T^{k,3}I_\alpha M^{b-b_{2B}}T^{k,4}(f)(x)\\
&\leqslant C\|b\|_{BMO(\omega)}\omega(x)^{1-\alpha/n}\Big(M_{\alpha,r,\omega}(T^{k,4}f)(x)+M_{\alpha,1,\omega}(T^{k,4}(f))(x)\Big)\\
&\quad + C\|b\|_{BMO(\omega)}\omega(x)M_{\alpha,1}(T^{k,4}(f))(x).
\end{aligned}\tag{3.5}$$

Write $T^{k,3}I_\alpha M^{b-b_{2B}}T^{k,4}(f)$ as

$$\begin{aligned}
&T^{k,3}I_\alpha M^{b-b_{2B}}T^{k,4}(f)(y)\\
&=T^{k,3}I_\alpha M^{(b-b_{2B})\chi_{2B}}T^{k,4}(f)(y)+T^{k,3}I_\alpha M^{(b-b_{2B})\chi_{(2B)^c}}T^{k,4}(f)(y)\\
&=V_1(y)+V_2(y).
\end{aligned}$$

Taking $c = \mathrm{V}_2(x_0)$, then

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B\Big||T^{k,3}I_\alpha M^{b-b_{2B}}T^{k,4}(f)(y)|^\delta-|c_2|^\delta\Big|dy\right)^{1/\delta}\\
&\leqslant C\left(\frac{1}{|B|}\int_B|\mathrm{V}_1(y)|^\delta dy\right)^{1/\delta}+C\left(\frac{1}{|B|}\int_B|\mathrm{V}_2(y)-\mathrm{V}_2(x_0)|^\delta dy\right)^{1/\delta}\\
&=N_1+N_2.
\end{aligned}$$

Since $T^{k,3}=\pm I$, by Lemma 2.9 and Hölder's inequality, we deduce that

$$\begin{aligned}
N_1&=\frac{C}{|B|}\int_B|I_\alpha M^{(b-b_{2B})\chi_{2B}}T^{k,4}(f)(y)|dy\\
&\leqslant \frac{C}{|B|^{1-\alpha/n}}\int_{R^n}|M^{(b-b_{2B})\chi_{2B}}T^{k,4}(f)(y)|dy\\
&\leqslant \frac{C}{|B|^{1-\alpha/n}}\left(\int_{2B}|b(y)-b_{2B}|^{r'}\omega(y)^{1-r'}dy\right)^{1/r'}\left(\int_{2B}|T^{k,4}(f)(y)|^r\omega(y)dy\right)^{1/r}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\|b\|_{BMO(\omega)}M_{\alpha,r,\omega}(T^{k,4}f)(x)\left(\frac{\omega(B)}{|B|}\right)^{1-\alpha/n}\\
&\leqslant C\|b\|_{BMO(\omega)}\omega(x)^{1-\alpha/n}M_{\alpha,r,\omega}(T^{k,4}f)(x).
\end{aligned}
$$

For any $y\in B$ and $z\in(2B)^c$, we have $|y-z|\sim|x_0-z|$. Thus, by Hölder's inequality,

$$
\begin{aligned}
N_2 &\leqslant \frac{C}{|B|}\int_B |T^{k,3}I_\alpha M^{(b-b_{2B})\chi_{(2B)^c}}T^{k,4}(f)(y)-T^{k,3}I_\alpha M^{(b-b_{2B})\chi_{(2B)^c}}T^{k,4}(f)(x_0)|dy\\
&\leqslant \frac{C}{|B|}\int_B\int_{(2B)^c}|b(z)-b_{2B}|\left|\frac{1}{|y-z|^{n-\alpha}}-\frac{1}{|x_0-z|^{n-\alpha}}\right||T^{k,2}(f)(z)|dzdy\\
&\leqslant \frac{C}{|B|}\int_B\int_{(2B)^c}|b(z)-b_{2B}\frac{|x_0-y|}{|x_0-z|^{n-\alpha+1}}|T^{k,2}(f)(z)|dzdy\\
&\leqslant C\sum_{j=1}^{\infty}\frac{r_B}{(2^jr_B)^{n-\alpha+1}}\int_{2^{j+1}B}|b(z)-b_{2B}||T^{k,2}(f)(z)|dz\\
&\leqslant C\sum_{j=1}^{\infty}2^{-j}|b_{2^{j+1}B}-b_{2B}|\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|T^{k,2}(f)(z)|dz\\
&\quad+C\sum_{j=1}^{\infty}2^{-j}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}||T^{k,2}(f)(z)|dz\\
&=N_{21}+N_{22}.
\end{aligned}
$$

Note

$$|b_{2^{j+1}B}-b_{2B}|\leqslant Cj\|b\|_{BMO(\omega)}\omega(x),$$

then

$$
\begin{aligned}
N_{21}&=C\sum_{j=1}^{\infty}2^{-j}|b_{2^{j+1}B}-b_{2B}|\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|T^{k,4}(f)(z)|dz\\
&\leqslant C\|b\|_{BMO(\omega)}\omega(x)\sum_{j=1}^{\infty}j2^{-j}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|T^{k,4}(f)(z)|dz\\
&\leqslant C\|b\|_{BMO(\omega)}\omega(x)M_{\alpha,1}(T^{k,4}(f))(x).
\end{aligned}
$$

By Hölder's inequality,

$$
\begin{aligned}
N_{22}=&C\sum_{j=1}^{\infty}2^{-j}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}||T^{k,2}(f)(z)|dz\\
&\leqslant C\sum_{j=1}^{\infty}2^{-j}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\left(\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}|^{r'}\omega(z)^{1-r'}dz\right)^{1/r'}\\
&\quad\times\left(\int_{2^{j+1}B}|T^{k,2}(f)(z)|^r\omega(z)dz\right)^{1/r}
\end{aligned}
$$

$$\begin{aligned}
&\leqslant C\|b\|_{BMO(\omega)}\sum_{j=1}^{\infty}2^{-j}\left(\frac{\omega(2^{j+1}B)}{|2^{j+1}B|}\right)^{1-\alpha/n}\\
&\quad\times\left(\frac{1}{\omega(2^{j+1}B)^{1-\alpha r/n}}\int_{2^{j+1}B}|T^{k,2}(f)(z)|^{r}\omega(z)dz\right)^{1/r}\\
&\leqslant\ C\|b\|_{BMO(\omega)}\omega(x)^{1-\alpha/n}M_{\alpha,r,\omega}(T^{k,4}(f))(x).
\end{aligned}$$

Combining the estimates for N_1 and N_2, the proof of (3.5) is completed. Since $\omega^{q/p}\in A_1$, by Lemma 2.7 we get

$$\begin{aligned}
&\left\|\omega(\cdot)M_{\alpha,1}(T^{k,4}(f))\right\|_{L^{q,\kappa q/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&=C\left\|M_{\alpha,1}(T^{k,4}(f))\right\|_{L^{q,\kappa q/p}(\omega^{q/p})}\\
&\leqslant C\|T^{k,4}(f))\|_{L^{p,\kappa}(\omega)}\\
&\leqslant C\|f\|_{L^{p,\kappa}(\omega)}.
\end{aligned}$$

By Lemma 2.5, we get

$$\begin{aligned}
&\left\|\omega(\cdot)^{1-\alpha/n}(M_{\alpha,\omega,r}(T^{k,4}f)+M_{\alpha,\omega,1}(T^{k,4}(f))\right\|_{L^{q,\kappa q/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&=\left\|M_{\alpha,r,\omega}(T^{k,4}f)+M_{\alpha,1,\omega}(T^{k,4}(f))\right\|_{L^{q,\kappa q/p}(\omega^{q/p},\omega)}\\
&\leqslant C\|T^{k,4}f\|_{L^{p,\kappa}(\omega)}\\
&\leqslant C\|f\|_{L^{p,\kappa}(\omega)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left\|\sum_{k=1}^{Q}T^{k,3}I_{\alpha}M^{b-b_{2B}}T^{k,4}(f)\right\|_{L^{q,\kappa q/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&\leqslant C\left\|\sum_{k=1}^{Q}M_{\delta}^{\sharp}T^{k,3}I_{\alpha}M^{b-b_{2B}}T^{k,4}(f)\right\|_{L^{q,\kappa q/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&\leqslant C\sum_{k=1}^{Q}\left\|\omega(\cdot)^{1-\alpha/n}\left(M_{\alpha,r,\omega}(T^{k,4}f)+M_{\alpha,1,\omega}(T^{k,4}(f)\right)\right\|_{L^{q,\kappa q/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&\quad+C\sum_{k=1}^{Q}\left\|\omega(\cdot)M_{\alpha}(T^{k,4}(f))\right\|_{L^{q,\kappa q/p}(\omega^{1-(1-\alpha/n)q},\omega)}\\
&\leqslant C\|f\|_{L^{p,\kappa}(\omega)}.
\end{aligned}\tag{3.6}$$

Combined with (3.1), (3.4) and (3.6), it finishes the proof of Theorem 1.1. □

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